Competition Math Week 4 Homework:
Email sui.gabriel8@gmail.com or belmontfreelessons@gmail.com if you have any questions

**Review:**
Vieta's with quadratics:

Find the sum and product of the roots of the following quadratics:
$x^2 + 5x - 18$
Sum = -5
Product = -18

$x^2 + 14x + 8$
Sum = -14
Product = 8

Given that the following quadratic has roots r and s
$x^2 + ax + b$
And rs = 4, r+s = -7
Find a and b

Notice a = -(r+s) = 7
b = rs = 4

Given that the following quadratic has roots 4 and r
$x^2 + ax + 8$
Find r and a

4+r = -a
4r = 8 -> r = 2
a = -6

Polynomials:
Divide the following expressions using synthetic division
$x^3 + 2x^2 - 5x - 6$ divided by x-2

| | | | | |
|---|---|---|---|---|
| 2 | 1 | 2 | -5 | -6 |
| | | 2 | 8 | 6 |
| | 1 | 4 | 3 | 0 |

-> $x^2 + 4x + 3$

$x^3 + 4x^2 - 16x - 15$ divided by $x-3$

| | | | | |
|---|---|---|---|---|
| 3 | 1 | 4 | -16 | -15 |
| | | 3 | 21 | 15 |
| | 1 | 7 | 5 | 0 |

-> $x^2 + 7x + 5$

$x^4 + x^3 - 18x^2 + 13x + 15$ divided by $x+5$

| | | | | | |
|---|---|---|---|---|---|
| -5 | 1 | 1 | -18 | 13 | 15 |
| | | -5 | 20 | -10 | -15 |
| | 1 | -4 | 2 | 3 | 0 |

-> $x^3 - 4x^2 + 2x + 3$

Factor the following polynomials using any method:

$x^3 + 8x^2 + x - 42 = 0$

$(x-2)(x+3)(x+7)$

$x^4 - 3x^3 - 27x^2 -13x + 42 = 0$

$(x-1)(x+2)(x+3)(x-7)$

$4x^3 - 9x^2 - 81x - 54 = 0$

$(x+3)(x-6)(4x+3)$

List all possible rational roots in the following polynomials:

$x^3 + 18x^2 - 14x + 92 = 0$

±factors of 92

= ±1,2,4,23,46,92

$4x^5 + 3x^3 - 12x + 5 = 0$

±factors of 5 / factors of 4

= ±(1,5)/(1,2,4)

= ±1,½,¼,5,5/2,5/4

$3x^4 + 2x^3 - 5x^2 + 10x - 14 = 0$

±factors of 14 / factors of 3

= ±(1,2,7,14)/(1,3)

= ±1,⅓,2,⅔,7,7/3,14,14/3

Higher order Vietas:

Given
$f(x) = x^3 + 2x^2 - 6x + 9$
With roots r,s, and t
Find rst, r+s+t, and rs+st+rt

$rst = -9$
$r+s+t = -2$
$rs+st+rt = -6$

**Challenge:**

Given:
$f(x) = x^3 + 8x^2 + 5x - 10$
Has roots r,s, and t
Find:
$r^2 + s^2 + t^2$
$1/r + 1/s + 1/t$
$r^3 + s^3 + t^3$
*hard* $(r+1/r)(s+1/s)(t+1/t)$

Notice: $(-8)^2 = (r+s+t)^2 = r^2 + s^2 + t^2 + 2(rs+st+rt) = r^2 + s^2 + t^2 + 2(5)$
So, $r^2 + s^2 + t^2 = (-8)^2 - 2*5 = 54$

Likewise, notice $1/r + 1/s + 1/t = (st + rt + rs) / (rst) = 5 / 10 = ½$

We use the same method in $r^2 + s^2 + t^2$ to solve for $r^3 + s^3 + t^3$
Notice $(-8)^3 = (r+s+t)^3 = r^3 + s^3 + t^3 + 3rs^2 + 3sr^2 + 3rt^2 + 3tr^2 + 3\ st^2 + 3ts^2 + 6rst$
Notice by adding 3rst, we get:
$r^3 + s^3 + t^3 + 3(rs^2 + sr^2 + rst) + 3(rt^2 + tr^2 + rst) + 3(st^2 + ts^2 + rst)$
$= r^3 + s^3 + t^3 + 3rs(r+s+t) + 3rt(r+s+t) + 3st(r+s+t)$
$= r^3 + s^3 + t^3 + 3(r+s+t)(rs+rt+st)$
And of course, we have to subtract back the 3rst, to get:
$(-8)^3 = (r+s+t)^3 = r^3 + s^3 + t^3 + 3(r+s+t)(rs+rt+st) - 3rst = r^3 + s^3 + t^3 + 3(-8)(5) + 3*10$
So
$r^3 + s^3 + t^3 = (-8)^3 - 3(-8)(5) + 3*10 = -512 + 120 + 30 = -362$

Finally, notice that
(r+1/r)(s+1/s)(t+1/t) = rst + 1/rst + r/st + s/rt + t/rs + rs/t + rt/s + st/r
= 10 + 1/10 + (r/st + s/rt + t/rs) + (rs/t + rt/s + st/r)
Looking at r/st + s/rt + t/rs, we see:
r/st + s/rt + t/rs = $(r^2 + s^2 + t^2)$ / rst = 54 / 10
Looking at rs/t + rt/s + st/r, we see:
rs/t + rt/s + st/r = [ $(rs)^2 + (rt)^2 + (st)^2$ ] / rst
Seeing the sum of squares of rs, rt, st, leads us to want to square (rs + rt + st) which we know the value of:
$(rs + rt + st)^2 = 5^2 = (rs)^2 + (rt)^2 + (st)^2 + 2(r^2st) + 2(rs^2t) + 2(rst^2)$
Notice:
$2(r^2st) + 2(rs^2t) + 2(rst^2)$ = 2(r + s + t)(rst) = 2(-8)(10) = -160
So,
$(rs)^2 + (rt)^2 + (st)^2$ = 25 + 160 = 185
And,
rs/t + rt/s + st/r = 185 / 10
So,
(r+1/r)(s+1/s)(t+1/t) = 10 + 1/10 + 54 / 10 + 185 / 10 = 34

(2020 AIME II #11) **very hard**

Let $P(x) = x^2 - 3x - 7$, and let $Q(x)$ and $R(x)$ be two quadratic polynomials also with the coefficient of $x^2$ equal to $1$. David computes each of the three sums $P + Q$, $P + R$, and $Q + R$ and is surprised to find that each pair of these sums has a common root, and these three common roots are distinct. If $Q(0) = 2$, then $R(0) = \frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Find $m + n$.

Solution Linked here