Aljabar Linear

Contoh Kasus & Penyelesaian dengan
R, Python, Maple, & Matlab

Prana Ugiana Gio

Aljabar Linear: Contoh Kasus dan

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https://statcal.com/ | https://statkomat.com/

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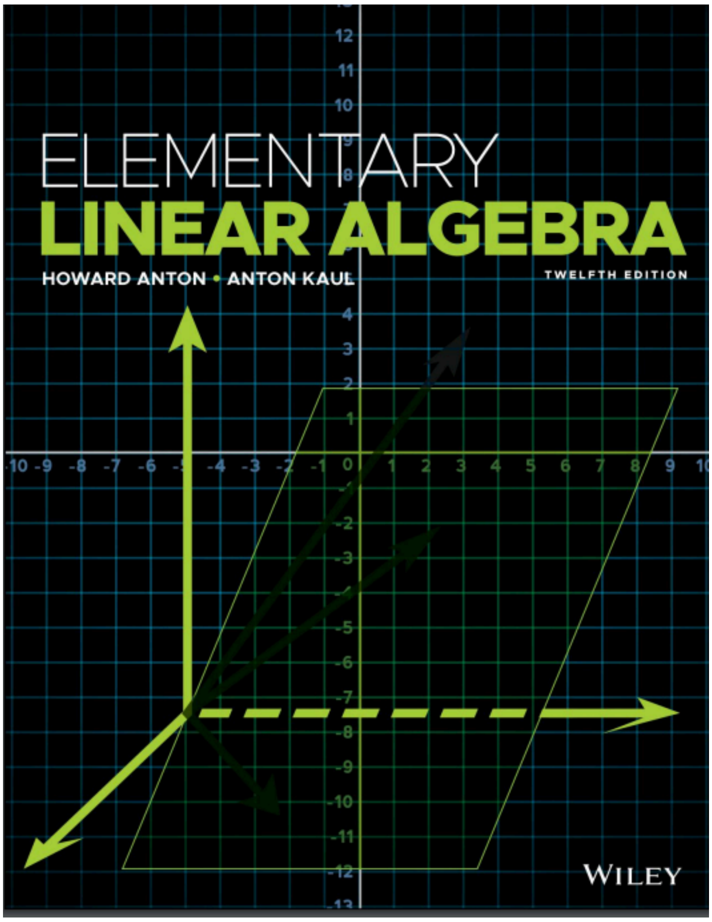

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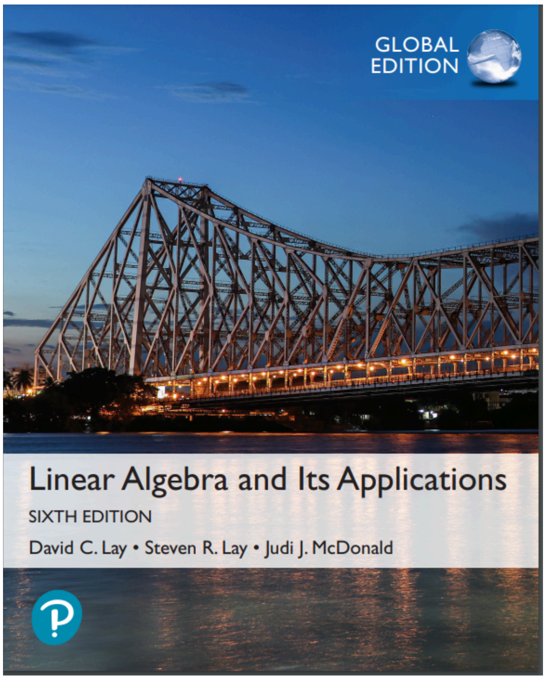

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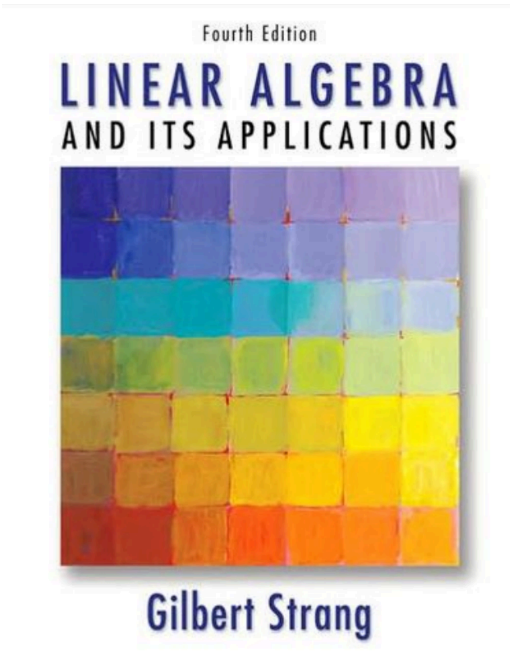

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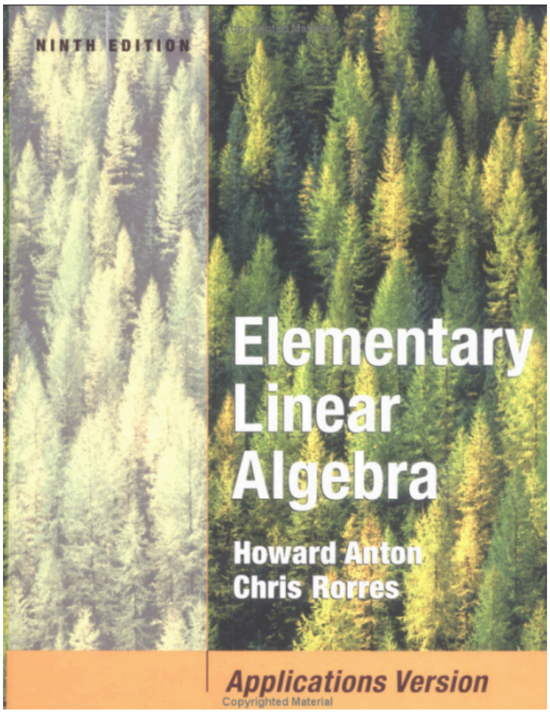

KONTEN

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YOUTUBE

- Part 1: Penjelasan 1.1 sampai 1.3, Link: https://youtu.be/Su1jbJnnYAs

Bab 1

Sistem Persamaan Linier dan Matriks

[1.1] Mengubah Sistem Persamaan Linear ke dalam Bentuk Matriks

Contoh kasus berikut diambil dari [1] halaman 1. Diberikan sistem persamaan linier sebagai berikut.

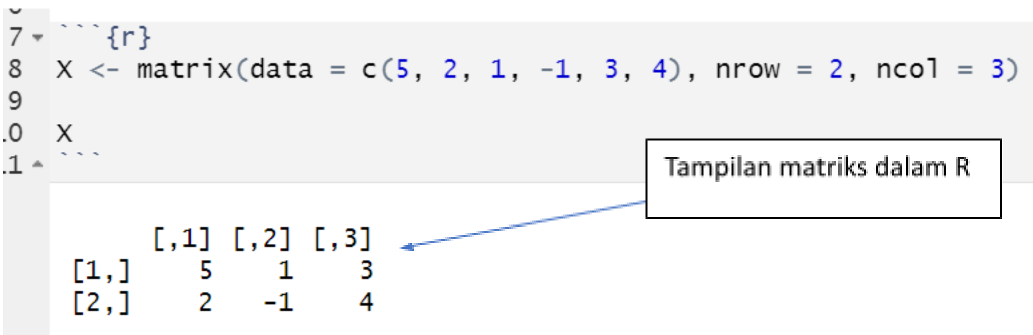
$$5x + y = 3$$

$$2x - y = 4$$

Dari sistem persamaan linier di atas, jika diubah ke dalam bentuk matriks, akan menjadi sebagai berikut.

$$[5\ 1\ 3\ 2\ \ -\ 1\ 4\ ]$$

Gambar 1 disajikan tampilan matriks dalam R.



Gambar 1. Tampilan Matriks dalam R

[1.2] Solusi dari Sistem Persamaan Linier

Contoh kasus berikut diambil dari [1] halaman 5. Diberikan sistem persamaan linier sebagai berikut.

$$x - y = 1$$

$$2x + y = 6$$

Berikut akan dihitung solusi dari sistem persamaan linier di atas. Diketahui

$$x - y = 1$$

$$x = 1 + y$$

Kita substitusikan ke dalam $2x + y = 6$, yakni

$$2(1 + y) + y = 6$$

$$2 + 2y + y = 6$$

$$3y = 4$$

$$y = \frac{4}{3}$$

$$x = \frac{7}{3}$$

Gambar 2 dan Gambar 3 disajikan penyelesaian dengan R.

```{r}
koefisien <- matrix(data = c(1, 2, -1, 1), nrow = 2, ncol = 2)
koefisien

```

```
     [,1] [,2]
[1,]    1   -1
[2,]    2    1
```

```{r}
konstanta <- matrix(data = c(1, 6), nrow = 2, ncol = 1)
konstanta
```

```
     [,1]
[1,]    1
[2,]    6
```

```{r}
solve(koefisien, konstanta)

```

```
         [,1]
[1,] 2.333333
[2,] 1.333333
```

Diperoleh solusi , dan .

Gambar 2. Penyelesaian dengan Fungsi solve( )

```{r}
X <- matrix(data = c(1, 2, -1, 1, 1, 6), nrow = 2, ncol = 3)

X
```

```
     [,1] [,2] [,3]
[1,]    1   -1    1
[2,]    2    1    6
```

```{r}
pracma::rref(X)

```

```
     [,1] [,2]     [,3]
[1,]    1    0 2.333333
[2,]    0    1 1.333333
```

Diperoleh solusi , dan .

Gambar 3. Penyelesaian dengan Fungsi rref( )

[1.3] Perbedaan antara *Row Echelon Form* dan *Reduced Row Echelon Form*; Eliminasi Gauss-Jordan

Contoh kasus berikut diambil dari [1] halaman 11 dan 12. Berikut kutipan dari isi buku tersebut.

EXAMPLE 1 | Row Echelon and Reduced Row Echelon Form

The following matrices are in reduced row echelon form.

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Semua matriks ini dalam bentuk ***reduced row echelon form***.

Semua matriks ini dalam bentuk ***row echelon form***, TAPI BUKAN ***reduced row echelon form***. Alasannya karena

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The following matrices are in row echelon form but not reduced row echelon form.

$$\begin{bmatrix} 1 & 4 & -3 & 7 \\ 0 & 1 & 6 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 2 & 6 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

EXAMPLE 2 | More on Row Echelon and Reduced Row Echelon Form

As Example 1 illustrates, a matrix in row echelon form has zeros below each leading 1, whereas a matrix in reduced row echelon form has zeros below *and above* each leading 1. Thus, with any real numbers substituted for the $*$'s, all matrices of the following types are in row echelon form:

$$\begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 1 & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

All matrices of the following types are in reduced row echelon form:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

Sekarang, akan diberikan kasus untuk menyelesaikan sistem persamaan linier dengan metode eliminasi Gauss-Jordan. Referensinya ada pada [1] halaman 15.

The algorithm we have just described for reducing a matrix to reduced row echelon form is called ***Gauss–Jordan elimination***. It consists of two parts, a ***forward phase*** in which zeros are introduced below the leading 1's and a ***backward phase*** in which zeros are introduced above the leading 1's. If only the forward phase is used, then the procedure produces a row echelon form and is called ***Gaussian elimination***. For example, in the preceding computations a row echelon form was obtained at the end of Step 5.

Contoh kasusnya saya ambil pada halaman 23. Yang akan kita kerjakan nomor 5.

In Exercises 5–8, solve the system by Gaussian elimination.

5.
$$\begin{aligned} x_1 + x_2 + 2x_3 &= 8 \\ -x_1 - 2x_2 + 3x_3 &= 1 \\ 3x_1 - 7x_2 + 4x_3 &= 10 \end{aligned}$$

6.
$$\begin{aligned} 2x_1 + 2x_2 + 2x_3 &= 0 \\ -2x_1 + 5x_2 + 2x_3 &= 1 \\ 8x_1 + x_2 + 4x_3 &= -1 \end{aligned}$$

7.
$$\begin{aligned} x - y + 2z - w &= -1 \\ 2x + y - 2z - 2w &= -2 \\ -x + 2y - 4z + w &= 1 \\ 3x \qquad\qquad - 3w &= -3 \end{aligned}$$

8.
$$\begin{aligned} -2b + 3c &= 1 \\ 3a + 6b - 3c &= -2 \\ 6a + 6b + 3c &= 5 \end{aligned}$$

Sistem persamaan linier pada soal nomor 5 disajikan ke dalam bentuk matriks sebagai berikut.

$$A = [1\ 1\ 2\ 8\quad -1\ \ -2\ 3\ 1\ \ 3\ \ -7\ 4\ 10\ ]$$

```
```{r}
X <- matrix(data = c(1, -1, 3, 1, -2, -7, 2, 3, 4, 8, 1, 10), nrow = 3, ncol = 4)
X

```
```

```
     [,1] [,2] [,3] [,4]
[1,]    1    1    2    8
[2,]   -1   -2    3    1
[3,]    3   -7    4   10
```

$$A = [1\ 1\ 2\ 8\ \ 0\ \ -1\ 5\ 9\ \ 3\ \ -7\ 4\ 10\ ]$$

```
Untuk Baris Ke-2: b1 + b2
```{r}
X[2,] =  X[1,] + X[2,]
X
```
```

```
     [,1] [,2] [,3] [,4]
[1,]    1    1    2    8
[2,]    0   -1    5    9
[3,]    3   -7    4   10
```

$$A = [1\ 1\ 2\ 8\ \ 0\ \ -1\ 5\ 9\ \ 0\ 10\ 2\ 14\ ]$$

```
Untuk Baris Ke-3: 3*b1 - b3
```{r}
X[3,] =  3*X[1,] - X[3,]
X
```
```

```
     [,1] [,2] [,3] [,4]
[1,]    1    1    2    8
[2,]    0   -1    5    9
[3,]    0   10    2   14
```

$$A = [1\ 1\ 2\ 8\ \ 0\ 1\ \ -5\ \ -9\ \ 0\ 10\ 2\ 14\ ]$$

```
Untuk Baris Ke-2: -1*b2
```{r}
X[2,] =  -1*X[2,]
X
```
```

```
     [,1] [,2] [,3] [,4]
[1,]    1    1    2    8
[2,]    0    1   -5   -9
[3,]    0   10    2   14
```

$$A = [1\ 1\ 2\ 8\ \ 0\ 1\ \ -5\ \ -9\ \ 0\ 0\ \ -52\ \ -104\ ]$$

```
Untuk Baris Ke-3: 10*b2 - b3
```{r}
X[3,] =  10*X[2,] - X[3,]
X
```
```

```
     [,1] [,2] [,3] [,4]
[1,]    1    1    2    8
[2,]    0    1   -5   -9
[3,]    0    0  -52 -104
```

$$A = [1\ 1\ 2\ 8\ 0\ 1\ -5\ -9\ 0\ 0\ 1\ 2\ ]$$

```
Untuk Baris Ke-3: -1/52*b3
```{r}
X[3,] =  -1/52*X[3,]
X
```
```

```
     [,1] [,2] [,3] [,4]
[1,]    1    1    2    8
[2,]    0    1   -5   -9
[3,]    0    0    1    2
```

$$A = [1\ 0\ 7\ 17\ 0\ 1\ -5\ -9\ 0\ 0\ 1\ 2\ ]$$

```
Untuk Baris Ke-1: b1 - b2
```{r}
X[1,] =  X[1,] - X[2,]
X
```
```

```
     [,1] [,2] [,3] [,4]
[1,]    1    0    7   17
[2,]    0    1   -5   -9
[3,]    0    0    1    2
```

$$A = [1\ 0\ 0\ 3\ 0\ 1\ -5\ -9\ 0\ 0\ 1\ 2\ ]$$

```
Untuk Baris Ke-1: b1 - 7*b3
```{r}
X[1,] =  X[1,] - 7*X[3,]
X
```
```

```
     [,1] [,2] [,3] [,4]
[1,]    1    0    0    3
[2,]    0    1   -5   -9
[3,]    0    0    1    2
```

$$A = [1\ 0\ 0\ 3\ \ 0\ 1\ 0\ 1\ \ 0\ 0\ 1\ 2\ ]$$

```
Untuk Baris Ke-2: b2 + 5*b3
```{r}
X[2,] =  X[2,] + 5*X[3,]
X
```
```

```
     [,1] [,2] [,3] [,4]
[1,]    1    0    0    3
[2,]    0    1    0    1
[3,]    0    0    1    2
```

Sehingga diperoleh solusi $x_1 = 3,\ x_2 = 1,\ x_3 = 2.$

```
```{r}
X <- matrix(data = c(1, -1, 3, 1, -2, -7, 2, 3, 4, 8, 1, 10), nrow = 3, ncol = 4)
X
cat(sprintf("\n\n"))
pracma::rref(X)
```
```

```
     [,1] [,2] [,3] [,4]
[1,]    1    1    2    8
[2,]   -1   -2    3    1
[3,]    3   -7    4   10


     [,1] [,2] [,3] [,4]
[1,]    1    0    0    3
[2,]    0    1    0    1
[3,]    0    0    1    2
```

Dengan menggunakan fungsi **rref( )** dalam paket R **pracma**, untuk melakukan eliminasi Gauss-Jordan. Diperoleh solusi .

[1.4] Invers dari Suatu Matriks

Contoh kasus berikut diambil dari [1] halaman 44 dan 45. Berikut kutipan dari isi buku tersebut terkait invers dari suatu matriks.

Definition 1

If A is a square matrix, and if there exists a matrix B of the same size for which $AB = BA = I$, then A is said to be ***invertible*** (or ***nonsingular***) and B is called an ***inverse*** of A. If no such matrix B exists, then A is said to be ***singular***.

Misalkan diberikan suatu matriks

$$A = [2 \ - 5 \ - 1\ 3\,]\ dan\ B = [3\ 5\ 1\ 2\,]$$

Perhatikan bahwa

$$AB = [2 \ - 5 \ - 1\ 3\,][3\ 5\ 1\ 2\,] = [1\ 0\ 0\ 1\,] = I$$

adalah matriks identitas.

$$AB = [3\ 5\ 1\ 2\,][2 \ - 5 \ - 1\ 3\,] = [1\ 0\ 0\ 1\,] = I$$

Yakni, $AB = BA = I$, A dikatakan ***invertible (*****atau** ***nonsingular)*** dan B dikatakan ***invers*** **dari matriks A.**

```
```{r}
A <- matrix(data = c(2, -1, -5, 3), nrow = 2, ncol = 2)

A
```

     [,1] [,2]
[1,]   2   -5
[2,]   -1   3

```{r}
B <- matrix(data = c(3, 1, 5, 2), nrow = 2, ncol = 2)

B
```

     [,1] [,2]
[1,]   3   5
[2,]   1   2

```{r}
A %*% B
```

     [,1] [,2]
[1,]   1   0
[2,]   0   1

```{r}
B %*% A
```

     [,1] [,2]
[1,]   1   0
[2,]   0   1
```

[1.5] Matriks Singular

Contoh kasus berikut diambil dari [1] halaman 45. Berikut kutipan dari isi buku tersebut terkait matriks singular.

EXAMPLE 6 | A Class of Singular Matrices

A square matrix with a row or column of zeros is singular. To help understand why this is so, consider the matrix

$$A = \begin{bmatrix} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{bmatrix}$$

To prove that A is singular we must show that there is no 3×3 matrix B such that

$$AB = BA = I$$

For this purpose let $\mathbf{c}_1, \mathbf{c}_2, \mathbf{0}$ be the column vectors of A. Thus, for any 3×3 matrix B we can express the product BA as

$$BA = B[\mathbf{c}_1 \quad \mathbf{c}_2 \quad \mathbf{0}] = [B\mathbf{c}_1 \quad B\mathbf{c}_2 \quad \mathbf{0}] \qquad \textbf{[Formula (6) of Section 1.3]}$$

The column of zeros shows that $BA \neq I$ and hence that A is singular.

As in Example 6, we will frequently denote a zero matrix with one row or one column by a boldface zero.

Jadi teman-teman, misalkan ada suatu matriks $A = [1\ 4\ 0\ 2\ 5\ 0\ 3\ 6\ 0\ ]$.

Nah, **suatu matriks persegi (*a square matrix*)** dengan ***row or column zeros***, itu dinamakan **matriks singular.** Jadi matriks A di atas merupakan matriks singular.

Nah, pada matriks singular, yakni si A, **there is no 3 x 3 matriks B**, **sehingga memenuhi**

$$AB = BA = I$$

Artinya, matriks singular GAK PUNYA INVERS HE HE HE HE

[1.5] Matriks Singular

```{r}

A <- matrix(data = c(1, 2, 3, 4, 5, 6, 0, 0, 0), nrow = 3, ncol = 3)

A

```

```
     [,1] [,2] [,3]
[1,]    1    4    0
[2,]    2    5    0
[3,]    3    6    0
```

```{r}
solve(A) #Fungsi solve digunakan untuk menentukan invers dari matriks A

```

```
Error in solve.default(A) :
 Lapack routine dgesv: system is exactly singular: U[3,3] = 0
```

Matriks A tidak memiliki invers, karena matriks A merupakan matriks singular, pada kolom ke-3, seluruh elemennya bernilai 0.

[1.6] Determinan suatu Matriks; Contoh Perhitungan Invers Suatu Matriks

Contoh kasus berikut diambil dari [1] halaman 46. Berikut kutipan dari isi buku tersebut terkait determinan suatu matriks.

In the next section we will develop a method for computing the inverse of an invertible matrix of any size. For now we give the following theorem that specifies conditions under which a 2×2 matrix is invertible and provides a simple formula for its inverse.

The quantity $ad - bc$ in Theorem 1.4.5 is called the ***determinant*** of the 2×2 matrix A and is denoted by

$$\det(A) = ad - bc$$

or alternatively by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Theorem 1.4.5

The matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible if and only if $ad - bc \neq 0$, in which case the inverse is given by the formula

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \tag{2}$$

EXAMPLE 7 | Calculating the Inverse of a 2×2 Matrix

In each part, determine whether the matrix is invertible. If so, find its inverse.

$(a)\ A = \begin{bmatrix} 6 & 1 \\ 5 & 2 \end{bmatrix} \qquad (b)\ A = \begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix}$

Solution (a) The determinant of A is $\det(A) = (6)(2) - (1)(5) = 7$, which is nonzero. Thus, A is invertible, and its inverse is

$$A^{-1} = \frac{1}{7}\begin{bmatrix} 2 & -1 \\ -5 & 6 \end{bmatrix} = \begin{bmatrix} \frac{2}{7} & -\frac{1}{7} \\ -\frac{5}{7} & \frac{6}{7} \end{bmatrix}$$

We leave it for you to confirm that $AA^{-1} = A^{-1}A = I$.

Solution (b) The matrix is not invertible since $\det(A) = (-1)(-6) - (2)(3) = 0$.

Misalkan diberikan matriks $A = [6\ 1\ 5\ 2\ ]$, akan dicari invers dari matriks A. Namun, kita perlu memeriksa terlebih dahulu apakah matriks A bersifat ***invertible*** (punya invers gak si A). Maka dari itu, kita cek dulu nilai determinan A.

$$\det det\ (A) = ad\ -\ bc\ = (6)(2) - (1)(5) = \ 12\ -\ 5\ =\ 7,\ yakni \neq 0$$

Maka matriks A ***invertible*** (punya invers). Sekarang baru kita cari invers dari matriks A.

$$A^{-1} = \frac{1}{ad-bc}[d \ - b \ - c \, a \,]$$

$$A^{-1} = \frac{1}{7}[2 \ - 1 \ - 5 \, 6\,] = \left[\frac{2}{7} \ - \frac{1}{7} \ - \frac{5}{7} \, \frac{6}{7} \right]$$

Sekarang akan kita perlihatkan $AA^{-1} = A^{-1}A = I$.

$$AA^{-1} = [6 \, 1 \, 5 \, 2\,]\left[\frac{2}{7} \ - \frac{1}{7} \ - \frac{5}{7} \, \frac{6}{7} \right] = [1 \, 0 \, 0 \, 1\,]$$

$$A^{-1}A = \left[\frac{2}{7} \ - \frac{1}{7} \ - \frac{5}{7} \, \frac{6}{7} \right][6 \, 1 \, 5 \, 2\,] = [1 \, 0 \, 0 \, 1\,]$$

```
[1.6] Determinan suatu Matriks; Contoh Perhitungan Invers Suatu Matriks


```{r}
A <- matrix(data = c(6, 5, 1, 2), nrow = 2, ncol = 2)

A
```

     [,1] [,2]
[1,]   6   1
[2,]   5   2

```{r}
det(A) #menghitung determinan dari A

```

[1] 7
```

```{r}
B <- matrix(data = c(2/7, -5/7, -1/7, 6/7), nrow = 2, ncol = 2)

B

```

```
          [,1]       [,2]
[1,]  0.2857143 -0.1428571
[2,] -0.7142857  0.8571429
```

```{r}
A %*% B
```

```
              [,1] [,2]
[1,]  1.000000e+00    0
[2,] -2.220446e-16    1
```

```{r}
B %*% A
```

```
     [,1] [,2]
[1,]    1    0
[2,]    0    1
```

```{r}
solve(A) #Menghitung invers dari matriks A

```

```
          [,1]       [,2]
[1,]  0.2857143 -0.1428571
[2,] -0.7142857  0.8571429
```

[1.7] Contoh Matriks Singular

Contoh kasus berikut diambil dari [1] halaman 46.

EXAMPLE 7 | Calculating the Inverse of a 2×2 Matrix

In each part, determine whether the matrix is invertible. If so, find its inverse.

$$(a)\ A = \begin{bmatrix} 6 & 1 \\ 5 & 2 \end{bmatrix} \qquad (b)\ A = \begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix}$$

Solution (a) The determinant of A is $\det(A) = (6)(2) - (1)(5) = 7$, which is nonzero. Thus, A is invertible, and its inverse is

$$A^{-1} = \frac{1}{7}\begin{bmatrix} 2 & -1 \\ -5 & 6 \end{bmatrix} = \begin{bmatrix} \frac{2}{7} & -\frac{1}{7} \\ -\frac{5}{7} & \frac{6}{7} \end{bmatrix}$$

We leave it for you to confirm that $AA^{-1} = A^{-1}A = I$.

Solution (b) The matrix is not invertible since $\det(A) = (-1)(-6) - (2)(3) = 0$.