Individual Round Shortlisted Questions

Section A (8 qustions)

1. **How many 3-digit multiples of 9 are perfect squares?**

   $10^2 = 100$ and $32^2 > 1000$ so there are 22 perfect squares in the range. $\frac{22}{3} = 7.333$ so there are 7 such numbers.

2. **Points A, B, C and D lie on a line, in that order, with $AB = CD$ and $BC = 12$. Point E is not on the line, and $BE = CE = 10$. The perimeter of ΔAED is twice the perimeter of ΔBEC. Find AB.**

   Let M be the foot of the altitude from E to BC. Then $MB = MC = 6$ because $\triangle BEC$ is isosceles. By the Pythagorean triple $(6, 8, 10)$ the altitude is 8. Since $(8, 15, 17)$ is the only primitive Pythagorean triple with leg 8 we test $AE = DE = 17, AM = DM = 15$. Since $2(10 + 10 + 12) = (17 + 17 + 2 \cdot 15)$ this works, giving us $AB = 15 - 6 = \boxed{\text{(D) } 9}$.

3. **Suppose that x and y are positive reals such that $x - y^2 = 3, x^2 + y^4 = 13$. Find x.**

   Squaring both sides of $x - y^2 = 3$ gives $x^2 + y^4 - 2xy^2 = 9$. Subtract this equation from twice the second given to get $x^2 + 2xy^2 + y^4 = 17 = \Rightarrow x + y^2 = \pm 17$. Combining this equation with the first given, we see that $x = \frac{3 \pm \sqrt{17}}{2}$. Since x is a positive real, x must be $\frac{3+\sqrt{17}}{2}$.

4. **Suppose that $p(x)$ is a polynomial and that $p(x) - p'(x) = x^2 + 2x + 1$. Compute $p(5)$.**

   Observe that $p(x)$ must be quadratic. Let $p(x) = ax^2 + bx + c$. Comparing coefficients gives $a = 1,\ b - 2a = 2,\ and\ c - b = 1$. So $b = 4,\ c = 5,\ p(x) = x2 + 4x + 5$ and $p(5) = 25 + 20 + 5 = 50$.

5. **In how many ways can 8 people be arranged in a line if Alice and Bob must be next to each other, and Carol must be somewhere behind Dan?**

   Let us place Alice and Bob as a single person; there are then 7! = 5040 different arrangements. Alice can be in front of Bob or vice versa, multiplying the number of possibilities by 2, but Carol is behind Dan in exactly half of those, so that the answer is just 5040.

6. **Two sides of a triangle have lengths 10 and 15. The length of the altitude to the third side is the average of the lengths of the altitudes to the two given sides. How long is the third side?**

   Let a and b be the length of the altitudes to the two given sides and c be the third side. Calculating the area of the triangle using the three sides as base we get,

$15a = 10b = \frac{a+b}{2} * c$. Solving gives $c = 12$

7. **Two non-zero real numbers, a and b satisfy $ab = a - b$. Find $\frac{a}{b} + \frac{b}{a} - ab$.**

$$\frac{a}{b} + \frac{b}{a} - ab = \frac{a^2+b^2}{ab} - (a-b) = \frac{a^2+b^2}{a-b} - \frac{(a-b)^2}{(a-b)} = \frac{2ab}{a-b} = \frac{2(a-b)}{a-b} = 2 \Rightarrow \boxed{\text{E}}$$

8. **What is the smallest positive integer with six positive odd integer divisors and twelve positive even integer divisors?**

Dividing the greatest power of 2 from n, we have an odd integer with six positive divisors, which indicates that it either is ($6 = 2 \cdot 3$) a prime raised to the 5th power, or two primes, one of which is squared. The smallest example of the former is $3^5 = 243$, while the smallest example of the latter is $3^2 \cdot 5 = 45$.

Suppose we now divide all of the odd factors from n; then we require a power of 2 with $\frac{18}{6} = 3$ factors, namely $2^{3-1} = 4$. Thus, our answer is $2^2 \cdot 3^2 \cdot 5 = \boxed{180}$.

9. **In year N, the 300^{th} day of the year is a Tuesday. In year $N+1$, the 200^{th} day is also a Tuesday. On what day of the week did the 100^{th} day of year $N-1$ occur?**

There are either $65 + 200 = 265$ or $66 + 200 = 266$ days between the first two dates depending upon whether or not year $N+1$ is a leap year (since the February 29th of the leap year would come between the 300th day of year N and 200th day of year $N+1$). Since 7 divides into 266 but not 265, for both days to be a Tuesday, year N must be a leap year.

Hence, year $N-1$ is not a leap year, and so since there are $265 + 300 = 565$ days between the date in years $N, N-1$, this leaves a remainder of 5 upon division by 7. Since we are subtracting days, we count 5 days before Tuesday, which gives us $\boxed{(\textbf{A})\ \text{Thursday.}}$

10. **Find the smallest positive integer that is twice a perfect square and three times a perfect cube.**

Let n be such a number. If n is divisible by 2 and 3 exactly $e2$ and $e3$ times, then $e2$ is odd and a multiple of three, and $e3$ is even and one more than a multiple of three. The smallest possible exponents are $n2 = 3$ and $n3 = 4$. The answer is then $2^3 \cdot 3^4 = 648$.

11. **Compute the largest positive integer n such that $\frac{2007!}{2007^n}$ is an integer.**

Note that 2007 = 32 · 223. Using the fact that the number of times a prime p divides n! is given by the Legendre's formula, it follows that the answer is 9.

12. **Three real numbers x, y, and z are such that $\frac{x+4}{2} = \frac{y+9}{z-3} = \frac{x+5}{z-5}$. Determine the value of $\frac{x}{y}$**

½ . Because the first and third fractions are equal, adding their numerators and denominators produces another fraction equal to the others: $\frac{(x+4)+(x+5)}{2+(z-5)} = \frac{2x+9}{z-3}$. Then $y + 9 = 2x + 9$, etc.

13. **How many 5-digit numbers $abcde$ where a, b, c, d, e are the digits of the 5-digit number, exist such that digits b and d are each the sum of the digits to their immediate left and right? (That is, $b = a + c$ and $d = c + e$)**

    Chosing a, c, e fixes b and d. Take cases c = 0 to 8, to get total = 330

14. **For non-negative integers A, C and U, it is given that $A + C + U = 10$. Find the maximum value of $ACU + AC + AU + CU$.**

    Use AM-GM to get $ACU \leq \left(\frac{10}{3}\right)^3$. And

    $AC + CU + AU \leq A^2 + C^2 + U^2 = (A + C + U)^2 - 2(AC + CU + AU)$. So $AC + CU + AU \leq \frac{(A+C+U)^2}{3}$. For $ACU = 37$, it must be case of (1,1,37) not possible. (4,3,3) is a construction for the max value giving 69.

15. **Let a and b be integer solutions to $17a + 6b = 13$. What is the smallest possible positive value for $a - b$?**

    First group as $17(a - b) + 23b = 13$. Taking this equation modulo 23, we get $-6(a - b) \equiv -10 \ (mod\ 23)$. Since -4 is an inverse of -6 modulo 23, then we multiply to get $(a - b) \equiv 17 \ (mod\ 23)$. Therefore, the smallest possible positive value for $(a - b)$ is 17. This can be satisfied by $a = 5,\ b = -12$.

Section B (2 questions)

1. **Find all positive real number(s) x satisfying the equation $\left\{(x + 1)^3\right\} = x^3$, where {y} denotes the fractional part of y, for example {3.1416….}= 0.1416…..**

$$\left\{(x + 1)^3\right\} = \left\{x^3 + 3x^2 + 3x + 1\right\} = \{x^3 + 3x^2 + 3x\}$$

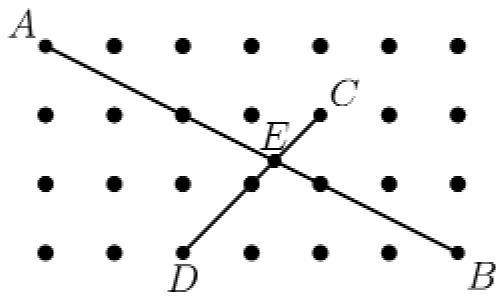
   $x^3 < 1$ since it is equal to a fractional part.

   So, $\left\{x^3\right\} = x^3$ thus $3x^2 + 3x$ must be an integer.

   So, $x^2 + x = 0, \frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}, \frac{5}{3}$

   Solving gives $x = 0, \frac{\sqrt{21}-3}{6}, \frac{\sqrt{33}-3}{6}, \frac{\sqrt{5}-1}{2}, \frac{\sqrt{57}-3}{6}, \frac{\sqrt{69}-3}{6}$

2. **The diagram shows 28 lattice points, each one unit from its nearest neighbors. Segment AB meets segment CD at E. Find the length of segment AE.**



Let l_1 be the line containing A and B and let l_2 be the line containing C and D. If we set the bottom left point at $(0,0)$, then $A=(0,3)$, $B=(6,0)$, $C=(4,2)$, and $D=(2,0)$.

The line l_1 is given by the equation $y=m_1x+b_1$. The y-intercept is $A=(0,3)$, so $b_1=3$. We are given two points on l_1, hence we can compute the slope, m_1 to be $\frac{0-3}{6-0}=-\frac{1}{2}$, so l_1 is the line $y=\frac{-1}{2}x+3$

Similarly, l_2 is given by $y=m_2x+b_2$. The slope in this case is $\frac{2-0}{4-2}=1$, so $y=x+b_2$. Plugging in the point $(2,0)$ gives us $b_2=-2$, so l_2 is the line $y=x-2$.

At E, the intersection point, both of the equations must be true, so

$$y=x-2, y=\frac{-1}{2}x+3 \Rightarrow x-2=\frac{-1}{2}x+3$$
$$\Rightarrow x=\frac{10}{3}$$
$$\Rightarrow y=\frac{4}{3}$$

We have the coordinates of A and E, so we can use the distance formula here:

$$\sqrt{\left(\frac{10}{3}-0\right)^2+\left(\frac{4}{3}-3\right)^2}=\frac{5\sqrt{5}}{3}$$

3. Find all real numbers x such that $x^2+\left[\frac{x}{2}\right]+\left[\frac{x}{3}\right]=10$, where $[x]$ represents the greatest integer less than or equal to x.

Evidently x^2 must be an integer. Well, there aren't that many things to check, are there? Among positive x, $\sqrt{8}$ is too small and $\sqrt{9}$ is too big; among negative x, $-\sqrt{15}$ is too small and $-\sqrt{13}$ is too big.