MANGALORE INSTITUTE OF TECHNOLOGY & ENGINEERING

Kinematics and Dynamics for Robotics

MODULE 2

Forward Kinematics

Detailed
Exam-Oriented Notes

Course: Kinematics and Dynamics for Robotics
Module: 2 - Forward Kinematics
Coverage: Syllabus Module 2 only
Textbook scope: Sections 4.1.1-4.1.3 and 4.2

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1 Module 2 at a Glance

The syllabus specifies Module 2 as **Forward Kinematics**, with the following topics: *Product of Exponentials Formula; First Formulation - Screw Axes in the Base Frame; Second Formulation - Screw Axes in the End-Effector Frame; and the Universal Robot Description Format*. The supplied syllabus maps these topics to textbook sections 4.1.1-4.1.3 and 4.2.[1]

> **Key idea. Forward kinematics** means calculating the position and orientation of the robot end-effector frame from the joint coordinates θ. For an open-chain robot, the Product of Exponentials (PoE) formula describes this configuration using the robot's home configuration and its joint screw axes.

2 Forward Kinematics: Basic Idea

Consider an open-chain robot with joint variables

$$\theta = (\theta_1, \theta_2, \ldots, \theta_n).$$

The end-effector configuration is a homogeneous transformation

$$T(\theta) = \begin{bmatrix} R(\theta) & p(\theta) \\ 0\,0\,0 & 1 \end{bmatrix} \in \mathrm{SE}(3),$$

where R(θ) gives orientation and p(θ) gives position.

For a simple planar 3R arm with link lengths L_1, L_2, L_3, the direct trigonometric form is

$$x = L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2) + L_3 \cos(\theta_1 + \theta_2 + \theta_3),$$

$$y = L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2) + L_3 \sin(\theta_1 + \theta_2 + \theta_3),$$

$$\phi = \theta_1 + \theta_2 + \theta_3.$$

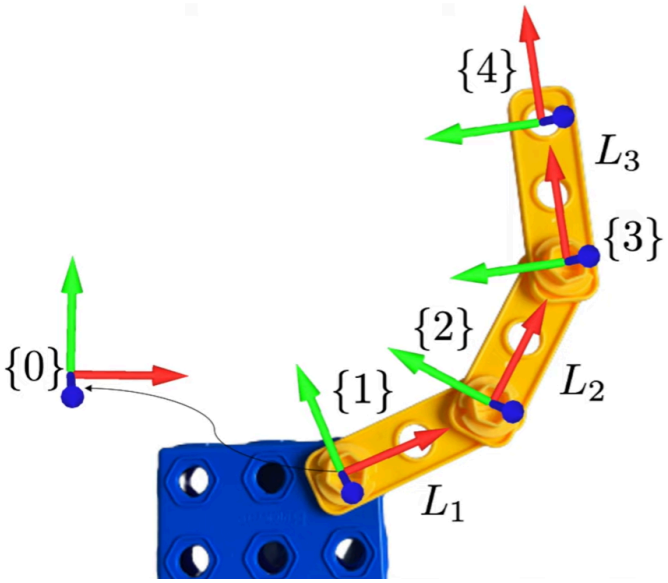


Figure 1: 3R planar open chain and its reference frames, reproduced from the supplied Module 2 PPT.

[1]Source: supplied KDR syllabus.

2.1 Homogeneous transformation viewpoint

The same 3R planar chain can be represented by frame-to-frame homogeneous transformations:

$$T_{04} = T_{01}T_{12}T_{23}T_{34}.$$

A useful set of matrices is

$$T_{01} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$T_{12} = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & L_1 \\ \sin\theta_2 & \cos\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad T_{23} = \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 & L_2 \\ \sin\theta_3 & \cos\theta_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$T_{34} = \begin{bmatrix} 1 & 0 & 0 & L_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Key idea. The matrix-product approach works well for small examples, but spatial robots quickly become cumbersome if every link frame is treated separately. The PoE formulation provides a systematic alternative in which the main kinematic data are the home configuration and joint screw axes.

3 Product of Exponentials Formula

3.1 Home configuration

Let $\mathbf{M} \in \mathrm{SE}(3)$ denote the end-effector frame configuration when all joint variables are zero. This is the robot's home, or zero, configuration.

For the 3R planar arm,

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & L_1 + L_2 + L_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

3.2 Screw axis of a joint

A one-degree-of-freedom joint is represented by a six-vector

$$\mathbf{S}_i = \begin{bmatrix} \omega_i \\ v_i \end{bmatrix}.$$

The angular part is ω_i and the linear part is v_i.

For a revolute joint:

- ω_i is a unit vector along the positive joint axis.
- Choose any point q_i on the joint axis.
- Then

$$v_i = -\omega_i \times q_i.$$

For a prismatic joint:

$$\omega_i = 0,$$
$$v_i = \text{unit vector in the positive translation direction.}$$

The joint variable is an angle for a revolute joint and a displacement for a prismatic joint.

3.3 Matrix representation of a screw axis

For

$$\mathbf{S} = \begin{bmatrix} \omega \\ v \end{bmatrix}, \quad \omega = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix},$$

the skew-symmetric matrix is

$$[\omega] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}.$$

The corresponding 4×4 matrix is

$$[\mathbf{S}] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 & v_1 \\ \omega_3 & 0 & -\omega_1 & v_2 \\ -\omega_2 & \omega_1 & 0 & v_3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \in \mathfrak{se}(3).$$

3.4 Meaning of the matrix exponential

The matrix exponential

$$e^{[\mathbf{S}]\theta}$$

represents the finite rigid-body displacement generated by the screw axis **S** through joint displacement θ. For a revolute joint, this is a pure rotation about the joint axis (zero pitch). For a prismatic joint, it is a pure translation along the joint axis.

4 First Formulation: Screw Axes in the Base Frame

4.1 Idea

Choose:

1. a fixed base frame {s},
2. an end-effector frame {b},

3. the home configuration **M**,
4. the screw axis $\mathbf{S}_i$ of every joint, expressed in the fixed base frame when the robot is in its home position.

Each joint displacement can be viewed as applying a screw motion to all outward links. Starting from the most distal joint and then including the more proximal joints gives the full forward-kinematics expression.

4.2 Space-form PoE formula

For an n-joint open chain,

$$T(\theta) = e^{[\mathbf{S}_1]\theta_1} e^{[\mathbf{S}_2]\theta_2} \cdots e^{[\mathbf{S}_n]\theta_n} \mathbf{M}$$

This is the **space form** because the screw axes are expressed in the fixed space/base frame.

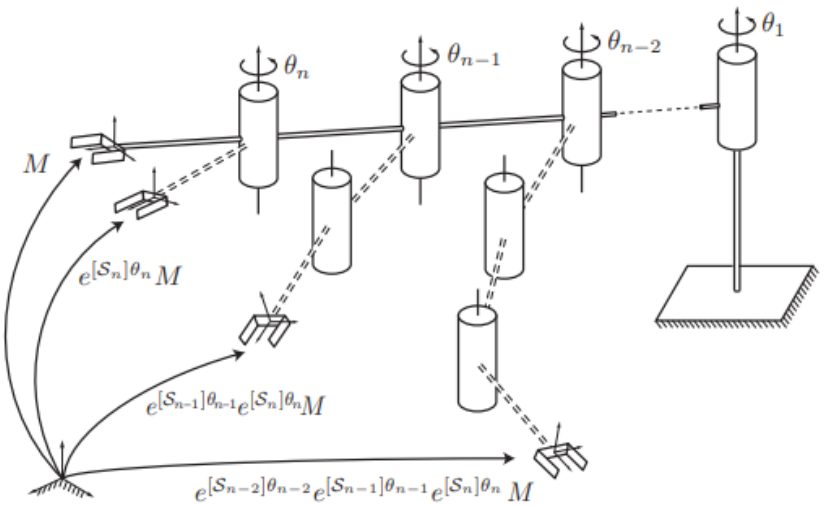


Figure 2: Illustration of the PoE formula for an n-link spatial open chain.

4.3 Data required to use the space form

The three essential ingredients are:

1. **M**: end-effector configuration at the home position;
2. $\mathbf{S}_1, \ldots, \mathbf{S}_n$: joint screw axes expressed in the fixed base frame at home;
3. $\theta_1, \ldots, \theta_n$: current joint variables.

No link-by-link reference-frame chain is required just to write the PoE formula.

4.4 Detailed procedure for a revolute joint

To construct $\mathbf{S}_i$:

1. Inspect the positive joint-axis direction and write the unit vector ω_i.
2. Select any convenient point q_i on that axis.
3. Compute

$$v_i = -\omega_i \times q_i.$$

4. Stack the two vectors:

$$\mathbf{S}_i = \begin{bmatrix} \omega_i \\ v_i \end{bmatrix}.$$

5. Repeat for every joint.
6. Multiply the exponentials in joint order and finish with $\mathbf{M}$.

4.5 Example: 3R spatial open chain

For the 3R spatial arm in the supplied material, the home configuration is

$$\mathbf{M} = \begin{bmatrix} 0 & 0 & 1 & L_1 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & -L_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The three screw axes expressed in the fixed frame are

$$\mathbf{S}_1 = (0, 0, 1,\ 0, 0, 0)^T,$$

$$\mathbf{S}_2 = (0, -1, 0,\ 0, 0, -L_1)^T,$$

$$\mathbf{S}_3 = (1, 0, 0,\ 0, -L_2, 0)^T.$$

Hence,

$$T(\theta) = e^{[\mathbf{S}_1]\theta_1} e^{[\mathbf{S}_2]\theta_2} e^{[\mathbf{S}_3]\theta_3} \mathbf{M}.$$

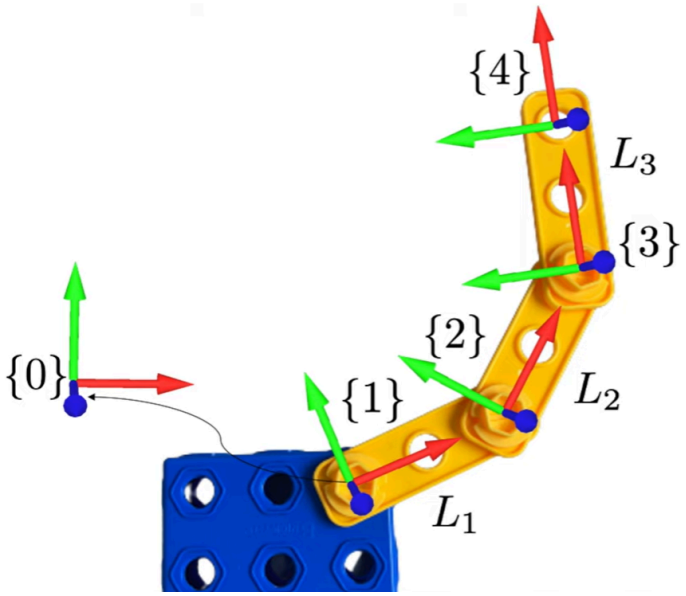


Figure 3: A representative 3R robot with frames and joint axes.

4.6 Example: 3R planar open chain

For the planar arm,

$$\mathbf{S}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{S}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -L_1 \\ 0 \end{bmatrix}, \quad \mathbf{S}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -(L_1 + L_2) \\ 0 \end{bmatrix}.$$

Using the space-form PoE formula,

$$T(\theta) = e^{[\mathbf{S}_1]\theta_1} e^{[\mathbf{S}_2]\theta_2} e^{[\mathbf{S}_3]\theta_3} \mathbf{M},$$

where

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & L_1 + L_2 + L_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

4.7 Example: 6R spatial open chain

A six-degree-of-freedom arm has the minimum number of joints needed for a general rigid-body end-effector pose in SE(3), subject to the robot's workspace and joint limits.

For the 6R arm shown in the supplied material, the home configuration is

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3L \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

and the space screw axes are

i	ω_i	v_i
1	(0, 0, 1)	(0, 0, 0)
2	(0, 1, 0)	(0, 0, 0)
3	(−1, 0, 0)	(0, 0, 0)
4	(−1, 0, 0)	(0, 0, L)
5	(−1, 0, 0)	(0, 0, 2L)
6	(0, 1, 0)	(0, 0, 0)

The forward kinematics is still simply

$$T(\theta) = e^{[\mathbf{S}_1]\theta_1} e^{[\mathbf{S}_2]\theta_2} e^{[\mathbf{S}_3]\theta_3} e^{[\mathbf{S}_4]\theta_4} e^{[\mathbf{S}_5]\theta_5} e^{[\mathbf{S}_6]\theta_6} \mathbf{M}.$$

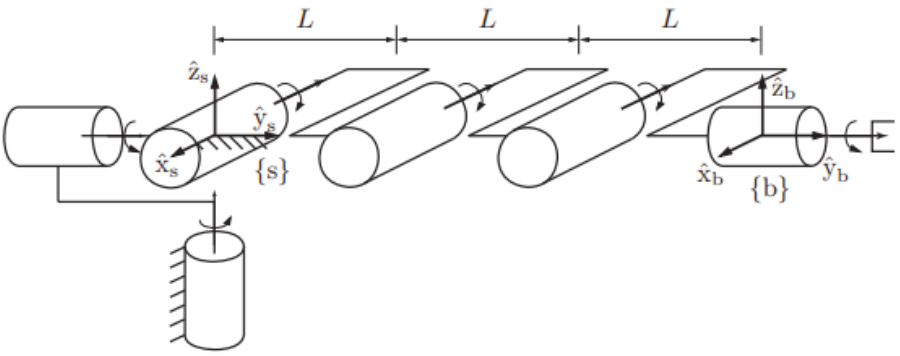


Figure 4: 6R spatial open chain used to illustrate the PoE formulation.

4.8 Example: RRPRRR spatial open chain

For a prismatic joint, remember that $\omega = 0$. In the supplied RRPRRR example, the third joint is prismatic, so

$$\mathbf{S}_3 = (0, 0, 0,\ 0, 1, 0)^T.$$

The screw-axis data given in the source are

i	ω_i	v_i
1	(0, 0, 1)	(0, 0, 0)
2	(1, 0, 0)	(0, 0, 0)
3	(0, 0, 0)	(0, 1, 0)
4	(0, 1, 0)	(0, 0, 0)
5	(1, 0, 0)	(0, 0, $-L_1$)
6	(0, 1, 0)	(0, 0, 0)

This example is useful for remembering that PoE handles mixed revolute-prismatic chains without changing the overall structure of the formula.

4.9 Example: UR5 6R robot arm

The supplied material uses the UR5 as a practical 6R example. The geometry is specified by W_1, W_2, L_1, L_2, H_1, H_2, and the space screw axes are tabulated in the source.

For the stated joint configuration

ɔn

$$\theta_2 = -\frac{\pi}{2}, \qquad \theta_5 = \frac{\pi}{2},$$

with the other joint variables zero,

ɛro,

$$T(\theta) = e^{-[S_2]\pi/2} e^{[S_5]\pi/2} \mathbf{M}.$$

The supplied example evaluates this to

$$T(\theta) = \begin{bmatrix} 0 & -1 & 0 & 0.095 \\ 1 & 0 & 0 & 0.109 \\ 0 & 0 & 1 & 0.988 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

with linear dimensions expressed in metres.

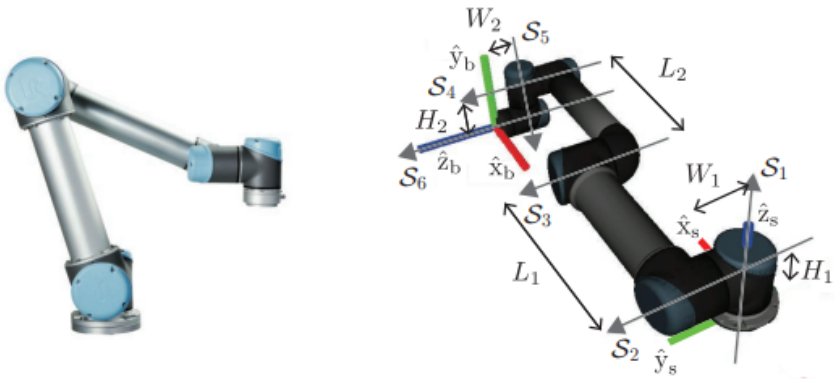


Figure 5: UR5 6R robot arm and its space-frame screw axes.

5 Second Formulation: Screw Axes in the End-Effector Frame

5.1 Why a second form?

The same robot can be described using the joint screw axes expressed in the end-effector (body) frame at the home position. Let these body-frame screw axes be

$$\mathbf{B}_i = \begin{bmatrix} \omega_i^b \\ v_i^b \end{bmatrix}.$$

The body formulation is especially useful when the joint axes are more convenient to express from the end-effector side.

5.2 Space form and body form

Starting with

$$T(\theta) = e^{[\mathbf{S}_1]\theta_1} \cdots e^{[\mathbf{S}_n]\theta_n}\mathbf{M},$$

use the similarity transformation identity to move $\mathbf{M}$ to the left. The result is

$$\boxed{T(\theta) = \mathbf{M}e^{[\mathbf{B}_1]\theta_1}e^{[\mathbf{B}_2]\theta_2}\cdots e^{[\mathbf{B}_n]\theta_n}}$$

This is the **body form** of the PoE formula.

The individual body screw axes are related to the space screw axes by

$$[\mathbf{B}_i] = \mathbf{M}^{-1}[\mathbf{S}_i]\mathbf{M},$$

or equivalently,

$$\boxed{\mathbf{B}_i = [\mathrm{Ad}_{\mathbf{M}^{-1}}]\mathbf{S}_i.}$$

5.3 Adjoint representation used in the conversion

For a homogeneous transformation

$$T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix},$$

the 6×6 adjoint matrix has the block form

$$[\mathrm{Ad}_T] = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix}.$$

Therefore,

$$\mathbf{B}_i = [\mathrm{Ad}_{\mathbf{M}^{-1}}]\mathbf{S}_i.$$

This relation converts a screw-axis representation from the space frame to the body frame at the home position.

5.4 How the order should be understood

- In the space form, the rightmost joint transformation acts first on **M**; the effect is then propagated inward through the more proximal joints.
- In the body form, **M** is on the left and the product progresses from the first joint toward the more distal joints.
- In either representation, the screw axes are determined at the home position. A distal joint displacement does not alter the fixed-frame description needed for the space form, and a proximal joint displacement does not alter the body-frame description needed for the body form.

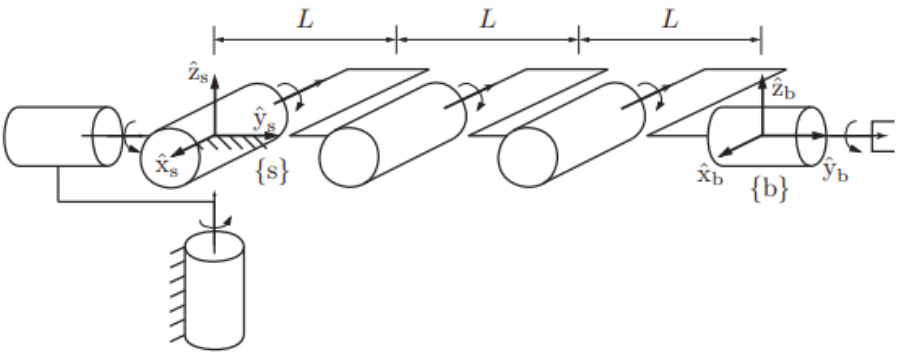


Figure 6: A 6R chain can be described with either space-frame or body-frame screw axes.

5.5 Example: 6R chain in body form

For the supplied 6R example, the body screw axes are

i	ω_i	v_i
1	(0, 0, 1)	(−3L, 0, 0)
2	(0, 1, 0)	(0, 0, 0)
3	(−1, 0, 0)	(0, 0, −3L)
4	(−1, 0, 0)	(0, 0, −2L)
5	(−1, 0, 0)	(0, 0, −L)
6	(0, 1, 0)	(0, 0, 0)

Hence,

$$T(\theta) = \mathbf{M} e^{[\mathbf{B}_1]\theta_1} e^{[\mathbf{B}_2]\theta_2} e^{[\mathbf{B}_3]\theta_3} e^{[\mathbf{B}_4]\theta_4} e^{[\mathbf{B}_5]\theta_5} e^{[\mathbf{B}_6]\theta_6}.$$

5.6 Example: Barrett WAM 7R robot arm

The WAM has seven revolute joints, so its joint space has one more dimension than the six-dimensional end-effector pose space SE(3). The supplied source notes that this extra degree of freedom makes the arm kinematically redundant for general end-effector positioning.

At the home configuration, the body screw axes listed in the source are

$\mathbf{i}$	$\boldsymbol{\omega}_\mathbf{i}$	$\mathbf{v}_\mathbf{i}$
1	(0, 0, 1)	(0, 0, 0)
2	(0, 1, 0)	($L_1 + L_2 + L_3$, 0, 0)
3	(0, 0, 1)	(0, 0, 0)
4	(0, 1, 0)	($L_2 + L_3$, 0, W_1)
5	(0, 0, 1)	(0, 0, 0)
6	(0, 1, 0)	(L_3, 0, 0)
7	(0, 0, 1)	(0, 0, 0)

The home configuration is

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & L_1 + L_2 + L_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

ırce’s illustrative configuration

For the source’s illustrative configuration

$$\theta_2 = 45^\circ, \qquad \theta_4 = -45^\circ, \qquad \theta_6 = -90^\circ,$$

with all other joint variables zero,

$$T(\theta) = \mathbf{M} e^{[\mathbf{B}_2]\pi/4} e^{-[\mathbf{B}_4]\pi/4} e^{-[\mathbf{B}_6]\pi/2}$$

$$T(\theta) = \begin{bmatrix} 0 & 0 & -1 & 0.3157 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0.6571 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

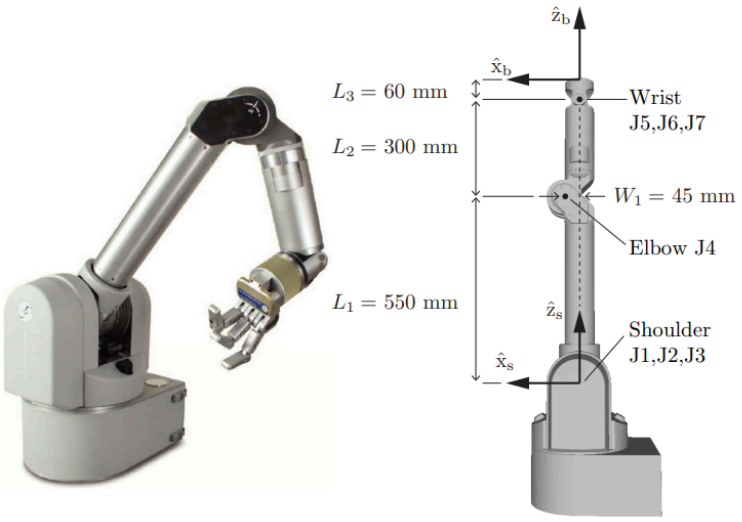


Figure 7: Barrett WAM 7R arm and its zero-configuration geometry.

6 Universal Robot Description Format (URDF)

6.1 Definition and purpose

URDF (Universal Robot Description Format) is an XML-based file format used in ROS to describe a robot's **kinematics, link geometry, and inertial properties**. A URDF describes the robot as links connected by joints.

Key idea. URDF is mainly a **description format**. It does not replace the PoE equations; instead, it stores the robot model information from which kinematic and dynamic software can construct the robot model.

6.2 Joints in URDF

A URDF joint connects exactly two links:

- parent: the parent link;
- child: the child link;
- type: for example revolute, continuous, prismatic, or fixed;
- origin: the position and orientation of the child-link frame relative to the parent-link frame when the joint variable is zero;
- axis: a unit vector giving the positive rotation direction for a revolute joint or the positive translation direction for a prismatic joint.

A representative revolute joint entry is

```
<joint name="joint2" type="continuous">
          <parent link="link1"/>
          <child link="link2"/>
<origin rpy="0.0 1.5708 0.0" xyz="0.0 0.13585 0.0"/>
          <axis xyz="0 1 0"/>
          </joint>
```

6.3 Joint origin

The origin element specifies the relative pose of the child frame at the zero joint position. It is written with:

```
xyz="x y z",  rpy="roll pitch yaw".
```

The supplied material describes the RPY sequence as:

1. roll about the fixed $\hat{x}_a$ axis,
2. pitch about the fixed $\hat{y}_a$ axis,
3. yaw about the fixed $\hat{z}_a$ axis.

6.4 Joint axis

For a revolute joint, the axis is a unit vector expressed in the child-link frame. The direction defines the positive sense of rotation.

For a prismatic joint, the same axis field specifies the positive direction of translation.

6.5 Links in URDF

A link contains the physical properties of a rigid body. In the supplied material, a link description includes:

- mass;
- an origin frame locating and orienting the center-of-mass frame relative to the link frame;
- an inertia matrix represented by the six independent entries of the symmetric 3 × 3 inertia tensor.

A typical inertial block has the form

```
<inertial>
  <mass value="..."/>
  <origin rpy="..." xyz="..."/>
  <inertia ixx="..." ixy="..." ixz="..."
           iyy="..." iyz="..." izz="..."/>
</inertial>
```

6.6 Tree structure

A URDF represents a robot using a tree structure: links are the nodes and joints are the connecting edges. Serial-chain robot arms and robot hands can be represented this way.

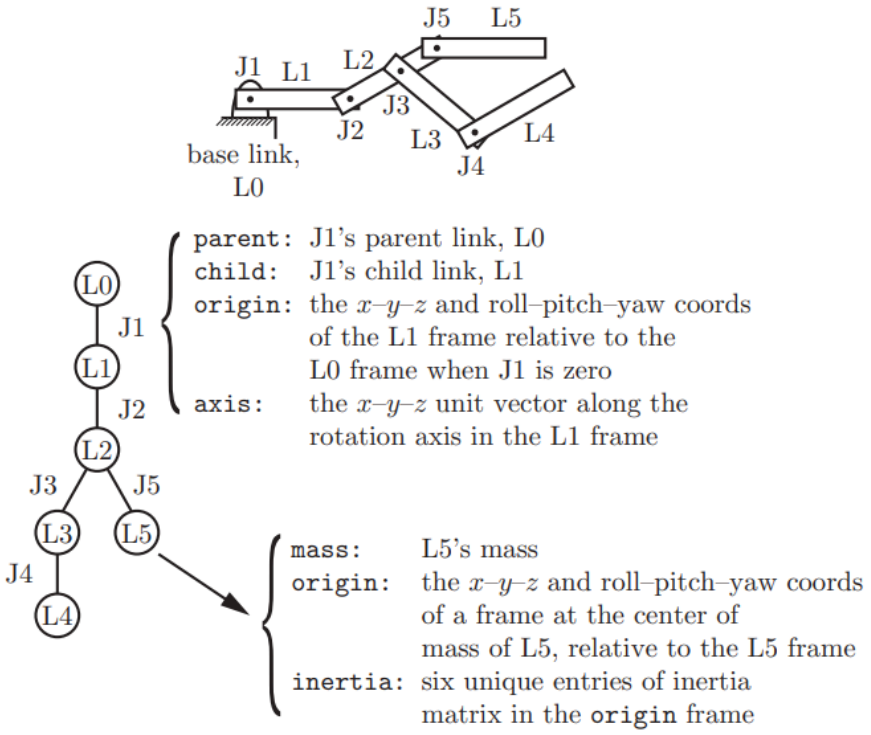


Figure 8: Five-link robot represented as a tree with joints as edges and links as nodes.

The supplied textbook explicitly notes that standard URDF describes tree-structured robots and does not directly represent closed-loop mechanisms such as a Stewart platform.

6.7 UR5 in URDF

The supplied textbook uses the UR5 to demonstrate how joint and link data are encoded. The kinematic portion contains:

- a fixed world-to-base joint;
- joints 1 through 6;
- parent and child link assignments;
- a relative origin for each joint;
- the joint axis vector;
- a fixed end-effector attachment joint.

The same URDF file also contains link inertial properties. The source notes that the listed mass and inertial data are illustrative rather than exact.

7 Comparison of the Two PoE Formulations

Feature	Space form	Body form
Screw-axis frame	Fixed base/space frame	End-effector/body frame at home
Main formula	$T = e^{[S_{1}]\theta_1} \cdots e^{[S_{n}]\theta_n} M$	$T = M e^{[B_{1}]\theta_1} \cdots e^{[B_{n}]\theta_n}$
Home configuration	M on the right	M on the left
Axis conversion	Start directly from S_i in space frame	$B_i = Ad_{M^{-1}} S_i$

Feature	Space form	Body form
Geometric viewpoint	Proximal transformations successively act on outward links	Start from home pose and work outward through the joints
Key requirement	S_i determined at home in the space frame	B_i determined at home in the body frame

8 PoE and D-H: What to Remember from this Module

The supplied textbook briefly contrasts PoE with the Denavit-Hartenberg representation. This module does not require a detailed D-H treatment, but the conceptual distinction is important.

- D-H uses link reference frames and a compact parameterization.
- PoE uses screw axes and the home configuration.
- PoE does not require a chain of link frames to write the forward-kinematics formula.
- For an n-joint robot, PoE stores six quantities per screw axis, while D-H uses a smaller parameter set for the robot geometry.

9 Illustrative Worked Calculations

These examples stay within the Module 2 topics and are intended to make the formulas easier to apply in examination problems.

9.1 Worked calculation 1: 3R planar position

Given

$$L_1 = 1.0, \quad L_2 = 0.8, \quad L_3 = 0.5,$$
$$\theta_1 = 30^\circ, \quad \theta_2 = 45^\circ, \quad \theta_3 = -20^\circ,$$

$$\theta_1 + \theta_2 = 75^\circ, \qquad \theta_1 + \theta_2 + \theta_3 = 55^\circ.$$

$$x = 1\cos 30^\circ + 0.8\cos 75^\circ + 0.5\cos 55^\circ \approx 1.262,$$
$$y = 1\sin 30^\circ + 0.8\sin 75^\circ + 0.5\sin 55^\circ \approx 1.670,$$
$$\phi = 55^\circ.$$

9.2 Worked calculation 2: Revolute screw axis

Let

$$\omega = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad q = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}.$$

Then

$$\omega \times q = \begin{bmatrix}0\\0\\1\end{bmatrix} \times \begin{bmatrix}2\\3\\0\end{bmatrix} = \begin{bmatrix}-3\\2\\0\end{bmatrix}.$$

Hence

$$v = -\omega \times q = \begin{bmatrix}3\\-2\\0\end{bmatrix},$$

so the screw axis is

$$\mathbf{S} = \begin{bmatrix}0\\0\\1\\3\\-2\\0\end{bmatrix}.$$

9.3 Worked calculation 3: Prismatic screw axis

For a prismatic joint translating in the positive x direction,

$$\omega = \begin{bmatrix}0\\0\\0\end{bmatrix}, \qquad v = \begin{bmatrix}1\\0\\0\end{bmatrix}.$$

Thus

$$\mathbf{S} = \begin{bmatrix}0\\0\\0\\1\\0\\0\end{bmatrix}.$$

If the joint displacement is d, the screw exponential produces a translation of distance d along x.

9.4 Worked calculation 4: Writing the PoE model

For a 3R open chain with known M, S_1, S_2, and S_3, the correct space-form expression is

$$T(\theta) = e^{[S_1]\theta_1} e^{[S_2]\theta_2} e^{[S_3]\theta_3} M.$$

The important exam step is not to multiply blindly first. First identify:

$$M, \qquad S_1, \qquad S_2, \qquad S_3,$$

then form the ordered product.

9.5 Worked calculation 5: Space-to-body conversion

Suppose M and a space screw axis S_i are known. The body screw axis is obtained from

$$[B_i] = M^{-1}[S_i]M,$$

or

$$B_i = \mathrm{Ad}_{M^{-1}} S_i.$$

The procedure is therefore:

1. write M;
2. calculate M^{-1};
3. form $M^{-1}[S_i]M$;
4. read the resulting 4 × 4 matrix back as the twist matrix of B_i.

10 Exam-Oriented Step-by-Step Methods

10.1 Method A: Solve a PoE forward-kinematics problem in space form

1. Draw or inspect the robot in its home position.
2. Set all joint variables to zero.
3. Assign the space/base frame and end-effector frame.
4. Determine the home matrix M.
5. For each joint, write ω_i in the positive axis direction.
6. For a revolute joint choose a point q_i and compute $v_i = -\omega_i \times q_i$.
7. For a prismatic joint set $\omega_i = 0$ and choose unit v_i along the positive translation direction.
8. Build $[S_i]$ for each joint if matrix exponentials are required.
9. Substitute the joint variables into

$$T(\theta) = e^{[S_1]\theta_1} \cdots e^{[S_n]\theta_n} M.$$

10. Simplify the final 4 × 4 homogeneous matrix.

10.2 Method B: Solve using body-frame screw axes

1. Start from the home matrix M.
2. Determine the space screw axes S_i if they are the easiest quantities to obtain.
3. Convert using

$$B_i = \mathrm{Ad}_{M^{-1}} S_i.$$

4. Form

$$T(\theta) = M e^{[B_1]\theta_1} \cdots e^{[B_n]\theta_n}.$$

10.3 Method C: Read a URDF kinematic model

When given a URDF file, identify the following in order:

1. robot name;
2. base/world link;

3. every joint and its type;
4. parent and child links;
5. joint origin (x, y, z) and RPY orientation;
6. joint axis;
7. fixed end-effector attachment;
8. link mass and inertia, when required.

11 Common Mistakes to Avoid

1. **Using the wrong joint-axis direction.** The sign of ω_i must match the chosen positive joint motion.
2. **Forgetting the minus sign in a revolute screw axis.** Use $v_i = -\omega_i \times q_i$.
3. **Mixing frames.** A space screw axis and a body screw axis are not interchangeable without the adjoint transformation.
4. **Putting M in the wrong position.** Space form has M on the right; body form has M on the left.
5. **Ignoring the prismatic case.** For a prismatic joint, $\omega = 0$ and v is the unit translation direction.
6. **Treating URDF as a closed-loop mechanism description.** The supplied source describes URDF as a tree representation and does not directly represent closed loops.

12 Quick Formula Sheet

Homogeneous transformation

$$T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \in \mathrm{SE}(3).$$

3R planar direct kinematics

$$x = L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2) + L_3 \cos(\theta_1 + \theta_2 + \theta_3),$$

$$y = L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2) + L_3 \sin(\theta_1 + \theta_2 + \theta_3),$$

$$\phi = \theta_1 + \theta_2 + \theta_3.$$

Revolute screw axis

$$S = \begin{bmatrix} \omega \\ v \end{bmatrix}, \qquad v = -\omega \times q, \qquad \|\omega\| = 1.$$

Prismatic screw axis

$$\omega = 0, \qquad \|v\| = 1.$$

Screw matrix

$$[S] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix}.$$

Space form of PoE

$$\boxed{T(\theta) = e^{[S_1]\theta_1} \cdots e^{[S_n]\theta_n} M}.$$

Body form of PoE

$$\boxed{T(\theta) = M e^{[B_1]\theta_1} \cdots e^{[B_n]\theta_n}}.$$

Space-to-body screw-axis conversion

$$[B_i] = M^{-1}[S_i]M,$$

$$\boxed{B_i = \mathrm{Ad}_{M^{-1}} S_i}.$$

Adjoint matrix

$$T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \quad \Longrightarrow \quad [\mathrm{Ad}_T] = \begin{bmatrix} R & 0 \\ [p]\,R & R \end{bmatrix}.$$

13 Module 2 Revision Checklist

Before an examination, make sure you can do all of the following without referring to notes:

- Explain forward kinematics in one or two sentences.
- Write the 3R planar position and orientation equations.
- Define the home configuration M.
- Define a screw axis S = (ω, v).
- Derive v = −ω × q for a revolute joint.
- State the space-form PoE formula.
- List the three pieces of data needed for the space form.
- Construct screw axes for revolute and prismatic joints.
- State and use the body-form PoE formula.
- Convert S_i to B_i using the adjoint relationship.
- Explain the difference between space and body formulations.
- Define URDF and list the main joint and link fields.
- Explain parent, child, origin, axis, mass, and inertia in URDF.
- Explain why URDF represents tree structures and not general closed loops.

14 Source Alignment

These notes are restricted to the Module 2 topics listed in the supplied syllabus and use the supplied Module 2 textbook PDF and PPT as the basis. The core organization follows the textbook sequence around Sections 4.1.1-4.1.3 and 4.2, while the figures and slide presentation were also checked against the supplied Module 2 PPT.

End of Module 2 - Forward Kinematics