Example: Consider the differential equation y''-y=0
a) Verify that y=e^x is a solution to this differential equation.
b)
c)
d)

a) Verify that **y=e^x** is a solution to this differential equation.
What do we need to do? Find y', y'', plug in to the original:
y=e^x
y'=e^x
y''=e^x
Substitute into y'' - y =0
e^x - e^x =? 0
0=0 YES y=e^x is a solution.

b) Verify that **y=e^(-x)** is a solution.
y'= - e^(-x)
y''=e^(-x)
Substitute: y'' - y =0
e^(-x) - e^(-x)=? 0
0 = 0 YES it's a solution.

The general solution is all functions of the form:
y = ce^x+de^(-x) (c,d are constants) -- "linear combination of the two basic solutions y1=e^x and y2=e^(-x)"

Second order linear differential equations typically have two basic solutions y1 and y2, and the general solution is just a linear combination of those:
y=cy1+dy2

d) find the specific solution to: y'' - y = 0, y(0)=1, y'(0)=3
General solution: **y = ce^x+de^(-x)**
Plug in the initial conditions to find constants:
Plug in y(0)=1: **y = ce^x+de^(-x)**
1 = ce^0+de^(-0)
1 = c+d
Plug in y'(0)=3:
First, find y' = ce^x - de^(-x)
3 = ce^0 - de^(-0)
3 = c - d

We have two equations (in orange) with two unknowns - use them to find c and d.

1=c+d
3=c-d
--------- Add these equations together ----
4=2c+0
4=2c
c=2
1=2+d
d=-1

Specific solution (plug in the values of c and d into the general solution): y=2e^x-e^(-x)

Recall: second order linear differential equation: y''+p(x)y'+q(x)y=f(x)
We are going to look at simpler ones: first, they will be homogeneous (so f(x)=0). Second, we will ONLY allow constants in front of the y' and y (no x's).
We call these:
"Second order linear homogeneous differential equations with constant coefficients"

ay'' + by' + cy = 0

(**we give special permission to allow a constant in front of y" in these equations**)

> Brief aside: 2y''+4y'+6y=0 -- we can easily get rid of the coefficient of y'', by dividing through by it:
> y''+2y'+3y=0)

STEPS TO SOLVE:
STEP 1: Guess a solution of the form y = e^(rx) where r is a constant.
STEP 2: let's plug our guess into the differential equation.
STEP 3: Solve for r (r1 and r2 -- two solutions because we solve a quadratic equations)
STEP 4: should give two solutions, y1=e^(r1x) and y2=e^(r2x)
STEP 5: Combine to get the general solution: y = ce^(r1x)+de^(r2x)

$$y = ce^{(r_1x)} + de^{(r_2x)}$$

EXAMPLE: Find the general solution: y''+6y'+5y=0. Now find the particular solution that satisfies y(0)=3, y'(0)=-1

STEP 1: Guess solution of form y=e^(rx), r is some constant.
STEP 2: Plug in to the differential equation y''+6y'+5y=0

First find y' and y" for our guess:
y'=re^(rx)
y" = r^2 e^(rx)

$$y'' = r^2 e^{(rx)}$$

Now substitute:
r^2 e^(rx) + 6 r e^(rx) + 5 e^(rx) =0 ←- can we solve this for r?

$$r^2 e^{(rx)} + 6r\ e^{(rx)} + 5e^{(rx)} = 0$$

Factor out e^(rx):
e^(rx)(r^2+6r+5)=0

Use "zero product rule" to split this up:
Either e^(rx)=0 or r^2+6r+5=0
The first one is never zero, so it must be that
r^2+6r+5=0 ← this is called the **characteristic polynomial** for this differential equation
Let's solve this: $r^2 + 6r + 5 = 0$
(r + 5)(r+1)=0 GREAT!
What are the solutions for r?
r=-5 and r=-1

This gives us two different basic solutions:
y=e^(-5x) and y=e^(-x)

STEP 5:
What is the general solution to y"+6y'+5y=0?
General solution: y=ce^(-5x)+de^(-x), where c,d are constants.

Find the specific solution satisfying y(0)=3, y'(0)=-1
Plug in y(0)=3 into **y=ce^(-5x)+de^(-x),**
3=ce^0 + de^0
3=c+d

Need to plug in y'(0)=-1,
First need to find y' if **y=ce^(-5x)+de^(-x),**
y'=-5ce^(-5x) - de^(-x)
-1 = -5ce^0 - de^0
-1 = -5c - d

Solve this system of two equations:
3=c+d
-1 = -5c - d
------ add them -----
2 = -4c

c=-1/2
3=-1/2+d
d=7/2

$y = -\frac{1}{2}e^{-5x} + \frac{7}{2}e^{-x}$ $\leftarrow$ **particular solution satisfying the initial conditions.**

3=c+d
5=2c-11d