Analysis Lesson 12
MAT320/MAT640 Analysis
with Professor Sormani
Spring 2022

Epsilon Delta proofs of lim as x to c of f(x) is L

Be sure to spend a full 6 hours on each lesson.

Your work for today's lesson will go in a googledoc you create entitled MAT320S22-Lesson12-Lastname-Firstname with your last name and your first name. The googledoc will be shared with the professor sormanic@gmail.com as an editor. Put any questions you have inside your doc and email me to let me know it is there. Be sure to complete one page of HW on paper and take a selfie holding up a few pages.

This Lesson has four parts with lots of classwork and ten homework problems.

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Part 1 Limits of Functions

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Watch Playlist limf-1to4

11:10 PM Mon Mar 8

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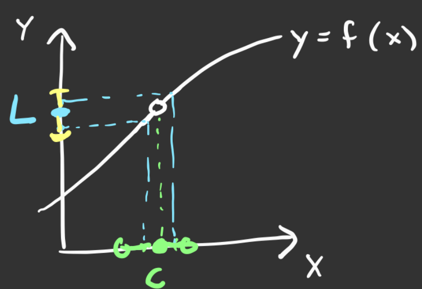
Limits $\lim_{x \to c} f(x) = L$

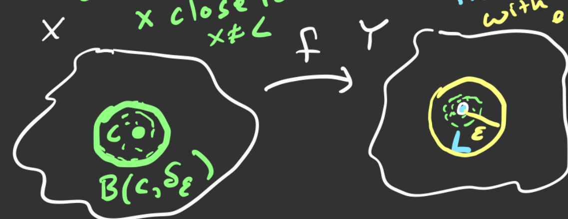
$$\forall \varepsilon > 0 \ \exists \delta_\varepsilon > 0 \text{ s.t. } 0 < d(x,c) < \delta_\varepsilon \Rightarrow d(f(x), L) < \varepsilon$$

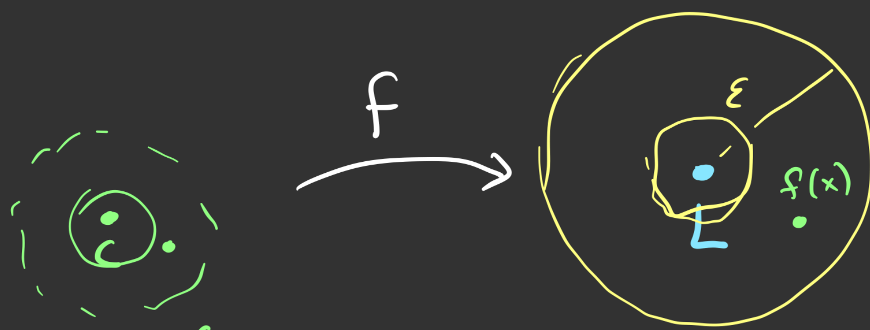
error around the target limit

we can find an estimate about c

$$x \neq c \qquad \underbrace{x \in B(c, \delta_\varepsilon)}_{\text{how carefully must we aim or estimate } x \text{ close to } c,\ x \neq c} \Rightarrow \underbrace{f(x) \in B(L, \varepsilon)}_{\text{to guarantee } f(x) \text{ lands in the target with error } < \varepsilon.}$$







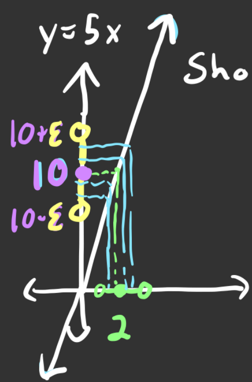
x near c

$|x-c| < \delta_\varepsilon$

If we choose a smaller $\varepsilon > 0$

then we take δ_ε smaller too.

Classwork: Prove that $\lim_{x\to 2} 5x = 10$



Show: $\forall \varepsilon>0\ \exists \delta_\varepsilon>0$ s.t. $0<|x-2|<\delta_\varepsilon \Rightarrow |5x-10|<\varepsilon$

$10-\varepsilon < 5x < 10+\varepsilon$

Structure of the Proof:

① Given any $\varepsilon>0$

Choose $\delta_\varepsilon = \boxed{} > 0$

① Must justify why $\delta_\varepsilon > 0$

② Whenever $0<|x-2|<\delta_\varepsilon$

we have ...

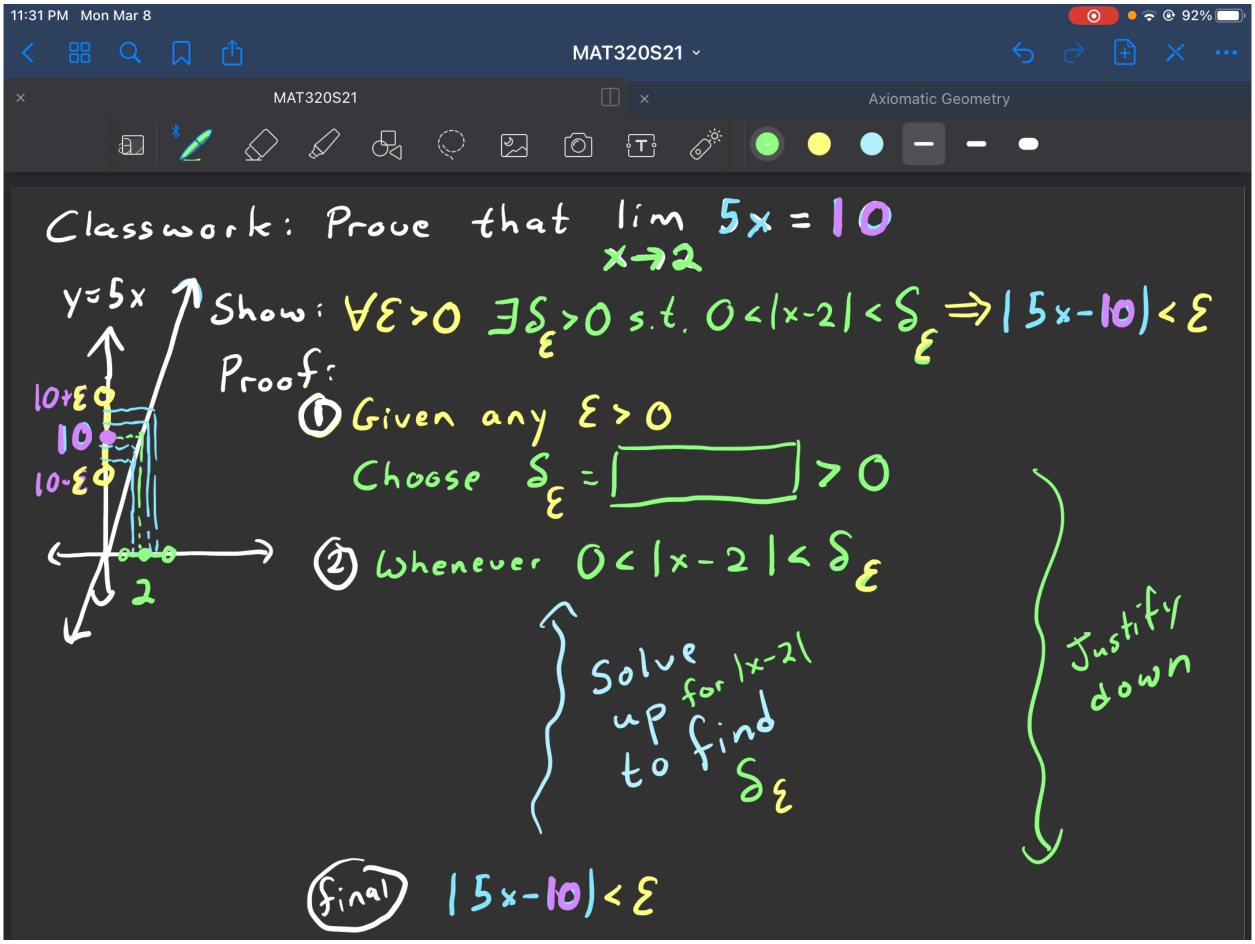
$\Downarrow$

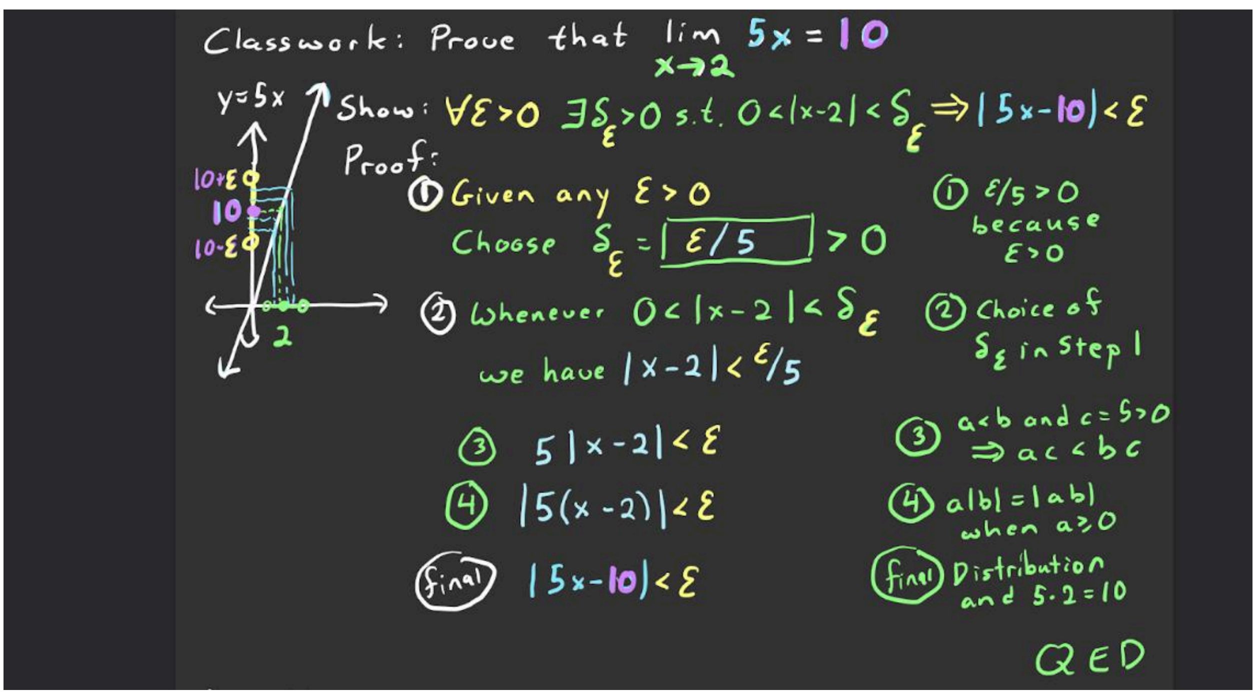
Final $|5x-10|<\varepsilon$

② Use choice of δ_ε

Solve upwards to find δ_ε

Justify down





HW1: What happens if you try to prove that the limit as x goes to 4 of 5x is 20?

Start by setting up the structure and solve upwards. Does the proof work?

HW2: What happens if you try to prove that the limit as x goes to 2 of 5x is 12 instead of 10?   Start by setting up the structure with 12 instead if 10 and solve upwards. Where does the proof fail? Or does the proof work?

HW3: What happens if you try to prove that the limit as x goes to 4 of 5x is 10?   Start by setting up the structure and solve upwards. Where does the proof fail? Or does the proof work?

**HW4: What happens if you try to prove that the limit as x goes to 4 of 5x is 20?
Start by setting up the structure and solve upwards. Where does the proof fail?   Or does the proof work?**

HW5: Now prove that the limit as x goes to 2 of 4x+6 is 14.   This has more steps solving upwards because the function is a little more complicated.

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Part 2: Sums, Products and Quotients

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Watch [Playlist limf-5to9]

Classwork: If $\lim\limits_{t\to c} f(t) = F$ and $\lim\limits_{t\to c} g(t) = G$

ε epsilon δ delta $\cancel{\in}$

then $\lim\limits_{t\to c}\left(2f(t) + 4g(t)\right) = 2F + 4G$

Given: $\forall \varepsilon > 0 \quad \exists \delta_\varepsilon^f > 0$ s.t $0 < |t - c| < \delta_\varepsilon^f \Rightarrow |f(t) - F| < \varepsilon$

$\forall \varepsilon > 0 \quad \exists \delta_\varepsilon^g > 0$ s.t $0 < |t - c| < \delta_\varepsilon^g \Rightarrow |g(t) - G| < \varepsilon$

Show: $\forall \varepsilon > 0 \quad \exists \delta_\varepsilon > 0$ s.t $0 < |t - c| < \delta_\varepsilon \Rightarrow |(2f(t) + 4g(t)) - (2F + 4G)| < \varepsilon$

need parentheses

Set up the proof using the show line

✱

Given: $\forall \varepsilon > 0 \quad \exists \delta_\varepsilon^f > 0$ s.t $0 < |t-c| < \delta_\varepsilon^f \Rightarrow |f(t) - F| < \varepsilon$

$\forall \varepsilon > 0 \quad \exists \delta_\varepsilon^g > 0$ s.t $0 < |t-c| < \delta_\varepsilon^g \Rightarrow |g(t) - G| < \varepsilon$

Show: $\forall \varepsilon > 0 \quad \exists \delta_\varepsilon > 0$ s.t $0 < |t-c| < \delta_\varepsilon \Rightarrow |(2f(t)+4g(t)) - (2F+4G)| < \varepsilon$

need parentheses

Proof: ① Given any $\varepsilon > 0$

Choose $\delta_\varepsilon = \boxed{} > 0$

② Whenever $0 < |t-c| < \delta_\varepsilon$

we have

Solve up

Justify Down

final $|(2f(t)+4g(t)) - (2F+4G)| < \varepsilon$

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need parentheses

Proof: (1) Given any $\varepsilon > 0$

Choose $\delta_\varepsilon = \boxed{} > 0$

(2) Whenever $0 < |t-c| < \delta_\varepsilon$

we have $|f(t)-F| < \frac{\varepsilon}{4}$ AND $|g(t)-G| < \varepsilon/8$

solve up

$2|f(t)-F| < \varepsilon/2$ AND $4|g(t)-G| < \varepsilon/2$

$|2f(t)-2F| < \frac{\varepsilon}{2}$ AND $|4g(t)-4G| < \frac{\varepsilon}{2}$

$|2f(t)-2F| + |4g(t)-4G| < \varepsilon$

$\vee$| triangle ineq

$|2f(t)-2F+4g(t)-4G| < \varepsilon$

$|2f(t)+4g(t)-2F-4G| < \varepsilon$

final $|(2f(t)+4g(t))-(2F+4G)| < \varepsilon$

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Classwork: If $\lim_{t\to c} f(t) = F$ and $\lim_{t \to c} g(t) = G$

ε epsilon δ delta

then $\lim_{t\to c} (2f(t) + 4g(t)) = 2F + 4G$

Given: $\forall \varepsilon > 0 \quad \exists \delta_\varepsilon^f > 0$ s.t $0 < |t-c| < \delta_\varepsilon^f \Rightarrow |f(t) - F| < \varepsilon$

$\forall \varepsilon > 0 \quad \exists \delta_\varepsilon^g > 0$ s.t $0 < |t-c| < \delta_\varepsilon^g \Rightarrow |g(t) - G| < \varepsilon$

Show: $\forall \varepsilon > 0 \quad \exists \delta_\varepsilon > 0$ s.t $0 < |t-c| < \delta_\varepsilon \Rightarrow |(2f(t) + 4g(t)) - (2F + 4G)| < \varepsilon$

need parentheses

Proof: ① Given any $\varepsilon > 0$ Choose $\delta_\varepsilon = \boxed{\min\{\delta^f_{\varepsilon/4}, \delta^g_{\varepsilon/8}\}} > 0$

② Whenever $0 < |t-c| < \delta_\varepsilon$

we have $0 < |t-c| < \delta^f_{\varepsilon/4}$ AND $0 < |t-c| < \delta^g_{\varepsilon/8}$

③ $|f(t) - F| < \frac{\varepsilon}{4}$ AND $|g(t) - G| < \varepsilon/8$

④ $2|f(t) - F| < \varepsilon/2$ AND $4|g(t) - G| < \varepsilon/2$

⑤ $|2f(t) - 2F| < \frac{\varepsilon}{2}$ AND $|4g(t) - 4G| < \frac{\varepsilon}{2}$

⑥ $|2f(t) - 2F| + |4g(t) - 4G| < \varepsilon$

VI triangle ineq

⑦ $|2f(t) - 2F + 4g(t) - 4G| < \varepsilon$

⑧ $|2f(t) + 4g(t) - 2F - 4G| < \varepsilon$

final $|(2f(t) + 4g(t)) - (2F + 4G)| < \varepsilon$

solve up

Justify down

① Given $\delta^f_{\varepsilon/4} > 0$ and $\delta^g_{\varepsilon/8} > 0$

② by choice of δ_ε in step 1 and defn of min

③ Given

④ $a < b$ and $c = 2 > 0 \Rightarrow ac < bc$ also $c = 4 > 0$

⑤

⑥ $\frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

⑦ triangle ineq

⑧ + ⑨ algebra

QED

Thm If $\lim\limits_{x\to x_0} a(x) = A$ and $\lim\limits_{x\to x_0} b(x) = B$

then $\lim\limits_{x\to x_0} a(x)\cdot b(x) = A\cdot B$

Given $\forall \varepsilon>0\ \exists \delta_\varepsilon^a > 0$ s.t. $0<|x-x_0|<\delta_\varepsilon^a \Rightarrow |a(x)-A|<\varepsilon$

$\forall \varepsilon>0\ \exists \delta_\varepsilon^b > 0$ s.t $0<|x-x_0|<\delta_\varepsilon^b \Rightarrow |b(x)-B|<\varepsilon$

Show $\forall \varepsilon>0\ \exists \delta_\varepsilon>0$ s.t $0<|x-x_0|<\delta_\varepsilon \Rightarrow |a(x)b(x)-A\cdot B|<\varepsilon$

Proof (1) Given any $\varepsilon>0$

Choose $\delta_\varepsilon = \boxed{} > 0$

(2) Whenever $0<|x-x_0|<\delta_\varepsilon$

we have:

Break into two parts $\varepsilon/2$ and triangle inequality

$$|(a(x)-A)\,b(x) + A\,(b(x)-B)| < \varepsilon$$

Key Trick! $$|a(x)b(x) - A\,b(x) + A\,b(x) - A\cdot B| < \varepsilon$$

final $$|a(x)b(x) - A\cdot B| < \varepsilon$$

$x \to x_0$

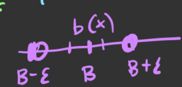
Given $\forall \varepsilon > 0 \ \exists \delta_\varepsilon^a > 0$ s.t. $0 < |x - x_0| < \delta_\varepsilon^a \Rightarrow |a(x) - A| < \varepsilon$

$\forall \varepsilon > 0 \ \exists \delta_\varepsilon^b > 0$ s.t. $0 < |x - x_0| < \delta_\varepsilon^b \Rightarrow |b(x) - B| < \varepsilon$

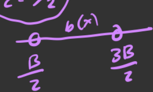
Show $\forall \varepsilon > 0 \ \exists \delta_\varepsilon > 0$ s.t. $0 < |x - x_0| < \delta_\varepsilon \Rightarrow |a(x)b(x) - A \cdot B| < \varepsilon$

Proof ① Given any $\varepsilon > 0$
Choose $\delta_\varepsilon = \boxed{} > 0$

② Whenever $0 < |x - x_0| < \delta_\varepsilon$
we have:



$\varepsilon = B/2$



$|x - c| < \delta^b_{B/2}$

$b(x) \to B$
So $|b(x)| > |B/2|$ for x close to c
Warning δ cannot depend on x

$0 < |x - c| < \delta^b_{\frac{\varepsilon}{2|A|}}$

$|a(x) - A| < \frac{\varepsilon}{2|B/2|}$

$|a(x) - A| < \frac{\varepsilon}{2|b(x)|}$ AND $|b(x) - B| < \frac{\varepsilon}{2|A|}$ ← be careful $|A| \neq 0$, $|b(x)| \neq 0$

$|a(x) - A| \cdot |b(x)| < \frac{\varepsilon}{2}$ AND $|A| \cdot |b(x) - B| < \frac{\varepsilon}{2}$

$|(a(x) - A)b(x)| < \frac{\varepsilon}{2}$ AND $|A(b(x) - B)| < \varepsilon/2$

$|(a(x) - A)b(x)| + |A(b(x) - B)| < \varepsilon$

$|(a(x) - A)b(x) + A(b(x) - B)| < \varepsilon$

Key Trick! $|a(x)b(x) - Ab(x) + Ab(x) - A \cdot B| < \varepsilon$

final $|a(x)b(x) - A \cdot B| < \varepsilon$

Thm If $\lim\limits_{x\to x_0} a(x) = A > 0$ and $\lim\limits_{x\to x_0} b(x) = B > 0$

then $\lim\limits_{x\to x_0} a(x)\cdot b(x) = A\cdot B$

Given $\forall \varepsilon>0\ \exists \delta^a_\varepsilon>0$ s.t. $0<|x-x_0|<\delta^a_\varepsilon \Rightarrow |a(x)-A|<\varepsilon$

$\forall\varepsilon>0\ \exists\delta^b_\varepsilon>0$ s.t. $0<|x-x_0|<\delta^b_\varepsilon\Rightarrow |b(x)-B|<\varepsilon$

Show $\forall\varepsilon>0\ \exists\delta_\varepsilon>0$ s.t. $0<|x-x_0|<\delta_\varepsilon \Rightarrow |a(x)b(x)-A\cdot B|<\varepsilon$

Proof (1) Given $\varepsilon>0$

Choose $\delta_\varepsilon = \boxed{\min\{\ ,\ ,\ \}}$

$b(x)$ $B-\varepsilon$ B $B+\varepsilon$

$\varepsilon = B/2$

$b(x)$ $\frac{B}{2}$ $\frac{3B}{2}$

(2) Whenever $0<|x-c|<\delta_\varepsilon$ we have

$0<|x-c|<\delta^a_{\frac{\varepsilon}{2|B/2|}}$ AND $0<|x-c|<\delta^b_{B/2}$ AND $0<|x-c|<\delta^b_{\frac{\varepsilon}{2|A|}}$

$b(x)\to B$ So $|b(x)|>|B/2|$ for x close to c

Warning δ cannot depend on x

(3) $|a(x)-A|<\frac{\varepsilon}{2|B/2|}$

(4) $|a(x)-A|<\frac{\varepsilon}{2|b(x)|}$ AND $|b(x)-B|<\frac{\varepsilon}{2|A|}$ ← be careful $|A|\neq 0$, $|b(x)|\neq 0$

(5) $|a(x)-A|\cdot|b(x)|<\frac{\varepsilon}{2}$ AND $|A|\cdot|b(x)-B|<\frac{\varepsilon}{2}$

(6) $|(a(x)-A)b(x)|<\frac{\varepsilon}{2}$ AND $|A(b(x)-B)|<\varepsilon/2$

(7) $|(a(x)-A)b(x)|+|A(b(x)-B)|<\varepsilon$

(8) $|(a(x)-A)b(x)+A(b(x)-B)|<\varepsilon$

(9) Key Trick! $|a(x)b(x)-Ab(x)+Ab(x)-A\cdot B|<\varepsilon$

final $|a(x)b(x)-A\cdot B|<\varepsilon$

Classwork

fill in justification

HW6 Fill in the justifications above

HW7 Prove that if $\lim\limits_{x\to x_0} a(x)=A$ and $\lim\limits_{x\to x_0} b(x)=B$

then $\lim\limits_{x\to x_0}(a(x)-b(x)) = A-B$

Extra Credit Prove that if $\lim\limits_{t\to c} f(t) = L\neq 0$

then $\lim\limits_{t\to c}\frac{1}{f(t)} = \frac{1}{L}$.

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Part 3: Sandwich Theorem and Trig Limits

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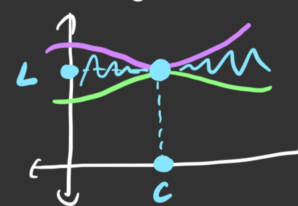
Watch Playlist limf-10to15

Classwork Prove the Sandwich Theorem

If $\lim_{x \to c} f(x) = L = \lim_{x \to c} h(x)$ and $f(x) \leq g(x) \leq h(x)$

Then $\lim_{x \to c} g(x) = L$



Given:

Show:

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Classwork Prove the Sandwich Theorem

If $\lim_{x\to c} f(x) = L = \lim_{x\to c} h(x)$ and $f(x) \le g(x) \le h(x)$

Then $\lim_{x\to c} g(x) = L$

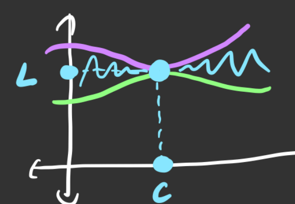
Given: $\forall \varepsilon > 0 \ \exists \delta_\varepsilon^f > 0$ s.t. $0 < |x-c| < \delta_\varepsilon^f \Rightarrow |f(x) - L| < \varepsilon$

$\forall \varepsilon > 0 \ \exists \delta_\varepsilon^h > 0$ s.t. $0 < |x-c| < \delta_\varepsilon^h \Rightarrow |h(x) - L| < \varepsilon$

$f(x) \le g(x) \le h(x)$

Show: $\forall \varepsilon > 0 \ \exists \delta_\varepsilon^g > 0$ s.t. $0 < |x-c| < \delta_\varepsilon^g \Rightarrow |g(x) - L| < \varepsilon$

Outline of proof using show



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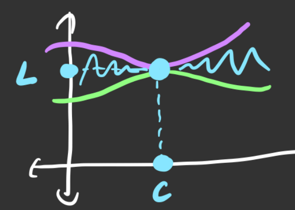
$f(x) \le g(x) \le h(x)$

Show: $\forall \varepsilon > 0 \ \exists \delta^g_\varepsilon > 0$ s.t. $0 < |x-c| < \delta^g_\varepsilon \Rightarrow |g(x) - L| < \varepsilon$



Proof: ① Given any $\varepsilon > 0$

Choose $\delta^g_\varepsilon = \boxed{} > 0$

② whenever $0 < |x-c| < \delta^g_\varepsilon$

we have ....

final $|g(x) - L| < \varepsilon$

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Classwork Prove the Sandwich Theorem

If $\lim_{x\to c} f(x) = L = \lim_{x\to c} h(x)$ and $f(x) \le g(x) \le h(x)$

Then $\lim_{x\to c} g(x) = L$

Given: $\forall \varepsilon > 0 \ \exists \delta_\varepsilon^f > 0$ s.t. $0 < |x-c| < \delta_\varepsilon^f \Rightarrow |f(x) - L| < \varepsilon$

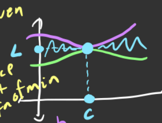
$\forall \varepsilon > 0 \ \exists \delta_\varepsilon^h > 0$ s.t. $0 < |x-c| < \delta_\varepsilon^h \Rightarrow |h(x) - L| < \varepsilon$

$f(x) \le g(x) \le h(x)$

Show: $\forall \varepsilon > 0 \ \exists \delta_\varepsilon^g > 0$ s.t. $0 < |x-c| < \delta_\varepsilon^g \Rightarrow |g(x) - L| < \varepsilon$

Proof: ① Given any $\varepsilon > 0$ — ① by given

Choose $\delta_\varepsilon^g = \boxed{\min\{\delta_\varepsilon^f, \delta_\varepsilon^h\}} > 0$



② whenever $0 < |x-c| < \delta_\varepsilon^g$ — ② by choice of δ_ε + defn of min

we have $0 < |x-c| < \delta_\varepsilon^f$ AND $0 < |x-c| < \delta_\varepsilon^h$

③ $|f(x) - L| < \varepsilon$ AND $|h(x) - L| < \varepsilon$ — ③ Given $\lim_{x\to c} f(x) = L$, $\lim_{x\to c} g(x) = L$

④ $L - \varepsilon < f(x) < L + \varepsilon$ AND $L - \varepsilon < h(x) < L + \varepsilon$ — ④ $|a-b| < c \Leftrightarrow b - c < a < b + c$

⑤ $L - \varepsilon < f(x) \le g(x) \le h(x) < L + \varepsilon$ — ⑤ step 4 and given $f(x) \le g(x) \le h(x)$

⑥ $L - \varepsilon < g(x) < L + \varepsilon$ — ⑥ by step 5

final $|g(x) - L| < \varepsilon$ — final $|a-b| < c \Leftrightarrow b - c < a < b + c$

QED

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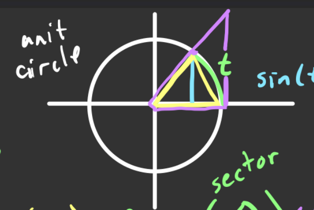
Classwork

Prove $\displaystyle\lim_{t\to 0}\frac{\sin(t)}{t}=1$



t = arclength on unit circle

sin(t) = height

$$\cos(t)\le\frac{\sin(t)}{t}\le 1 \quad \Leftarrow \quad \begin{cases} \text{Area}(\triangle)\le\text{Area}(\text{sector})\le\text{Area}(\triangle) \\ \frac{1}{2}\,1\cdot\sin(t)\le\left(\frac{t}{2\pi}\right)\pi 1^2\le\frac{1}{2}\cdot 1\cdot\tan(t) \\ \frac{\sin(t)}{t}\le \underbrace{1\le\frac{\sin(t)}{t}\,\frac{1}{\cos(t)}}_{\cos(t)\le\frac{\sin(t)}{t}} \end{cases}$$

Use Sandwich Theorem.

$$\lim_{t\to 0}\cos(t)=1=\lim_{t\to 0}1$$

Thus $\displaystyle\lim_{t\to 0}\frac{\sin(t)}{t}=1$

Prove $\lim_{t \to 0} \cos(t) = 1$

$\forall \varepsilon > 0 \ \exists \delta_\varepsilon > 0$ s.t. $0 < |t - 0| < \delta_\varepsilon \Rightarrow |\cos(t) - 1| < \varepsilon$

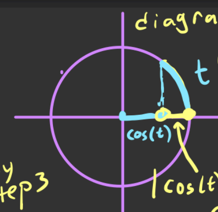
(1) Given any $\varepsilon > 0$ Choose $\delta_\varepsilon = \boxed{\varepsilon} > 0$ (1) $\delta_\varepsilon > 0$ because $\varepsilon > 0$

(2) Whenever $0 < |t - 0| < \delta_\varepsilon$

$|t| < \delta_\varepsilon = \varepsilon$ (2) $|t-0| = |t|$

(3) by defn of cosine and diagram

(3) $|\cos(t) - 1| \leq |t| < \varepsilon$



(4) final $|\cos(t) - 1| < \varepsilon$ (4) by step 3

QED

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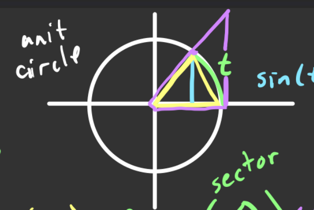
Classwork

Prove $\lim_{t \to 0} \frac{\sin(t)}{t} = 1$



$t =$ arclength on unit circle

$\sin(t) =$ height

$$\cos(t) \le \frac{\sin(t)}{t} \le 1 \Longleftarrow \begin{cases} \text{Area}(\triangle) \le \text{Area}(\overset{\text{sector}}{\triangle}) \le \text{Area}(\triangle) \\ \frac{1}{2} 1 \cdot \sin(t) \le \left(\frac{t}{2\pi}\right) \pi 1^2 \le \frac{1}{2} \cdot 1 \cdot \tan(t) \\ \frac{\sin(t)}{t} \le 1 \le \underbrace{\frac{\sin(t)}{t} \frac{1}{\cos(t)}}_{\cos(t) \le \frac{\sin(t)}{t}} \end{cases}$$

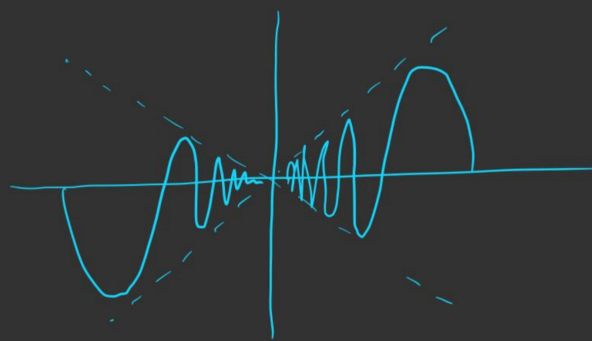
Use Sandwich Theorem.

$$\lim_{t \to 0} \cos(t) = 1 = \lim_{t \to 0} 1$$

Thus $\lim_{t \to 0} \frac{\sin(t)}{t} = 1$

$\boxed{\text{HW8}}$ Consider $f(x) = x \sin\left(\frac{1}{x}\right)$

which is defined when $x \neq 0$



Prove that $\lim\limits_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0$.

Hint: Use $-1 \le \sin(\theta) \le 1 \quad \forall \theta \in \mathbb{R}$

Sub in $\theta = \frac{1}{x}$ for any $x \neq 0$

Multiply by x

Apply the Sandwich Theorem.

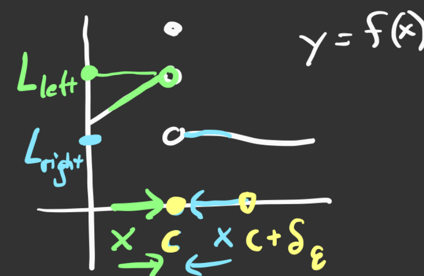
****

Part 4 Left and Right Limits

****

Watch [Playlist limf-16to18]

Left and Right limits



$$\lim_{x \to c_+} f(x) = L_{right}$$

$$\lim_{x \to c_-} f(x) = L_{left}$$

right side of x

$\forall \varepsilon > 0 \;\; \exists \delta_\varepsilon > 0$ s.t $c < x < c + \delta_\varepsilon \Rightarrow |f(x) - L_{right}| < \varepsilon$

* $\forall \varepsilon > 0 \;\; \exists \delta_\varepsilon > 0$ s.t $c - \delta_\varepsilon < x < c \Rightarrow |f(x) - L_{left}| < \varepsilon$

left side of x

4:06 AM Tue Mar 9 27%

MAT320S21

MAT320S21 | Axiomatic Geometry

Classwork: Let $f(x) = \frac{|x|}{x}$

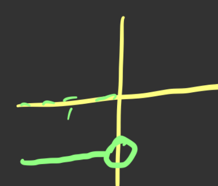
Show $\lim_{x \to 0_-} f(x) = -1$ left limit

Proof ① Given any $\varepsilon > 0$

Choose $\delta_\varepsilon = \boxed{\text{anything positive}} > 0$

② whenever $0 - \delta_\varepsilon < x < 0$

we have $x < 0$



(3) $0 < \varepsilon$

(4) $\left| -1 + 1 \right| < \varepsilon$

(5) $\left| \frac{-x}{x} + 1 \right| < \varepsilon$

final $\left| \frac{|x|}{x} - (-1) \right| < \varepsilon$

final $|x| = -x$ because $x<0$ in step 2

Fill in Justifications

Show $\lim\limits_{x \to 0^+} \frac{|x|}{x} = 1.$ ← Classwork

$\boxed{\text{HW9}}$ Let $f(x) = \begin{cases} 2x+3 & \text{for } x \geq 1 \\ 4x & \text{for } x < 1 \end{cases}$

Prove $\lim\limits_{x \to 1^+} f(x) = 5$

$\lim\limits_{x \to 1^-} f(x) = 4$

Thm: $\lim\limits_{x \to c^-} f(x) = L = \lim\limits_{x \to c^+} f(x)$

iff $\lim\limits_{x \to c} f(x) = L$

Consequence: $\lim\limits_{x \to c^-} f(x) \neq \lim\limits_{x \to c^-} f(x)$

then $\lim\limits_{x \to c} f(x)$ DNE

Does Not Exist

Thm: $\lim_{x \to c^-} f(x) = L = \lim_{x \to c^+} f(x)$

iff $\lim_{x \to c} f(x) = L$

$\boxed{\text{HW10}}$ Prove this theorem as follows

Part I: Given $\lim_{x \to c^-} f(x) = L$ and $\lim_{x \to c^+} f(x) = L$

Show $\lim_{x \to c} f(x) = L$

Given: $\forall \varepsilon > 0 \ \exists \delta_\varepsilon^- > 0$ s.t $c - \delta_\varepsilon^- < x < c \Rightarrow |f(x) - L| < \varepsilon$

[redacted]

$\forall \varepsilon > 0 \ \exists \delta_\varepsilon^+ > 0$ s t [redacted]

Show: $\forall \varepsilon > 0 \ \exists \delta_\varepsilon > 0$ s.t [redacted]

Proof:

① Given any $\varepsilon > 0$ Choose $\delta_\varepsilon = \min\{\delta_\varepsilon^-, \delta_\varepsilon^+\} > 0$ ① [redacted]

② whenever $0 < |x - c| < \delta_\varepsilon$ ② by defn of abs value

we have $-\delta_\varepsilon < x - c < 0$ OR $0 < x - c < \delta_\varepsilon$

③ $-\delta_\varepsilon^- \le -\delta_\varepsilon < x - c < 0$ OR $0 < x - c < \delta_\varepsilon \le \delta_\varepsilon^+$ ③ by defn of min and choice of δ_ε

④ $c - \delta_\varepsilon^- < x < c$ OR $c < x < c + \delta_\varepsilon^+$ ④ [redacted]

final $|f(x) - L| < \varepsilon$ OR $|f(x) - L| < \varepsilon$ final [redacted]

Part II : Given $\lim_{x \to c} f(x) = L$

Show $\lim_{x \to c^-} f(x) = L$

Given: $\forall \varepsilon > 0 \ \exists \delta_\varepsilon > 0$ [redacted]

Show: $\forall \varepsilon > 0 \ \exists \delta_\varepsilon^- > 0$ [redacted]

Proof:

① Given any $\varepsilon > 0$ choose $\delta_\varepsilon^- = \boxed{\delta_\varepsilon} > 0$ ① [redacted]

② whenever $c - \delta_\varepsilon^- < x < c$ we have. ② [redacted]

$-\delta_\varepsilon^- < x - c < 0$

③ $0 < |x - c| < \delta_\varepsilon^-$ ③ [redacted]

④ $0 < |x - c| < \delta_\varepsilon$ ④ by choice of δ_ε

final $|f(x) - L| < \varepsilon$ ③ by given

Part III : Given $\lim_{x \to c} f(x) = L$

Show $\lim_{x \to c^+} f(x) = L$

Given: [redacted]

Show [redacted]

Proof

① [redacted]

② [redacted]

③ [redacted]

④ [redacted]

final [redacted]

① [redacted]

② [redacted]

③ [redacted]

④ [redacted]

final [redacted]

Useful Theorems about limits you may wish to use to justify steps when proving theorems later

Sums of Limits $$\lim_{x \to a} f(x) + g(x) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

Differences of Limits $$\lim_{x \to a} f(x) - g(x) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$$

Products of Limits $$\lim_{x \to a} f(x) \cdot g(x) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$$

$$\lim_{x \to a} k f(x) = k \lim_{x \to a} f(x)$$

Quotients of Limits $$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} \quad \text{if } \lim_{x \to a} g(x) \neq 0$$

Sandwich $f(x) \le g(x) \le h(x)$ and $\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x)$ $\Longrightarrow \lim_{x \to a} g(x) = L$

Special Limit of sin(x) $$\lim_{x \to 0} \frac{\sin(x)}{x} = 1$$

Special Limit of cos(x) $$\lim_{x \to 0} \frac{1 - \cos(x)}{x} = 0$$