NOTE TO STUDENTS Spring 2019: These Lecture Notes are based on the MAT 2680 course I taught in Spring 2017. They will be updated class-by-class, throughout the semester.

DAY 1: SEC 1.2
First Order Equations (p.7–13, 16-17)
p.14: 1, 2(a-c,e-h), 4(a-f), 5, 6, [optional: p. 14: 9 and p. 21: 1-11]

SKILLS:
- find the order of a differential equation
- verify a given function is a solution to a differential equation (or initial value problem)
  * review of derivatives (trig fcns, product rule, chain rule)
- find all solutions (or solve initial value problem) of given differential equation (straight integration)
  * review of integration techniques

Welcome, Course Policies
WeBWorK 1 & 2 due next Tues

2-minute intros

STUFF YOU NEED TO REMEMBER:
All the derivatives! Exponents, exponential functions, logs, trig functions (ALL OF THEM!)
Integrals: substitution, integration by parts

BIG IDEA: I'm thinking of a function, y= ______ (<- something involving 'x')
YOU HAVE TO FIGURE IT OUT.

HINT 1: $y' = x^3$
HINT 2: y(1)=2

Defn. A differential equation is an equation that contains one or more derivatives of an unknown function. The **order** of a differential equation is simply the highest derivative that appears in the equation (compare to the *degree* of a polynomial).

Example 1: $y = \frac{x}{2} \cdot y'$.

QUESTION: What's the order? Is the function $y = x^2$ a solution?

RECALL NOTATION:
FUNCTION: y, f(x), y(x),
DERIVATIVE: dy/dx, f'(x), y', y'(x)
VARIABLE: Sometimes we use 't' instead of 'x' f(t)=t^3+2t+1

What is a solution? It's a function y that makes the equation work.

Example 2: $y' = x^3$
Find ALL solutions.
Now find one particular solution that satisfies y(1)=2?

When we add one or more conditions on the solution, we call it an **initial value problem**.

GROUP WORK:

Example 3: Is $y = x\sin x$ a solution to the differential equation
$y\sin x + y'\cos x - 1 = \sin x\cos x$?

Example 4: Find the solution to the differential equation
$y' = x\sqrt{x^2 + 8}$ satisfying $y(1) = 11$.

Example 5: Is $y = \frac{1}{2} + e^{-x^2}$ a solution to the initial value problem
$y' + 2xy = x,\ y(1) = \frac{3}{2}$?

Example 6: Find all solutions to $y' = xe^x$.

Example 7: Find the solution to $y'' = x^5 + \sqrt[3]{x^2} + x^{-2}$ satisfying
$y'(1) = \frac{23}{30}$ and $y(1) = 2$

SOLN:
Ex7: y(x) = (9 x^(8/3))/40 + x^7/42 + x - ln(x) + 631/840

DAY 2: Section 2.1
Linear First Order Equations (p.30–41)
p.41: 1-9 odd, 17-23 odd, 31-37 odd, 38, 40, 42
- find general solutions to a variety of linear first-order differential equations and initial value problems
    - homogeneous and non-homogeneous

HEADS UP: Very often we use "t" instead of "x" as our variable....

RECALL: Implicit differentiation - find the derivative $\frac{d}{dx}\ln|y|$

Defn: A first-order differential equation is **linear** if it has the form: $y' + p(x)y = f(x)$
We call such an equation **homogeneous** if $f(x) = 0$, that is: $y' + p(x)y = 0$

Example 1: Solve $y' - x^2y = 0$

Move "y" to RHS, divide by y to separate variables, now integrate. HINT: use "k" as constant!!
ANS: $y = ce^{x^3/3}$

CLASS
Example 2: Solve $xy' + y = 0$. Find the particular solution for which y(1)=3

GENERAL FORMULA FOR SOLVING HOMOGENEOUS LINEAR EQUATIONS
Let's figure out a general method to solve homogeneous linear equations of the form: $y' + p(x)y = 0$ (where p(x) is continuous on some interval (a,b)).
STRATEGY: Rewrite as $\frac{y'}{y} =- p(x)$. Integrate, raise *e* to each side, simplify.

Solution: $y = ce^{-P(x)}$, where $P(x) = \int p(x)dx$ is *any particular* antiderivative of p(x) on (a,b)

NON-HOMOGENOUS EQUATIONS AND VARIATION OF PARAMETERS
What about non-homogenous linear equations? $y' + p(x)y = f(x)$

First, consider the corresponding homogeneous equation (called the **complementary equation**):
$y' + p(x)y = 0$
Suppose we have a solution to the complementary equation - let's call it y_1 - so that
$y_1' + p(x)y_1 = 0$
We're going to modify y_1 to try to find a solution to the original equation.
A TIME-HONORED TRADITION IN DIFFY Qs: "the guess"
Let's guess that our solution has the form "something times y1", so $y = uy_1$, where u is some unknown function of x (we call u an **integrating factor**). Let's try to find u!

Why this guess? Good question - we want to be able to plug into the left side, and have most (but not all) parts cancel out, leaving behind exactly f(x)... playing around a bit shows that uy1 is a good candidate.

Substitute $y = uy_1$ into the equation. For the derivative y', what rule do we use?

$u'y_1 + u(y_1' + p(x)y_1) = f(x)$

Thus $u'y_1 = f(x)$, so $u' = \frac{f(x)}{y_1(x)}$.

Now integrate to find u, then substitute into $y = uy_1$ to find the solution y to the original equation.

Example 3: $y' + 2y = x^3e^{-2x}$

STRATEGY to solve $y' + p(x)y = f(x)$:
STEP 1: Solve the complementary equation $y' + p(x)y = 0$, let $y_1(x)$be a solution.
STEP 2: Look for a solution of the form $y = uy_1$. Substitute into the original equation.
STEP 3: Solve for u by isolating u' and integrating.
STEP 4: Substitute $y_1(x),\ u(x)$ to find the final solution: $y = uy_1$

Example: a) Find the general solution $y' + (\cot x)y = x \csc x$
b) Solve the initial value problem $y' + (\cot x)y = x \csc x$, $y(\pi/2) = 1$

DAY 3: Section 2.2
Separable Equations (p.45–52)
p.52: 2, 3, 6, 12, 17–27 odd, 28, 35, 37
Solve separable equations
- implicitly vs explicitly (solve for y)
- general vs particular solution/IVP

WEBWORK PROBLEMS 1&2: Don't recognize combined constants as a new constant, e.g. solution must be "+7c" instead of "+c", even though 7c is, indeed, a constant

Example 1: $y' = -\frac{x}{y}$

COULD DO A DISCUSSION OF: y' giving slope, drawing a slope field, then talk about separable and looking at possible solutions for different values of C??? USE DESMOS SLOPE FIELD: https://www.desmos.com/calculator/p7vd3cdmei

Defn: A first-order differential equation is **separable** if it can be written in the form $h(y) \cdot y' = g(x)$. *That is, we can separate the variables - x on one side, y on the other, with the y side multiplied by y'.*
NOTE: We can solve a separable equation by integrating both sides.

Example 1: a) Solve $y' = -\frac{x}{y}$.
b) Find the particular, explicit solution when y(1)=1. On what interval is the solution valid?
c) Find the particular, explicit solution when y(1)=-2. On what interval is the solution valid?

IMPLICIT SOLUTIONS
Note: Sometimes we cannot find an explicit solution (we can't "solve for y"), but we can still give an **implicit solution:** an equation in y and x that describes the solution.

Example 2. Show that the equation $\frac{dy}{dx} = \frac{x^2}{1-y^2}$ is separable, and then find the solution.
Is your solution implicit or explicit? Can you find an explicit solution?
ANS: $-x^3 + 3y - y^3 = c$

Example 3. Solve the initial value problem explicitly: $y' + y^4\cos(5x) = 0,\ y(0) = 1$. On what interval is the solution valid?
ANS: $y = -\sqrt[3]{\frac{-2}{2+\sin(6x)}}$, -infty<x<infty

DAY 4: Section 2.4

Transformation of Nonlinear Equations into Separable Equations (p.62–68)
p.68: 1–4, 7–11 odd, 15–18, 23–27 odd
- Bernoulli Equations

TODAY'S TRICK: Using substitutions to turn nonlinear, nonseparable equations into separable equations.

HEADS UP: In today's lecture let's use "t" as our independent variable instead of "x"...

Defn: A **Bernoulli Equation** has the form $y' + p(t)y = f(t)y^r$, where r is any real number except 0 or 1.
STEP 1: Let $y_1(t)$ be a solution to the complementary equation $y' + p(t)y = 0$.
STEP 2: Guess a solution of the form $y = uy_1$, where $u(t)$ is some (unknown) function of t.
STEP 3: Substitute into the original equation and separate variables. Then integrate to find u.

Example 1: Solve $y' - y = ty^2$

ANS: $y = \frac{e^t}{e^t(1-t)+c}$ or $y = -\frac{1}{t-1+ce^{-t}}$

Example 2: Solve the IVP $ty' + y = t^4y^4$, $y(1) = 1/2$

ANS: $y = \frac{1}{t(11-3t)^{1/3}}$

Defn: A differential equation is called **homogeneous** if it can be written in the form $y' = f(\frac{y}{x})$.
That is, every occurrence of the variables on the right is in the form of a fraction y/x.
ANNOYING NOTE: Yes, this meaning of "homogeneous" is entirely different than the way we used it previously ("=0").

STEP 1: In this case, we **always** use $y_1 = t$. MEMORIZE IT!
STEP 2: Guess a solution of the form $y = uy_1$ (so $y = ux$).
STEP 3: Substitute into the original equation, rearranging to use $u = \frac{y}{x}$ when necessary. Then integrate to find u.

Example 2: $y' = \frac{y+te^{-y/t}}{t}$

DAY 5: Section 2.5
Exact Equations (p.73–79)
p.79: 1–21 odd, 29, 30, 33, 34

Example 1. Solve: $6x^5 + 2xy^2 + (2yx^2 + 5y^4)\frac{dy}{dx} = 0$ (give the solution implicitly)
Verify that this equation is an implicit solution: $x^6 + x^2y^2 + y^5 = c$

How do we check? Take the derivative, and see if we get the original differential equation. NOTE: Is this a regular derivative (treating y as a function of x), or a partial derivative (treating y as a constant)? CHECK THE DERIVATIVE NOTATION IN THE EXAMPLE!

Note: The left side of the implicit solution will be important - it's a function of x and y, let's call it F(x,y): $F(x, y) = x^6 + x^2y^2 + y^5$
THE BIG QUESTION: How do we find the function $F(x, y) = x^6 + x^2y^2 + y^5$?

Example 2. Find the partial derivatives of the function $F(x, y) = x^6 + x^2y^2 + y^5$
RECALL: How many partial derivatives are there? When we take a partial derivative wrt one variable, how do we treat the other variable?

NOTE: The differential equation $6x^5 + 2xy^2 + (2yx^2 + 5y^4)\frac{dy}{dx} = 0$ can be broken into two parts: $M(x, y) + N(x, y)y' = 0$
M(x,y) is the result of taking the partial derivative of F with respect to x: $\frac{\partial}{\partial x}F(x, y) = M(x, y)$ *(We also write F_x)*
N(x,y) is the result of taking the partial derivative of F with respect to y: $\frac{\partial}{\partial y}F(x, y) = N(x, y)$ *(We also write F_y)*

Starting with M, N, how do we find F? Integrate M with respect to x to find F
BEWARE: When we take the partial derivative of F with respect to x, what "disappears"? *Anything that is a pure function of y.* So when we integrate, we have to put back "a function of y" (let's call it $\phi(y)$ -- this plays the role that a constant usually plays in standard integration.

$\int 6x^5 + 2x^2dx = x^6 + x^2y^2 + \phi(y)$.

This is our $F(x, y) = x^6 + x^2y^2 + \phi(y)$. BUT we have to figure out what $\phi(y)$ is. How do we do it? Take the partial derivative of F with respect to y -- this should equal M.
$F_y = 2x^2y + \phi'(y)$. This had better equal N, so $2x^2y + \phi'(y) = 2yx^2 + 5y^4$, so in this case $\phi'(y) = 5y^4$. Integrate to find $\phi(y) = y^5$.
Now, what is F(x,y)= ?

QUESTION:
Given a differential equation of the form: $M(x, y) + N(x, y)y' = 0$, how do we know that there is a function $F(x, y)$ that will work as in the example above? We have to verify the following condition:
EXACTNESS CONDITION: If such a $F(x, y)$ exists, then $F_{xy} = F_{yx}$, and vice versa. Therefore, to check if such a F exists, we simply have to check whether $M_y = N_x$.

Let's put all of this together.

SOLVING EXACT EQUATIONS:
Given a differential equation of the form: $M(x,y) + N(x,y)y' = 0$
1. Verify that the equation is **exact** by checking that $M_y = N_x$.
2. Integrate *M* with respect to x to obtain $F(x,y)$. *Treat y as a constant. Don't forget to add a "constant" term* $\phi(y)$.
3. Take the partial derivative F_y and set it equal to *N*, solve for $\phi'(y)$.
4. Integrate $\phi'(y)$ to find $\phi(y)$
5. The general solution to the differential equation is given implicitly by: $F(x,y) = c$.

Example 3. Solve $(y\cos x + 2xe^y) + (\sin x + x^2e^y - 1)y' = 0$
ANS: $y\sin x + x^2e^y - y = c$

Example 4. Solve $(3xy + y^2) + (x^2 + xy)y' = 0$
NOTE: $M_y \neq N_x$*, and so this equation is not exact - this method will not work.*

NOTE: I can realistically do ONE application in a class, and there are 2 days to cover 3 sections. In Spring 2019, I chose to focus on: Day 6: population, and Day 7: Newton's Law of Cooling (skipped Sec 4.3)

DAYS 6&7: Applications
: Section 4.1,4.2
4.1 Growth and Decay (p.130–137) p.138: 1–7 odd, 11, 13, 17
4.2 Cooling and Mixing (p.140–147) p.148: 1–11 odd, 15

4.1 - Radioactive decay - (half-life, initial amount, quantity over time Q(t))
- More applications (bread dough rising, candy consumption, water in a tank) in which change in quantity is related to quantity, plus other factors
- WeBWorK: rabbit populations, falling object with wind resistance
- WeBWorK OPL: radioactive decay, population, investments, drugs in bloodstream
4.2 - cooling problems (an object is moved from one temp to another, temperature over time. Questions about temperature at various times)
- WW OPL: not too many cooling problems - a few asking for numerical answers of various sorts
- mixing problems (tank with certain solution of water+salt, another solution added at a certain rate, mixture is drained at another rate)
- WW OPL: mixing problems.

How do we model real-world situations with differential equations?

MATHEMATICAL MODELS
Defn. A differential equation that describes some physical process is often called a mathematical model.

"If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is." --- John von Neumann

Discuss: simplifying assumptions. math describes "perfect world", if we're lucky we can find math model that gives a very good approximation to the "real world"

Example 1: A population of field mice inhabit a rural area. Some owls live in the area. Let's track the population p(t) over time (t in months).

- every month, each pair of mice produces one new mouse.
- every night, the owls eat 15 mice.

How many mice are killed each month?
How many mice are born each month?
Does the population go up, or down?
Depends on the population!

A. What if we start with 600 mice p(0)=600 - what is the population after one month? Two months?
B. What if we start with 1000 mice p(0)=1000 - how many after one month? Two months?

Let's translate this into a differential equation:
$\frac{dp}{dt} = 0.5p - 450$ "The rate of change of the population is 0.5 times the population minus 450"

Solve for p(t). (Use *c* as multiplicative constant.)

C. What is the particular solution for A. p(0)=800? What happens to this over time?
D. What is the particular solution for B. p(0)=1000? What happens over time?

In the first case, the initial population is too small - so the predation overcomes the reproduction rate, and the mice die off.
In the second case, the population is too large - so the reproduction rate overcomes the predation, and the mice population grows out of control.

Graph using slope field: https://bluffton.edu/homepages/facstaff/nesterd/java/slopefields.html
GRAPH: 0.5y-450
X: 0, 10
Y: 0,1200

Is there any "middle ground", an initial population for which the predation and reproduction are balanced and the population stays steady?

The *equilibrium solution* is p(0)=900.
The equilibrium solution can be obtained by setting the derivative $\frac{dp}{dt} = 0$ and solving for p.

GENERAL SETUP: $\frac{dp}{dt} = rp - k$

- Where t is time (measured in months), and p(t) is the population.
- r is the rate constant or growth rate (the rate at which reproduction occurs)
- k is the predation rate (the number killed each month by predators)

NEWTON'S LAW OF COOLING
You take a hot cup of coffee outside in the middle of winter. What happens to the temperature T(t) of the coffee over time?

Does it continue to cool forever?
Does it cool more quickly if the temperature outside is 22℉ (New York) or -40℉ (Alaska)?

"The rate of change of temperature is proportional to the DIFFERENCE between the temperature T of the object and the temperature T_m of the environment (or medium)"

NEWTON'S LAW OF COOLING:
$T' = -k(T - T_m)$

- T(t) is the temperature at time t
- T_m is the temperature of the medium (environment)
- k is a positive quantity, the *temperature decay constant* (depends on surface area of object and various properties of the environment)

NOTE: If T_0 is the initial value of T, then the general solution is $T = T_m + (T_0 - T_m)e^{-kt}$

Resource on coffee temperatures: https://driftaway.coffee/temperature/
Example (update): A n extra-hot cup of coffee at 180℉ is carried outside in 22℉ weather. After 5 minutes, the temperature of the coffee has dropped to 160℉. How long does it take the coffee to reach the perfect temperature of 130℉?

T=22+(158)e^(-0.027068t)
t=14.0559 minutes

Example (OLD): A cup of coffee at 200℉ is carried outside in 22℉ weather. After 5 minutes, the temperature of the coffee has dropped to 120℉. How long does it take the coffee to reach 75℉?
T=22+178e^(-0.11936t)
t=10.1499 minutes

SPRING 2019: Skipping Section 4.3, as I already spent 2 days on applications (population, and cooling).

~~**Day 7: Section 4.3**~~
~~4.3 Elementary Mechanics (p.151–160)~~
~~p.160: 3, 5, 7, 10~~
~~-objects in motion subject to gravity + air resistance~~

~~FALLING OBJECTS (turn to a friend, DISCUSS these three questions with them for 5 minutes, come up with some answer to each)~~

~~**Example 1:** Suppose that an object is falling in the atmosphere near sea level. Formulate a differential equation that describes the motion.~~
- ~~What kind of function are we trying to find? Position? Velocity? Acceleration?~~
- ~~Any guess as to what the solution will look like? Will it increase or decrease over time?~~
- ~~What are different factors that might influence the motion of the object?~~

~~SKETCH SOME OPTIONS - label v-axis, t-axis, etc~~

~~Thinking question: What are the different factors that might influence the motion of the object?~~
- ~~Newton: $F = ma$ (let us measure mass in kg, acceleration in m/s^2, force in Newtons)~~
- ~~Relationship between a(t) and v(t)? $a = \frac{dv}{dt}$~~
- ~~$F = m\left(\frac{dv}{dt}\right)$~~
- ~~Force #1: Gravity pulls down with force mg (g=acceleration due to gravity)~~
- ~~Force #2: Air resistance pushes up with force proportional to velocity γv (γ= constant = drag coefficient)~~

- Combine: $m\left(\frac{dv}{dt}\right) = mg - \gamma v$

  $m\left(\frac{dv}{dt}\right) = mg - \gamma v$
  m = mass (kg)
  v = velocity (m/s)
  g = acceleration due to gravity = $9.8 m/s^2$
  γ= gamma = drag coefficient

Example 2: Suppose the mass of the object in example 1 is 10kg, and the drag coefficient has been determined to be $\gamma = 2\ kg/s$. Write the differential equation describing the object's motion in the form $\frac{dv}{dt}$ = (that is, isolate $\frac{dv}{dt}$ on one side).

ANS: $\frac{dv}{dt} = 9.8 - \frac{v}{5}$

Example 3: Solve this differential equation - that is, find a formula for v(t).

solve for v: $v = 49 + ce^{-t/5}$

LOOK AT A SLOPE FIELD! What do you see?

What happens if c is: positive, negative, 0?

Example 4: Is there a solution in which the velocity is constant? What is it?

RULE: to find it exactly, set the derivative to 0.

Defn. A constant solution to a differential equation is called an **equilibrium solution**.
In the case of a falling object, it is often referred to as **terminal velocity**.

Example 6: Find the specific solution describing the the motion of an object if the initial velocity is $v_0 = 35\ m/s$:

Day 8: EXAM 1

Day 9: Section 3.1

3.1 Euler's Method (p.96–106)

p.106: 1–7 odd, 11–13, 15–19 odd, 20–22

- compute solutions to various IVPs using Euler's method. Compare approximate solution to actual value where possible.

NOTE: The movie Hidden Figures features a mention of "Euler's Method" in the key "chalkboard scene": https://www.youtube.com/watch?v=v-pbGAts_Fg While I'm not sure of the exact problem they are solving, I did find this publication about the re-entry problem (technical): https://www.faa.gov/other_visit/aviation_industry/designees_delegations/designee_types/ame/media/Section%20III.4.1.7%20Returning%20from%20Space.pdf

Example 1: Suppose y(x) is a solution to the initial value problem dy/dx=3-2x-0.5y, y(0)=1. Find the value of the function y at x=1 (find y(1)).
How do we do it?

Discuss
OPTION A:
1. Solve the differential equation.
2. Use the initial value y(0)=1 to find "c" and obtain the particular solution
$y = 14 - 4x - 13e^{-0.5x}$ *or y=14-4x-13e^(-0.5x)*
3. Substitute x=1 into the particular solution to find the value y(1).
NOTE: y(1)=2.11510

What if we can't solve the differential equation?
OPTION B:
Try to approximate the answer - numerical methods.
BENEFITS: It always works, even if we can't solve the differential equation!
DRAWBACKS: It only gives an approximate answer, not an exact answer.

Sketch.

Let's divide the x-interval up into 4 equal pieces.
How wide is each piece?
ANS: This important quantity is called the **step size** h. Here, h=0.25.
What are the x-coordinates?
x_0 = 0
x_1 = 0.25
x_2 = 0.5
x_3 = 0.75
x_4 = 1

Find (x_1,y_1)

Now, to find the y-value as we move from x_0 to x_1, we will use the slope of the function to determine whether we move up or down. What is the slope of the solution y at the point (0,1)?

ANS: y'=2.5

Are we moving up or down? *Sketch line. This is the **tangent line** to the function at the point (0,1).*

QUES: Could we find the equation of the tangent line, if we wanted?

ANS: YES - need a point and the slope, then use point-slope form y-y1 = m(x-x1)

How far up do we move, as we go from x_0=0 to x_1 =0.25? *Think about slope m = rise/run. If we know the "run" (the x-distance), we can calculate the "rise" (the y-distance) by multiplying run * slope.*

Y-distance is 0.25 * (2.5) = 0.625

The new y-coordinate is: old y-coordinate + y-distance = 1+0.625 = 1.625

Find (x_2,y_2)

x_2 = 0.5

y_2 = y_1 + h * (slope at (0.25, 1.625))

Continue, until you find the point (1,y_n). What is the final value of y? This is an approximation of y(1).

i	h	x_i	y_i	k = f(x_i,y_i)	y_(i+1)
0	0.25	0	1	2.5	1.625
1	0.25	0.25	1.625	1.6875	2.046875
2	0.25	0.5	2.046875	0.9765625	2.291015625
3	0.25	0.75	2.291015625	0.3544921875	2.379638672
4	0.25	1	**2.379638672**		

How close is our approximation to the "real" value? We would have to know the "real" answer to compare.

Here is the real answer: y(1) = 2.1151...

How do we make it better? Use more points / use a smaller step size!

What if h=0.1. How many points? Final approximation for y(1)?

Use spreadsheet here:
https://docs.google.com/spreadsheets/d/1DelFRN2CKpNk4oZ6UrUppaEFeyJlcj4fHbN_PsqaUMo/edit?usp=sharing

Euler's Method. Given the differential equation y' = f(x,y) with initial condition y(a)=b, find an approximate value of the solution at x=c using step size *h*.

- Find a sequence of points (x_0,y_0), (x_1,y_1), (x_2,y_2)...(x_n,y_n)
    - NOTE: the number of steps *n* is related to the step size *h* by: h = (c-a)/n

- The first point (x_0,y_0) is given by the initial condition y(a)=b, so x_0=a and y_0=b
- x-coordinate: x_(i+1)=x_i + h
  You can write down the x-coordinates immediately, since h is the difference between successive x-values:
- y-coordinate: y_(i+1) = y_i + h*f(x_i,y_i)
  To approximate the value of y at x_(i+1), we use the slope of y at x_i

I recommend making a table:

n	x_n	y_n	f(x_n,y_n)	y_(n+1)

SAVE THIS EXAMPLE:
Example2:
dy/dx=-2y+x^3 e^(-2x)
Solve

Example 3:
dy/dx = x^2-sin(x)*y^2, y(0.6)=3.5, estimate the value of y(1.8) using Euler's method with a step size of h=0.2.

"CORRECT" ANS: y(1.8) = 1.60733

i	h	x_i	y_i	k = f(x_i,y_i)	y_(i+1)
0	0.2	0.6	3.5	-6.556870299	2.18862594
1	0.2	0.8	2.18862594	-2.796195579	1.629386824
2	0.2	1	1.629386824	-1.234022515	1.382582321
3	0.2	1.2	1.382582321	-0.3416242858	1.314257464
4	0.2	1.4	1.314257464	0.2578596019	1.365829385
5	0.2	1.6	1.365829385	0.6953055316	1.504890491
6	0.5	1.8	1.504890491	1.03453176	2.022156371

WHAT DO YOU NEED TO BE ABLE TO DO?
1. Implement Euler's method by hand (with calculator) on the exam
2. Implement Euler's method and other numerical methods using your choice of technology

Day 8: Section 3.2
3.2 The Improved Euler Method and related Methods (p.109–116)
p.116: 1–7 odd, 11–13, 15–19 odd, 20–22

- compute solutions to various IVPs using Improved Euler's method. Compare approximate to actual where possible

Since yesterday's initial example ran long, I started today with a "Big Picture" - here's the setup that numerical methods can help with, and here's the 1-2-3 steps for Euler's method (and showed them using Example 1). Then did the same for Improved Euler's.

SETUP: y'=f(x,y), y(a)=b, approximate y(c) using step size h.
Find sequence of points, (x_0,y_0)=(a,b), (x_1, y_1), ... (x_n, y_n)
n = number of steps
h = (c-a)/n

EULERS METHOD
Start with (x_i, y_i)

- x_(i+1)=x_i + h
- y_(i+1) = y_i + h*f(x_i,y_i)

End with (x_(i+1), y_(i+1))
I recommend making a table:

i	x_i	y_i	f(x_i,y_i)	y_(i+1)

IMPROVED EULERS METHOD
Start with (x_i, y_i)

- x_(i+1) = x_i + h
- k1=f(x_i, y_i)
- z_(i+1) = y_i + h*k1
- k2 = f(x_(i+1), z_(i+1))
- y_(i+1)=y_i+h * [k1+k2]/2

End with (x_(i+1), y_(i+1))

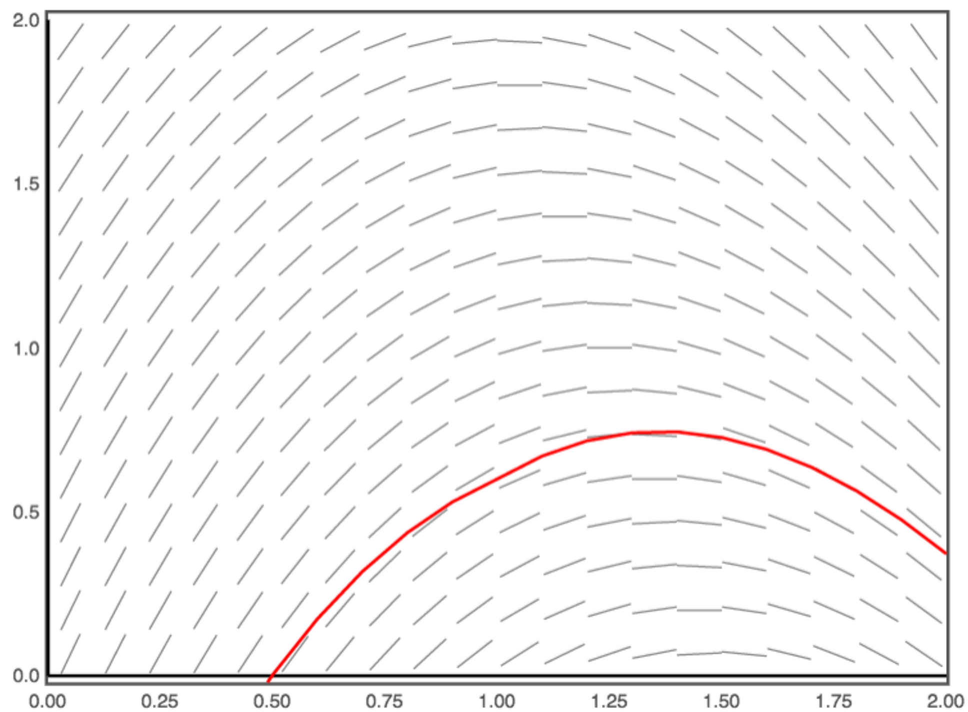
Example 1: Suppose y(x) is a solution to the initial value problem dy/dx=3-2x-0.5y, y(1)=0.6. Find an approximate value of y(2) using Euler's method with step size h=0.5.

NOTE: The solution to this initial value problem is:
y(x) = -4 x - 15.498 e^(-0.5 x) + 14
"CORRECT" ANS: y(2) = 0.298612

i	h	x_i	y_i	k = f(x_i,y_i)	y_(i+1)
0	0.5	1.0	0.6	0.7	0.95
1	0.5	1.5	0.95	-0.475	0.7125
2	0.3	2.0	**0.7125**		

PROBLEM: In Euler's method, when moving one point (x_i,y_i) to the next (x_(i+1),y_(i+1)) we use the slope at the first point to approximate the curve on the entire interval. If the slope varies across the interval, this may be inaccurate.

Discuss using slope field generator here:
https://bluffton.edu/homepages/facstaff/nesterd/java/slopefields.html?flags=0&ODE=x,y&SYS=t,x,y&dydx=3-2x-0.5y&dxdt=x+y&dydt=x*y-1&x=0,2,20&y=0,2,15&method=euler&h=0.1&pts0=%5B1,0.6%5D
or this screenshot:



We can improve the accuracy by using *the average of the slopes at the two points* (x_i,y_i) and (x_(i+1),y_(i+1)).

BASIC IDEA: Recall that the slope at a point is given by f(x,y). In Euler's method, we compute the next y-value by: y_(i+1)=y_i + h*f(x_i,y_i)

> To improve this, we want to use: y_(i+1)=y_i + h* [f(x_i,y_i)+f(x_(i+1),y(i+1)]/2
> PROBLEM: We don't yet know the y-value of the next point, y_(i+1). So instead, we use Euler's method to get an initial approximation of y_(i+1) -- call it z_(i+1) -- and then use the slope at (x_(i+1), z_(i+1)) in our calculation.

IMPROVED EULER FORMULA: yn+1=yn+f(tn,yn)+f(tn+h, yn+hfn)2h
NOTE: This is a "two step" method -- first we calculate z_(i+1) = y_i+h f(x_i,y_i), then we use that value to plug in and calculate y_(i+1).

IMPROVED EULER METHOD Given a point (x_i,y_i), how do we find the next point, (x_(i+1),y_(i+1))

Calculate:
Find x_(i+1) = x_i + h
Find k1=f(t_i, y_i)
Find z_(i+1) = y_i + h*k1
Find k2 = f(t_(i+1), z_(i+1))
Find y_(i+1)=y_i+h * [k1+k2]/2
Now we have (x_(i+1),y_(i+1))

Example 2: Suppose y(x) is a solution to the initial value problem dy/dx=3-2x-0.5y, y(1)=0.6. Find an approximate value of y(2) using the Improved Euler's method with step size h=0.5.

"CORRECT" ANS: y(2) = 0.298612

i	h	x_i	y_i	k1	z_(i+1)	k2	y_(i+1)
0	0.5	1	0.6	0.7	0.95	-0.475	0.65625
1	0.5	1.5	0.65625	-0.328125	0.4921875	-1.24609375	0.2626953125
2	0.5	2	**0.2626953125**				

Example 3:
dy/dx = x^2-sin(x)*y^2, y(0.6)=3.5, estimate the value of y(1.8) using Improved Euler's method with a step size of h=0.2.

"CORRECT" ANS: y(1.8) = 1.60733

i	h	x_i	y_i	k1	z_(i+1)	k2	y_(i+1)
0	0.2	0.6	3.5	-6.556870299	2.18862594	-2.796195579	2.564693412
1	0.2	0.8	2.564693412	-4.07851894	1.748989624	-1.574030043	1.999438514
2	0.2	1	1.999438514	-2.363994307	1.526639652	-0.7322369764	1.689815385
3	0.2	1.2	1.689815385	-1.221415276	1.44553233	-0.0991600018 6	1.557757858
4	0.2	1.4	1.557757858	-0.431301719 3	1.471497514	0.3956183461	1.55418952
5	0.2	1.6	1.55418952	0.1455248986	1.5832945	0.7987378463	1.648615795
6	0.2	1.8	**1.648615795**				

Day 8: Section 3.3
The Runge-Kutta Method (p.119–124)
p.124: 1–7 odd, 11–13, 15–19 odd, 20–22

- compute solutions to various IVPs using the Runge-Kutta method. Compare approximate solution to actual value where possible.

Today we are going to consider one final method. This method is the most complicated, but comes with a corresponding increase in precision - the solutions "get better quickly" as the step size decreases. This method is powerful enough to be used in many modern numerical methods applications - the Runge-Kutta Method.

BASIC IDEA. We will *not* carefully develop this method from scratch. However, I want to give you a flavor of the idea.

COMPARE: To approximate the solution curve y(x) on an interval between points, from x_i to x_(i+1):

- The basic idea of Euler's Method is to approximate the solution curve y(x) with a straight line. *SKETCH*
- The basic idea of the Runge-Kutta Method is to approximate the solution curve y(x) with a parabola instead. *SKETCH.*

FACT ABOUT PARABOLAS: On an interval [a,b], the slope of the secant line through the endpoints is equal to what? (any guesses?)

- NOT the average of the slopes at each end, but
- Use the slope at three points - the two endpoints, and the point in the middle. BUT weight the slope in the middle more heavily - count it 4 times.
- If the three slopes are m1, m2, and m3, then the slope of the secant line is:
  m = (m1 + 4m2 + m3)/6
- This is the idea in Runge-Kutta.
- PROBLEMS: We don't actually know the three points - only the first one. So we make a series of approximations, two in the middle and one on the right, and combine the slopes at these approximations in the correct way.

The classic Runge-Kutta method uses a weighted average of slopes to compute the next y-value:
y_(i+1)=y_i + h (k1+ 2*k2 + 2*k3 + k4)/6 $\quad y_{n+1} = y_n + h\left(\frac{k_1+2k_2+2k_3+k_4}{6}\right)$

NOTE: When I presented this in class in Spring 2017, I simplified the formulas by first finding each "approximate y-value" used in the subsequent evaluation of f(x,y). I labeled these z2,z3,z4 to match the corresponding k2,k3,k4 - here's the version I used in class:

RUNGE-KUTTA METHOD:
Start with: (x_i, y_i).

- x_(i+1) = x_i + h

- k1 = f(x_i, y_i)
- z2 = y_i + 0.5h*k1
- k2 = f(x_i + 0.5h, z2)
- z3 = y_i + 0.5h*k2
- k3 = f(x_i + 0.5h, z3)
- z4 = y_i + h*k3
- k4 = f(x_i + h, z4)
- y_(i+1)=y_i + h (k1+ 2*k2 + 2*k3 + k4)/6

End with (x_(i+1), y_(i+1))

RUNGE-KUTTA METHOD:
Start with: (x_i, y_i).

- x_(i+1) = x_i + h
- k1 = f(x_i, y_i)
- k2 = f(x_i + 0.5h, y_i + 0.5h*k1)
- k3 = f(x_i + 0.5h, y_i + 0.5h*k2)
- k4 = f(x_i + h, y_i + h*k3)
- y_(i+1)=y_i + h (k1+ 2*k2 + 2*k3 + k4)/6

End with (x_(i+1), y_(i+1))

Example 1: Consider the initial value problem y'+2y=x^3 e^(-2x), y(0)=1. Approximate the value of y(0.6) using a step size of 0.3.

Example 1:	y'+2y=x^3 e^(-2x), y(0)=1		find y(0.6) using 2 steps					
i	**h**	**x_i**	**y_i**	**k1 = f(x_i,y_i)**	**k2=f(x_i+.5h,y_i+.5hk1)**	**k3=f(x_i+.5h, y_i+.5hk2)**	**k4=f(x+h,y+hk3)**	**Runge-Kutta y_(i+1) = y_i + h*(k1+2k2+2k3+k4)/6**
0	0.3	0	1	-2	-1.397499739	-1.578249817	-1.038232196	0.5505134347
1	0.3	0.3	0.5505134347	-1.086208955	-0.7381155225	-0.8425435523	-0.5304427882	0.31161494
2	0.3	0.6	**0.31161494**					

Table 3.3.1. Numerical solution of $y' + 2y = x^3e^{-2x}$, $y(0) = 1$, by the Runge-Kuttta method and the improved Euler method.

x	$h = 0.1$	$h = 0.05$	$h = 0.1$	$h = 0.05$	Exact
0.0	1.000000000	1.000000000	1.000000000	1.000000000	1.000000000
0.1	0.820040937	0.819050572	0.818753803	0.818751370	0.818751221
0.2	0.672734445	0.671086455	0.670592417	0.670588418	0.670588174
0.3	0.552597643	0.550543878	0.549928221	0.549923281	0.549922980
0.4	0.455160637	0.452890616	0.452210430	0.452205001	0.452204669
0.5	0.376681251	0.374335747	0.373633492	0.373627899	0.373627557
0.6	0.313970920	0.311652239	0.310958768	0.310953242	0.310952904
0.7	0.264287611	0.262067624	0.261404568	0.261399270	0.261398947
0.8	0.225267702	0.223194281	0.222575989	0.222571024	0.222570721
0.9	0.194879501	0.192981757	0.192416882	0.192412317	0.192412038
1.0	0.171388070	0.169680673	0.169173489	0.169169356	0.169169104
	Improved Euler		Runge-Kutta		Exact

Day 9: Chapter 5 - Second Order Equations
Section 5.1 - 5.1 Homogeneous Linear Equations (p.194–203) - p.203: 1–5 odd, 9–21 odd
NOTE: I'm glossing over Sec 5.1, in part because we are running behind this semester. Instead, I'll give a quick intro to second order equations and then focus on constant coefficients
Section 5.2
Constant Coefficient Homogeneous Equations (p.210–217) - p.217: 1–17 odd, 18–21
NOTE: Today I'll cover only equations with two distinct real roots

DISCUSS: Second-order differential equations
Defn. A **linear second-order differential equation** can be written in the form: y"+p(x)y'+q(x)y=f(x)

NOTE: If the equation cannot be written in the form above, then it is nonlinear. Nonlinear second order equations are generally hard!

Example: y" - y = 0
a) What it the order? Is the equation linear? What are p(x), q(x), f(x)?
Defn: We call such an equation **homogeneous** if f(x)=0.
b) Verify that y1(x)=e^x is a solution.
Is this the only solution? *Try multiplying by a constant*
c) Verify that y2(x)=e^-x is a solution
NOTE: We can combine these two solutions to get the most general solution
d) Verify that y = c1 e^x + c2 e^-x is a solution
Discuss: How many constants? How many initial values will we need, to solve?
NOTE: y = c1 e^x + c2 e^-x is the **general solution** to the differential equation
d) Solve the initial value problem: y" - y = 0, y(0)=1, y'(0)=3

NOTE: What's the "hard part" of the above problem, which you were *NOT* asked to do? *Find the two basic solutions!*

ALTERNATE EXAMPLE - NONCONSTANT COEs - SKIP FOR NOW
Example: Given the differential equation: x^2 y" + x y' -4y = 0
a) What it the order? Is the equation linear? What are p(x), q(x), f(x)?
*Defn: We call such an equation **homogeneous** if f(x)=0.*
b) Verify that y1(x)=x^2 is a solution
c) Verify that y2(x)=1/x^2 is a solution
NOTE: We can combine these two solutions to get new solutions by: adding, multiplying by constant.
FACT: the general solution to this differential equation is y = c1 x^2 + c2/x^2
d) Verify that y = c1 x^2 + c2/x^2 is a solution
Discuss: How many constants? How many initial values will we need, to solve?
d) Solve the initial value problem: x^2 y" + x y' -4y = 0, y(1)=2, y'(1)=0

DISCUSS: In general, homogeneous linear second order equations have **two basic solutions y1 and y2** - and the **general solution is given by linear combination: y = c1y1+c2y2**
NOTE: Even linear second-order equations are often hard - so we will start simple.

Defn. The equation has **constant coefficients** if p(x) and q(x) are constant. In general, we allow a constant in front of the y" term as well:
Linear second-order homogeneous w/ constant coefficients: ay"+by'+cy=0

QUES: Did our example above fit this pattern?

SOLVING LINEAR SECOND-ORDER HOMOGENOUS EQUATIONS WITH CONSTANT COEFFICIENTS:

Given: ay"+by'+cy=0
IDEA: guess a solution of the form y=e^rx. How do we find the correct constant r?
Take derivatives and plug in.
(ar^2+br+c)e^rx = 0

Since e^rx is never zero, this equation has a solution only when ar^2+br+c=0

The **characteristic equation**: ar^2+br+c=0
The **characteristic polynomial** ar^2+br+c

How do we solve for r? What kind of equation is it? How many solutions?

FACT: If this equation has two real roots, r1 and r2, then the basic solutions are y1=e^r1x and y2=e^r2x, and the general solution to the differential equation is: y=c1e^r1x + c2e^r2x

EXAMPLE: Find the general solution: y"+6y'+5y=0. Now find the particular solution that satisfies y(0)=3, y'(0)=-1

GENERAL SOLUTION: y=c1e^-x+c2e^-5x
PARTICULAR SOLUTION: y=7/2 e^-x - ½ e^-5x

Day 10:
Section 5.2
Constant Coefficient Homogeneous Equations (p.210–217) - p.217: 1–17 odd, 18–21
Today I'll cover repeated roots & complex roots

SECOND ORDER LINEAR HOMOGENEOUS EQUATIONS WITH CONSTANT COEFFICIENTS
Given ay"+by'+cy=0:
STEP 1: GUESS a solution of the form y=e^rx
STEP 2: To find r, substitute and solve for r
SHORTCUT: This always leads to the characteristic equation: ar^2+br+c=0
REMINDER: the left side is called the characteristic polynomial
SOLVE WITH QUADRATIC FORMULA: r=[-b+-sqrt(b^2-4ac)]/2a, $r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
STEP 3: **If r1 and r2 are distinct real roots,** then the general solution is y=c1 e^r1x + c2 e^r2x
If r1=r2 is a repeated root, then the general solution is: _______?
If r1 and r2 are complex roots lambda+-omega*i $r_1, r_2 = \lambda \pm \omega i$, then the general solution is: _____?

Our goal for today is to fill in these two blanks!

QUESTION: Does every quadratic equation have two real roots?
NOPE! MUST CONSIDER a) repeated roots, b) complex roots

REPEATED ROOTS
Example. Find the most general solution: y"+6y'+9y=0

The characteristic equation has solutions r1=-3, r2=-3. BUT this would give a general solution y=c1e^-3x+c2e^-3x, and we can factor out e^-3x and combine constants to get: y=c3e^-3x -- *this is not the most general solution, since there are always TWO basic solutions. What is the other solution?*

STRATEGY: We have one solution, y1=e^-3x. GUESS the other solution has the form
y=uy1 = u e^-3x, where u is some unknown function.
Substitute into the original equation to find u (find y', y" first)
We get u"=0. Integrate with respect to x twice:
u'=c2
u=c2x+c1
y=e^-3x(c2x+c1) = c2xe^-3x + c1e^-3x
NOTE that if c2=0, we get our original solution. If c1=0, we get our second solution c2xe^-3x. Thus we have:

GENERAL SOLUTION: y=c1e^-3x + c2xe^-3x

RULE FOR REPEATED ROOTS:
RULE: If the characteristic equation has a repeated roots r1=r2, then the general solution is
y=c1e^r1x + c2xe^r1x

COMPLEX ROOTS

EXAMPLE. Find the general solution: y”+4y’+13y=0
The roots of the characteristic polynomial are: r1=-2+3i, r2=-2-3i.
GUESS BASIC SOLUTIONS: y1 = e^(-2+3i)x, y2=e^(-2-3i)x
Simplify: y1=e^-2x e^3ix
How do we make sense of e raised to a complex power?

EULER’S FORMULA: e^bi = cosb + i sinb

So y1 = e^-2x (cos 3x + i sin 3x)
And y2 = e^-2x (cos -3x + i sin -3x)

TRIG IDENTITIES: cos -b = cos b, sin -b = - sinb

Y2 = e^-2x (cos 3x - i sin 3x)

GENERAL SOLUTION: y= c1y1 + c2y2
y = c1e^-2x (cos3x + i sin 3x) + c2e^-2x (cos3x-isin3x)
y = e^-2x [(c1+c2)cos3x + (c1*i - c2*i)sin3x]
GENERAL SOLUTION: y = e^-2x (d1 cos 3x + d2 sin3x), where d1 and d2 are constants.

RULE: If the characteristic equation has complex roots r1=lamba+omega*i, r2=lambda-omega*i, then the general solution is y=e^lambda*x (c1 cos (omega*x) + c2 sin (omega*x))

Example: y”+6y’+10y=0, y(0)=1, y’(0)=0
GENERAL: y=e^-3x(c1 cosx + c2 sinx)
PARTICULAR: y=e^-3x (3sinx + cosx)

Day 11: Section 5.3 - Nonhomogeneous Linear Equations (p.221–227) - p.227: 1–5 odd, 9–13 odd, 16–20 even, 25–29 odd, 33–37 odd
Section 5.4 - The Method of Undetermined Coefficients I (p.229–235) - p.235: 1–29 odd
NOTE: These sections give a more theory-based presentation of this material (as opposed to Boyce & DiPrima) - I have not entirely followed Trench's exposition, choosing to focus more on working out examples.

NONHOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS
Theorem. Given a nonhomogeneous linear differential equation with constant coefficients:
$y''+p(x)y'+q(x)y = f(x)$
The general solution will be of the form:
$y(x) = c_1y_1(x) + c_2y_2(x) + y_p(x)$
Where:

- $c_1y_1(x) + c_2y_2(x)$ (sometimes called y_h) is the general solution to the the complementary (homogeneous) equation $y''+p(x)y'+q(x)y = 0$
- and $y_p(x)$ (sometimes called $Y(x)$) is any particular solution to the original (nonhomogeneous) equation.

Finding the solution involves two steps:

1. Solve the complementary equation to obtain $c_1y_1(x) + c_2y_2(x)$ *(using the techniques of the previous section)*
2. Find a single solution y_p to the original equation

It is step 2 that we will be focussing on this week. The basic idea is to guess a solution using the function f(x) on the right side as our guide - this method is called ***the method of undetermined coefficients***.

Example 1: $y''+9y=36$

Step 1: Solve the complementary/homogeneous equation $y''+9y=0$

SOLUTION: $y_h=c1 \sin(3x) + c2 \cos(3x)$

Step 2: Guess a solution to the original equation.

What kind of function is f(x)? Is it a constant function, an exponential function, a trig function, a polynomial, etc?

Since f(x) is a constant function (f(x)=36), let's guess our solution will also be constant:

GUESS $y_p = A$

Now find y_p' and y_p'', plug into original nonhomogeneous equation to find A.

PARTICULAR SOLUTION TO ORIGINAL EQ: $y_p = 4$

Step 3: Combine the solutions found in Steps 1 and 2 to obtain the general solution:

GENERAL SOLUTION: $y=c1 \sin(3x) + c2 \cos(3x) + 4$

Example 2: $y''-3y'-4y=3e^{2x}$, $y(0)=17/2$, $y'(0)=10$

Step 1: SOLUTION - COMPLEMENTARY: $y_h=c_1 e^{4x} + c_2 e^{-x}$

Step 2: What kind of function is $f(x)$?
GUESS solution: $y_p = Ae^{2x}$
PARTICULAR SOLUTION: $y_p = -1/2\, e^{2x}$
Step 3: GENERAL SOLUTION $y = c_1 e^{4x} + c_2 e^{-x} - 1/2\, e^{2x}$

Now substitute the initial conditions into y, y' to find c_1, c_2
PARTICULAR SOLUTION: $y = 4e^{4x} + 5e^{-x} - 1/2\, e^{2x}$

Save this example for later? IVP, f(x) is a polynomial
Example 2: *a) Find the general solution of* $y'' - 2y' + y = -3 - x + x^2$
b) Find the particular solution satisfying $y(0)=-2$, $y'(0)=1$

a) Part 1: $y_h = c_1 e^x + c_2 x e^x$
Part 2: What kind of function is f(x)? What degree?
GUESS a solution that is a polynomial of degree 2.
$y_p = A + Bx + Cx^2$
Take derivatives and substitute.
PARTICULAR SOLUTION: $y = 1 + 3x + x^2$
GENERAL SOLUTION: $y = 1 + 3x + x^2 + c_1 e^x + c_2 x e^x$

b) Find y'. Substitute initial conditions, solve for c1 and c2.
PARTICULAR SOLUTION: $y = 1 + 3x + x^2 - 3e^x + xe^x$

Day 12: Section 5.4 - The Method of Undetermined Coefficients I (p.229–235) - p.235: 1–29 odd
Section 5.5 - The Method of Undetermined Coefficients II (p238-244)

WARM UP
Example 1: y" - 9y' + 14y = 212 sin(2x)

STEP 1: General solution to complementary eq: y_h = c_1 e^(2x) + c_2 e^(7x)
STEP 2: What kind of function is f(x)? What should we guess for a solution y_p?
Whatever we guess for y_p, we will have to take two derivatives and substitute on the left side - after simplifying, we should get exactly 212 sin(2x). But for the sin(2x) will have derivatives involving both sin(2x) and cos(2x). How do we accommodate this? We guess that y_p is a combination of sines and cosines:
GUESS: y_p = A sin(2x) + B cos(2x)
Take derivatives, substitute, solve for A and B
PARTICULAR SOLUTION y_p = 5 sin(2x) + 9 cos(2x)
STEP 3: GENERAL SOLUTION: y = c_1 e^(2x) + c_2 e^(7x) + 5 sin(2x) + 9 cos(2x)

GREAT! What else can happen in these problems?

1. What if the right side f(x) contains a solution to the complementary equation?
2. What if the right side f(x) is a combination of functions?

Example 2: y"-7y'+12y = 5 e^(4x)

STEP 1: y_h = c_1 e^(4x) + c_2 e^(3x)
STEP 2: What should we guess for y_p?
NOTE: The obvious guess, y_p = Ae^(4x), won't work - because this is already a solution to the complementary equation. We need a function that is not equal to Ae^(4x), but has Ae^(4x) among the first and second derivatives.
GUESS: y_p = xe^(4x)
PARTICULAR SOLUTION: y_p = 5xe^(4x)
STEP 3: GENERAL SOLUTION: y = c_1 e^(4x) + c_2 e^(3x) + 5xe^(4x)

Example 3: WHAT GUESS SHOULD WE MAKE FOR y_p, BASED ON DIFFERENT f(x)?
a) ay" + by' + cy = 4e^(-5x)
GUESS: y_p = Ae^(-5x)
WHAT IF e^(-5x) is a solution to the complementary equation?
GUESS: y_p = Axe^(-5x)
WHAT IF xe^(-5x) is also a solution to the complementary equation?
GUESS: y_p = Ax^2e^(-5x)

b) ay" + by' + cy = 4x+3x^3
What kind of function is f(x)? What is the degree?
GUESS: y_p = Ax^3 + Bx^2 + Cx + D ("a polynomial of degree 3")

NOTE: For equations with constant coefficients, the complementary equation will never have a solution that is a polynomial - so this guess will work!

c) ay" + by' + cy = 4e^(2x)sin(6x)

What kind of function is f(x)?

GUESS: y_p = Ae^(2x)sin(6x)+Be^(2x)cos(6x)

Day 13: Section 5.6 Reduction of Order (p.248–252) p.253 1–3, 5, 9, 13, 17, 19, 25, 31

Given a single solution to a homogeneous equation, find the general solution using reduction of order

Today:
$P_0(x)y'' + P_1(x)y'+P_2(x)y=F(x)$
And
$P_0(x)y'' + P_1(x)y'+P_2(x)y=0$

What kind of equation is it? Second order, linear, (nonconstant coefficients), homogeneous or nonhomogeneous

How do we solve these? WHO KNOWS?
But today we'll learn a technique that works IF we have a hint.

Reduction of order
A method to find the general solution to

$P_0(x)y'' + P_1(x)y'+P_2(x)y=F(x)$
And
$P_0(x)y'' + P_1(x)y'+P_2(x)y=0$

PROVIDED you know ONE solution y_1 to the complementary equation.

STEP 1: Guess a solution to the original equation of the form $y=u\,y_1$. Take derivatives and plug in to the original equation, simplify.
STEP 2: The result should have only u'', u' (not u). Substitute $w=u'$, $w'=u$. Solve this first-order linear equation for w.
STEP 3: Now replace w with u'. Integrate to find u.
STEP 4: Combine u and y_1 to obtain the general solution $y=u\,y_1$.

Example 1: Solve $x^2y''-3xy'+3y=0$ given that $y_1=x$ is a solution.

HINT: What is the complementary equation? Same as original!
Step 1: $x^3u''-x^2u'=0 \rightarrow x^3w'-x^2w = 0$
Step 2: $w=C_1x$, $u=C_1x^2/2 + C_2$
Step 3: $y=ux=C_1x^3/2+C_2x$ *or* $y = c_1x^3 + c_2x$.

Example 2: Find the general solution of $xy''-(2x+1)y'+(x+1)y=x^2$,

Given that $y_1=e^x$ is a solution of the complementary equation $xy''-(2x+1)y'+(x+1)y=0$.
Step 1: $u''-u'/x=xe^{-x}$
Step 2: $w = -xe^{-x} + C_1x$, $u=(x+1)e^{-x} + C_1x^{22} + C_2$
Step 3: $y = ux = (x+1) + c_1x^2e^x + c_2 e^x$

5.7 Variation of Parameters (p. 255-262)
p.262: 1-5, 7, 11, 13, 31, 33, 34
NOTE: In Spring 2019 I gave this lecture on Day 23 (having rearranged the lecture order somewhat due to Spring Break & other factors)

Today we look at a method called variation of parameters:

- Used for finding a *particular solution* of $P_0(x)y'' + P_1(x)y' + P_2(x)y = F(x)$. We call the particular solution y_p.
- To use the method, we must already know the general solution to the complementary equation, $P_0(x)y'' + P_1(x)y' + P_2(x)y = 0$, which we call $y_h = c_1y_1 + c_2y_2$
- Once we find the particular solution to the original equation, the general solution of the original equation will be: $y = y_p + c_1y_1 + c_2y_2$.
- NOTE: We must assume that the leading coefficient, $P_0(x)$is nonzero on any interval we consider.

QUESTION: This method is similar to 'reduction of order', but in reduction of order we only need ONE solution to the complementary equation. Why do we need this method, too?
ANS:

- Usually simpler than reduction of order (provided we know two solutions to comp eq)
- This method is more general - the idea can used to solve more complicated differential equations (higher order equations, linear systems of equations).
- This method is a powerful tool used by researchers in differential equations! *If you study more diffy q's, you will a most certainly see it!*

BEWARE: Much of the method will seem familiar, but there will be one or two extra things to remember!
BEWARE: Things will get messy for a while, but they will get better!

THE IDEA BEHIND VARIATION OF PARAMETERS:
GUESS a solution of form $y_p = u_1y_1 + u_2y_2$
NOTE: We have TWO unknown functions, but only ONE equation to satisfy. This means we have some freedom to restrict u1 and u2.

Take derivative: $y_p' = u_1y_1' + u_1'y_1 + u_2y_2' + u_2'y_2$

Here is the "trick": Let's impose a condition on u1, u2 that will make this easier to solve.

LET US REQUIRE THAT: $u_1'y_1 + u_2'y_2 = 0$, ****IMPORTANT EQUATION 1.**

So $y_p' = u_1y_1' + u_2y_2'$
Take derivative again:
$y_p'' = u_1y_1'' + u_1'y_1' + u_2y_2'' + u_2'y_2'$

Plug y_p, y_p', y_p'' into the original equation, and group together terms with u1, u2:

- $u_1(P_0y_1'' + P_1y_1' + P_2y_1) + u_2(P_0y_2'' + P_1y_2' + P_2y_2) + P_0(u_1'y_1' + u_2'y_2') = F(x)$

NOTE: what are y1, y2? *Solutions to complementary equation.* When we plug them into LHS of original equation we get 0. This means the coefficients of u1 and u2 are 0!
Thus we have: $P_0(u_1'y_1' + u_2'y_2') = F(x)$, or

$u_1'y_1' + u_2'y_2' = \frac{F}{P_0}$ ** **IMPORTANT EQUATION II**

By combining the two "important equations", we can solve for u1' and u2', then integrate to find u1, u2 (since we are looking for a particular solution yp, take the constants of integration to be 0 for simplicity).

VARIATION OF PARAMETERS:
TO SOLVE: $P_0(x)y'' + P_1(x)y' + P_2(x)y = F(x)$, given y_1, y_2 (independent) solutions to the complementary equation:

1. GUESS A PARTICULAR SOLUTION: $y_p = u_1y_1 + u_2y_2$
2. WRITE DOWN THE SYSTEM OF EQUATIONS:

   $u_1'y_1 + u_2'y_2 = 0$

   $u_1'y_1' + u_2'y_2' = \frac{F}{P_0}$
3. Solve the system of equations for u_1', u_2'
4. Integrate to find u_1, u_2 (let constants of integration =0)
5. Substitute into y_p
6. The general solution is $y = y_p + c_1y_1 + c_2y_2$

Example 1: Find the general solution $x^2y'' - 2xy' + 2y = x^{9/2}$, given that $y_1 = x$ and $y_2 = x^2$ are solutions of the complementary equation $x^2y'' - 2xy' + 2y = 0$.

ANS: $y_p = \frac{4}{35}x^{9/2}$

Example 2: Find the general solution: $y'' + 3y' + 2y = \frac{1}{1+e^x}$

HINT: First find y_h, the general solution to the complementary equation.

ANS: $y_h = c_1e^{-x} + c_2e^{-2x}$, $y_p = (e^{-x} + e^{-2x})\ln(1 + e^x)$

NOTE: the term $-e^{-x}$ *in yp is a solution to the complementary equation, so it disappears.*

General solution: $y = (e^{-x} + e^{-2x})\ln(1 + e^x) + c_1e^{-x} + c_2e^{-2x}$

NOTE: NEED TO Add notes & WW assignments for the following two lectures, Spring 2019

6.1 Spring Problems I 6.2 Spring Problems II	268-277 279-284	p.277: 1, 3, 7-13 odd, 19, 21 p.288: 3, 4, 7-11 odd, 14-16
6.2 Spring Problems II (continued) 6.3 The RLC Circuit	284-287 290-295	p.288: 13, 17-20 p.295: 1-10

Day 14: Section 7.1 Review of Power Series (p.307–316) p.317: 1, 11, 13, 15–17
Section 7.2 Series Solutions Near an Ordinary Point I (p.320–328) p.329: 1, 3, 8, 11–13, 19–25 odd
7.1 Find radius of convergence of a power series. Simplify expressions involving series.
7.2 Find power series solutions to second-order linear ODEs with polynomial coefficients (either find first 7 terms, or give formula)

DISCUSSION: Follow-up to "flipped class" assignment.
- what is a power series? taylor series/maclaurin series?

RECALL: a **power series** is like a polynomial - but with infinitely many terms: a0+a1x+a2x^2+...

NOTE: We write this in sigma notation $\sum_{n=0}^{\infty} a_n x^n$

QUESTION: What is cos(0.7)? Can you find it without using any trig functions on your calculator, only +/-//?*

RECALL: Taylor and MacLaurin Series are just a particular kind of power series - used to solve a basic problem

BASIC IDEA: (Desmos)
Start with any function, y=cosx.
Pick a point on the graph, say x=0 -- this gives the point (0,1).
I am going to try to make a polynomial that matches my function at that point.
Here goes:
First try: y=1
Second try: y=1-1/2 x^2
Third try: y=1 - 1/2 x^2 + 1/24 x^4-1/720 x^6

QUESTION: What is cos(0.7)?
Find it without using any trig functions on your calculator! Plug into our third approximation.
NOTE: cos(0.7) == 0.764842187
Third approx gives y(0.7)=0.765841

NOTE: is our polynomial up to x^6 **exactly** equal to cos(x)? How could we make it better?

If we use infinitely many terms, the result is a power series that is:
- exactly equal to cosx
- equal to cosx not just at x=0 but for *any* x in the real numbers.

QUESTION: What makes one power series different from another? *The coefficients.*
QUESTION: How do we find the "right coefficients" so that our power series matches our function, in this case cosx? *That's the theory of Taylor Series:*

DEFN: If a function f(x) has derivatives of all orders at x=0, then the MacLaurin Series of f(x) is a0+a1x+a2x^2+a3x^3+a4x^4+...= $\sum_{n=0}^{\infty} a_n x^n$. If we consider a different point x=c, then we get the Taylor Series of f(x) at x=c, a0 + a1(x-c) + a2(x-c)^2+a3(x-c)^3+... $\sum_{n=0}^{\infty} a_n (x-c)^n$.

1. The coefficients a_n are given by: $a_n = \frac{f^{(n)}(c)}{n!}$
2. Amazingly, for most functions, the Taylor Series will be equal to the original function $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x-c)^n$

(TALK ABOUT INTERVAL OF CONVERGENCE?)

HOW DOES THIS FIT INTO DIFFERENTIAL EQUATIONS?

Sometimes we cannot solve a differential explicitly by finding a formula for y. However, if we think of a power series for y, we might be able to find the coefficients - this will give us an approximation for y (it won't match y exactly unless we use infinitely many terms, but it will be very close to correct - especially near the point x=c that we started with).

NOTE: Instead of the example below, I started the example from the following lecture:
Example 1. Find the series solution of the initial value problem y''-xy=0, y(0)=3, y'(0)=1. Estimate the value of y(2) using the first eight terms of the power series solution.

I only got the setup done - then, the following day, I finished it out and had a little more discussion about recurrence relations, etc. I never came back to the example below.

Example 1. Find the general solution of the differential equation y''+y=0 as a power series. *Give the coefficients in terms of constants a_0 and a_1.*

a. Write y as a power series about about x=0 $y = \sum_{n=0}^{\infty} a_n x^n$

b. Take derivatives of y and substitute into the differential equation.
c. equate coefficients on both sides to obtain a series of equations involving the an.
d. Take a0, a1 as given. our goal is to use these equations to EXPRESS LATER TERMS IN TERMS OF EARLIER ONES!
e. Make a list of a0, a1, a2, ... up to a7, all IN TERMS OF a0 and a1
f. put these back into the original power series to obtain an expression for y in terms of a0, a1

QUESTION: Normally we have two basic solutions, combined with two constants. How does that show up here? What are the basic solutions?

TWO DIRECTIONS WE CAN GO FROM HERE:

1. Solve IVP by finding values for an up to some point - get an approximate solution!
2. Simplify this expression, try to write it in sigma notation, relate it to other (known) series

ON BOARD: Write out y=a0+a1x+a2x^2+... (up to a5).
QUES: How many constants? (infinite)!
- Find y’, y”
- substitute y”, y into the differential equation
- collect like terms (terms with same power of x). Ask them to generate the next couple of terms (up to x^8) by pattern matching.
- equate left and right sides -- set each coefficient expression equal to 0.

Day 15: Section 7.3 Series Solutions Near an Ordinary Point II (p.335–338)

p.338: 1–5 odd, 19–23 odd, 33–37 odd, 41–45 odd

7.3 Find power series solutions to second-order linear ODEs with polynomial coefficients (examples with no explicit formula possible - calculate first few terms)

I had started this example the previous day, finished it today.

Example 1. Find the series solution of the initial value problem y''-xy=0, y(0)=3, y'(0)=1. Estimate the value of y(2) using the first eight terms of the power series solution.

PROCEED AS BEFORE
Do we know any of the an? (y=f(x), initial conditions: a0=y(0)/0!=3, a1=y'(0)/1!=-3)
Use these to find later terms (first eight). Use the resulting partial power series to estimate y(2).

RECURR. REL: a_(n+3)=1/(n+3)(n+2) a_n, with a_2=0.
TAYLOR: y=3+x+1/2 x^3+1/12 x^4+1/60 x^6+1/504 x^7+...
APPROX: y(2)= 11.654
ACTUAL ANSWER: y(2)=11.8037

*Did *not* cover this one:*

Example 2. Find the general solution to (1+8x^2)y"+2y=0 (give the first 6 terms in terms of constants a0 and a1)

nnghdy

RECURR. REL: a_(n+1)=-(8n(n-1)+2)/(n+2)(n+1) *a_n
a2=-2/2 a0
a3=-2/6 a1
a4=-18/12 a2
a5=-50/20 a3
a6=-98/30 a4
a7=-162/42 a5

Day 16: Section 7.4 Regular Singular Points Euler Equations (p.344–346) p.347: 1–12

*NOTE: This section does *not* use series to find solutions. Instead, the techniques are similar to those used second-order linear constant coefficient equations (solutions depend on roots of characteristic equation)*

Solve Euler equations for which the characteristic equation has 2 real roots, repeated root, or complex roots.

Are there any differential equations that cannot be solved by the series methods we’ve discussed so far? YES. For equations of the form:

P_0 (x)y” + P_1(x)y’+P_2(x)y=0

Our method will have problems if we try to create a series around a point x=c where the leading coefficient is 0 - that is, if P_0(c)=0.

NOTE: Today, we will just look at *one* kind of equation of this form. BUT this type of equation can actually be solved without series. Lucky!

NOTE: There are extensions of our series methods that will work in these cases.

Defn: Euler Equations: Can be written in the form

ax^2y”+bxy’+cy=0

Where a,b,c are constants and a is not=0.

NOTE: we will still have problems when x=0. Because of this, we will consider only solutions for x>0.

Example: Find the general solution x^2y”-xy’-8y=0

IDEA: Assume a solution of the form y=x^r (for some constant r).

Take derivatives and substitute. Factor out x^r.

r(r-1)-r-8=0

The result is like the characteristic equation, but here we call it the *indicial equation*.

Two real roots.

Two basic solutions:

y1=x^4

y2=x^-2

General solution: y = c1x^5 + c2x^-2

WHAT IF THERE IS A REPEATED ROOT OR COMPLEX ROOT?

REPEATED ROOT: r1=r2, BASIC SOLUTIONS: y1=x^r1, y2=ln(x)x^r1

COMPLEX ROOTS: r1=lamba+omega*i, r2=lambda-omega*i

BASIC SOLUTIONS: y1=x^lambda cos(omega ln(x)), y2=x^lambda sin(omega ln(x))

EXAMPLE 2: x^2y”-5xy’+9y=0, y(1)=3, y’(1)=5

ANS: y=3x^3-4x^3ln(x)

EXAMPLE 3: x^2y”+3xy’+2y=0

ANS: y=c1x^-1 cos(lnx)+c2x^-1 sin(lnx)

Day 25: Section 8.1 - Introduction to the Laplace Transform (p.394–402)
[NOTE: use table on p.463 of textbook for homework]
p.403: 1(a,b,d,e), 2(b,c,f,g,h,i), 4, 5, 18

NOTE: LAPLACE TRANSFORM WEBWORK CAN SOMETIMES GENERATE PROBLEMS THAT USE t*sine or t*cosine - ADD TO TABLE??

Overview: What is the Laplace transform all about? How do we use it/why do we study it?

- The Laplace transform of a function f(t) is another function called $\mathcal{L}$\{f(t)\}, or F(s).
  - NOTE: This is similar to the way that the derivative of a function f(t) is another function f'(t) or dfdt.
  - NOTE: The variable changes when we compute the Laplace transform - if the original f(t) is a function of t, then the Laplace transform is a function of another variable s.
- We use it to make solving differential equations easier, following this outline:
  1. Start with a differential equation.
  2. Take the Laplace transform of both sides.
     This replaces the differential equation with a much simpler (algebraic) equation.
  3. Solve the algebraic equation.
  4. Simplify the solution.*
     ** requires partial fraction decomposition*
  5. Take the inverse Laplace transform of the solution.
  6. This gives the solution to the original differential equation.
- Yes, but WHAT is the Laplace transform? Ask me about it sometime (go on...)

TRICKY PARTS: #2, #4, #5.
Today: #2 - how to take the Laplace transform.
- Finding the Laplace transform using the definition
- some basic Laplace transforms
- linearity of the Laplace transform
- Finding the Laplace transform using a table

Defn. If f(t) is a function** defined for t>0, then the Laplace transform of f(t) is:

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^{\infty} e^{-st} f(t)dt.$$

** NOTE: In fact, we also need the function f(t) to satisfy two additional conditions:
1. f(t) must be piecewise continuous on any interval [0,inf]
2. f(t) must be of exponential order: $|f(t)| \leq Ke^{at}$ *when* $t \geq M$ *(for some constants K, a, and M, with K and M positive).* This says that it cannot grow too quickly.

Example 1: Find the Laplace Transform of the constant function f(t)=1.
ANS: F(s)=1/s, s>0

NOTE: Interval of convergence!!

Example 2: Find the Laplace Transform of $f(t)=e^{at}$ where a is constant

ANS: $F(s)=1/(s-a)$, $s>a$

HANDOUT: Laplace Transforms for common functions (p463-464)

Day 25: 8.2 The Inverse Laplace Transform (p.405–412)

[NOTE: use table on p.463 of textbook for homework]

p.412: 1(a,b,d,e), 2(a–e), 3(a–d), 4(a,d,e), 6(a), 7(a), 8(a,d)

TODAY: taking Laplace transform, inverse Laplace transform using table

RECALL: Laplace transform of: $\mathcal{L}\{1\}$, $\mathcal{L}\{e^{at}\}$, $\mathcal{L}\{t^n\}$

LINEARITY RULE: For function *f,g* and constants *a,b*, $\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$

Ex: $\mathcal{L}\{5 + 6t^5 + 2e^{3t}\}$

GROUP WORKSHEET

(Submit answers on a separate sheet)

Part I. Find the Laplace Transform of each function, and determine the interval on which it is defined.

1. $t^2 + 4t^3 - 7t^6$
2. $\sin t + \cos 3t$
3. $5e^{2t} - 4\sin 3t$
4. $5t^2 + 3\sin 5t - 2e^{6t}\cos 2t$

Part II. Find the inverse Laplace Transform of each function.

1. $\frac{3}{s} + \frac{4!}{s^5} + \frac{1}{s-7}$, $s > 7$
2. $\frac{1}{s^2+25} + \frac{s}{s^2+25}$, $s > 0$
3. $\frac{1}{(s-4)^2+9}$, s>2
4. $\frac{1}{(s-6)^7} + \frac{5}{2s-7}$, $s > 6$
5. $\frac{5}{s^2-8s+41}$, $s > 4$

Day 27: Section 8.3 Solution of Initial Value Problems (p.414–419)
[NOTE: use table on p.463 of textbook for homework] p.419: 1–31 odd

Ex: Find the Inverse Laplace Transform:
a) $\frac{5}{(s-4)^2+25}$, s>4
b) $\frac{5}{s^2-8s+41}$, s>4
c) $\frac{1}{s-1}+\frac{4}{s+4}$, s>1
d) $\frac{5s}{s^2+3s-4}$, s>1

(HINT: a =b, c=d)
Starting from b), how do we get to a)? Sim. from d to c?

PARTIAL FRACTION DECOMPOSITION (discuss, do the example above).

Resources on Inverse Laplace and Partial Fractions:
Paul's Notes on Partial Fractions:
http://tutorial.math.lamar.edu/Classes/Alg/PartialFractions.aspx
OpenLab assignment from 2014 with videos on Inverse Laplace and Partial Fractions:
https://openlab.citytech.cuny.edu/2014-spring-mat-2680-reitz/?p=383
More details if the numerator has higher degree than the denominator:
https://openlab.citytech.cuny.edu/2015-spring-mat-2680-reitz/?tag=partial-fractions

SOLVING IVPs USING LAPLACE TRANSFORM
We will need to be able to take the Laplace transform of the derivative of a function.
Laplace transform of a derivative
Theorem. Suppose *y(t)* is continuous and of exponential order, and *f', f'' are* piecewise continuous.
Then the Laplace transform of y(t) is Y(s)
... of y'(t) is sY(s)-y(0)
... of y''(t) is s^2 Y(s)-sy(0)-y'(0)

Example 1. Use the Laplace transform to solve the differential equation $y'' - y' - 2y = 0$ with initial conditions $y(0) = 1,\ y'(0) = 0$.

ANS: $Y(s) = \frac{1/3}{s-2} + \frac{2/3}{s+1},\ y = 1/3\, e^{2t} + 2/3\, e^{-t}$

Example 2. Find the solution of the differential equation $y'' + y = \sin 2t$ satisfying initial conditions $y(0) = 2,\ y'(0) = 1$.

ANS: $Y(s) = \frac{2s}{s^2+1} + \frac{5/3}{s^2+1} - \frac{2/3}{s^2+4},\ y = 2\cos t + 5/3\sin t - 1/3\ \sin 2t$

Day 27: Revisiting first-order linear

Today, I am going go back over first-order linear equations and show how we can shortcut to the solution.

RECALL: First-order linear differential equations. TODAY: Shortcut!

EXAMPLES:

1. $y' + 2y = x^3 e^{-2x}$ (ANS: y=e^-2x (x^4/4 + C))
2. y'+4xy=x (ANS: y=¼ + Ce^(-2x^2))
3. $y' + (2/x)\,y = 2x^{-3} + x^{-1}$
4. $y' - x^2 y = 0$

SHORTCUT TO SOLVE LINEAR EQUATIONS:

Step 1: Rewrite the differential equation in the standard form: $y' + p(x)y = f(x)$

Step 2: Find u(x) by plugging in: $\mu(x) = e^{\int p(x)dx}$

NOTE: all you need is a single antiderivative - so no "C" in P(x)

Step 3: If the original equation is:

- not homogeneous, then the solution is $y = \frac{1}{\mu(x)} \int \mu(x) f(x) dx$
- homogeneous, the solution is: $y = C/\mu(x)$

RESOURCES

SIMIODE SIDE NOTE: There is a great "Day 1" activity on modeling death with M&Ms here: https://www.simiode.org/resources/1798/download/1-1-S-MM-DeathImmigration-StudentVersion.pdf

SLOPE FIELD GENERATOR (DESMOS): https://www.desmos.com/calculator/p7vd3cdmei
SLOPE FIELD GENERATOR (GEOGEBRA): https://www.geogebra.org/m/W7dAdgqc
SLOPE FIELD GENERATOR (this one works well when re-adjusting the window - e.g. for applications): https://bluffton.edu/homepages/facstaff/nesterd/java/slopefields.html

APPLICATIONS OF FIRST-ORDER DIFFERENTIAL EQUATIONS IN MECHANICAL ENGINEERING ANALYSIS
https://www.engr.sjsu.edu/trhsu/Chapter%203%20First%20order%20DEs.pdf

Fluid dynamics: design of containers and funnels
Heat conduction analysis: design of heat spreaders in microelectronics
Heat conduction & convection: heating and cooling chambers
Applications of rigid-body dynamic analysis

REQUIRED IN:

MECH 4730 Finite Element Methods
MECH 4760 **Vibration** and Advanced Dynamics

APPLICATIONS OF DIFFERENTIAL EQUATIONS IN COMPUTER ENGINEERING TECH

Electrical circuits

REQUIRED IN:

CET 4705 Component and Subsystem Design I
CET 4711 Computer-Controlled Systems Design I
CET 4762 Electromechanical Devices

OLD LECTURE NOTES (2015, Boyce and DiPrima text) BELOW:

*text: Boyce and DiPrima, **Elementary Differential Equations,** 10th edition, John Wiley & Sons, Inc.*

Day 1

Review, Section 1.1

Welcome, Course Policies
WeBWorK 1 & 2 due next Tues

2-minute intros

Defn. A **differential equation** is an equation that relates a function to one or more of its derivatives.
Ex: $y = \frac{x}{2} \cdot y'$. What is a solution? It's a function y that makes the equation work. Does the function $y = x^2$ satisfy the differential equation?

Differential equations have:
A mathematical / technical / abstract aspect - know your calculus and algebra!
and a conceptual / practical / real-world aspect - how it connects to solving real problems.
You will get the absolute most out of this course if you balance your attention to these two aspects!

RECALL: function y=f(t), or y(t) *NOTE: used to using x, we will often use t, sometimes other letter for our **independent variable**.*
- DERIVATIVES
- INTEGRALS

Ex: If $y = ln|t|$, find $\frac{dy}{dt}$.
Ex: Implicit differentiation: find $\frac{d}{dt}$ of $t^2 + y^2 = 0$
Ex: If $y' = e^{5t}$, find *y*.
Discuss constant - there is more than one such function y!

GROUP WORK. (Form groups of 3-4, choose one scribe)

REMEMBERING CALCULUS

Complete the questions below one at a time with your group.
Be sure the entire group understands the solution before you move on to the next question.
If your group gets stuck on a problem, leave it and move on.

Question 1 - Differentiation

a. write the equation for the tangent line to the function $y = t^3 + 3t$ at the point where $t = 1$.

For each function $y(t)$, find the derivative $\frac{dy}{dt}$.

b. $y = 5t^{12} + 2\sqrt{t} + t^{-3}$

c. $y = ln|t^2 + 7|$

d. $y = e^{5t+2}$

Question 2 - Integration

Find the most general antiderivative (or *indefinite integral*).

a. $\int t^2 + t^{-5} + \sqrt{t^3} + t^{-1}dt$

b. $\int \frac{3t^2+2}{t^3+2t}dt$ *(HINT: u-substitution)*

c. $\int te^t dt$ *(HINT: integration by parts)*

Question 3 - Differential Equations

a. Is $y = t^2 + 7t$ a solution to the differential equation $y - \frac{dy}{dt} \cdot t + t^2 = 0$? Why or why not?

(BONUS: Can you find another solution?)

b. Find **three different examples** of functions $y(t)$ such that $y' = t^2 + 2t$. Now give ONE specific example of such a function *y(t)* that also satisfies $y(0) = 5$.

Day 2

Section 1.1, 1.2

FROM LAST TIME:
a. Find deriv d/dt: $ln|2t - 600| + ln|5y| = 0$

b. Integrate: $\int \frac{y'}{y} dt$

MATHEMATICAL MODELS
Defn. A differential equation that describes some physical process is often called a **mathematical model**.

"If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is."
--- John von Neumann

Discuss: simplifying assumptions. math describes "perfect world", if we're lucky we can find math model that gives a very good approximation to the "real world"

You WILL be responsible for learning basic mathematical models of various types in this class, including falling objects, predator/prey populations, and more.

FALLING OBJECTS (turn to a friend, DISCUSS these three questions with them for 5 minutes, come up with some answer to each)

Example 1: Suppose that an object is falling in the atmosphere near sea level. Formulate a differential equation that describes the motion.

- *What kind of function are we trying to find? Position? Velocity? Acceleration?*
- *Any guess as to what the solution will look like? Will it increase or decrease over time?*
- *What are different factors that might influence the motion of the object?*

SKETCH SOME OPTIONS - label v-axis, t-axis, etc

Thinking question: What are the different factors that might influence the motion of the object?

- Newton: $F = ma$ (let us measure mass in kg, acceleration in m/s^2, force in Newtons)
- Relationship between a(t) and v(t)? $a = \frac{dv}{dt}$
- $F = m\left(\frac{dv}{dt}\right)$
- Force #1: Gravity pulls down with force mg (g=acceleration due to gravity)
- Force #2: Air resistance pushes up with force proportional to velocity γv (γ= constant = drag coefficient)
- Combine: $m\left(\frac{dv}{dt}\right) = mg - \gamma v$

  $m\left(\frac{dv}{dt}\right) = mg - \gamma v$
  m = mass (kg)

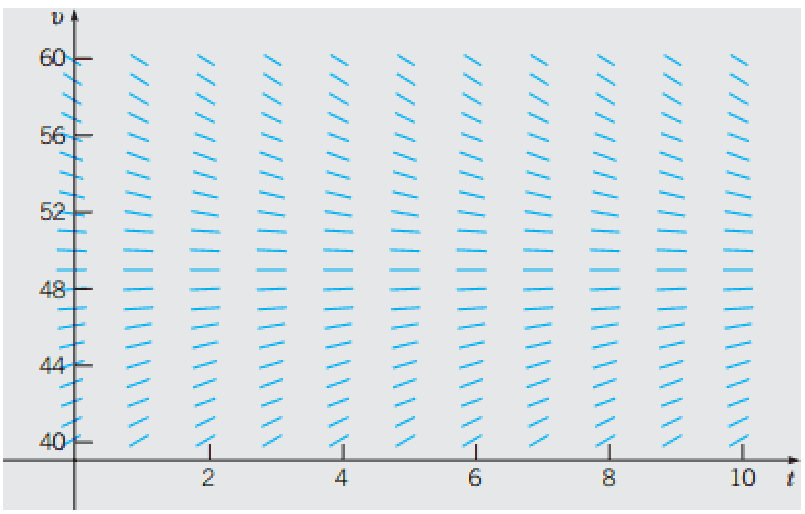
v = velocity (m/s)
g = acceleration due to gravity = $9.8 m/s^2$
γ= gamma = drag coefficient

Example 2: Suppose the mass of the object in example 1 is 10kg, and the drag coefficient has been determined to be $\gamma = 2\ kg/s$. Write the differential equation describing the objects motion in the form $\frac{dv}{dt} =$ (that is, isolate $\frac{dv}{dt}$ on one side).

ANS: $\frac{dv}{dt} = 9.8 - \frac{v}{5}$

Example 3: a. What is the slope when the v=0 m/s?
b. What is the slope when v=60 m/s?
c. What is the slope when v=40 m/s?

Draw a direction field (or slope field) for the differential equation in Example 3.



NOTE: A solution v(t) to the differential equation has to "follow the slope field" -- looking at it will give you a sense of the types of solutions you might get.
(Sketch some solutions)
Example 4: Is there a solution in which the velocity is constant? What is it?

RULE: to find it exactly, set the derivative to 0.

Defn. A constant solution to a differential equation is called an **equilibrium solution**.
In the case of a falling object, it is often referred to as **terminal velocity.**

Example 5: Solve this differential equation - that is, find a formula for v(t).

Basic idea: rearrange the equation $\frac{dv}{dt} = 9.8 - \frac{v}{5}$ so all occurrences of v and its derivatives are alone on one side, then integrate wrt t.
HINTS: common denom, then divide by 49-v. Easier if v is positive on bottom - multiply both sides by -1.
$\frac{dv/dt}{v-49} = -\frac{1}{5}$

integrate: $ln|v - 49| =- t/5 + C$
solve for v: $v = 49 + ce^{-t/5}$
This gives many solutions - how do we get a single specific solution?
What happens if c is: positive, negative, 0?

Example 6: Find the specific solution describing the the motion of an object if the initial velocity is $v_0 = 35\ m/s$.

FIELD MICE AND OWLS
Example 7: A population of field mice inhabit a rural area. The mice reproduce, which tends to drive the population upwards, but several owls prey on the mice, which tends to drive the population downwards. Write a differential equation describing the population of field mice over time.

Same questions:

- *What kind of function are we trying to find?*
- *Any guess as to what the solution will look like? Will it increase or decrease over time?*
- *What are different factors that might influence the motion of the object?*

$\frac{dp}{dt} = rp - k$
Where *t* is time (measured in months), and *p(t)* is the mouse population.
r is the **rate constant** or **growth rate** (the rate at which the mice reproduce) in mice-per-mice-per-month (new mice per old mice per month), or as the book has it, "per month"
k is the **predation rate** (the number of mice killed each month by predators) in mice/month

GROUPS:
Example 7: For a population of field mice, suppose the growth rate is 0.5, and the owls kill 15 mice per day.
a. Determine *r* and *k*, and sketch a direction field for the resulting differential equation (consider values of *v* between 800 and 1000).
b. Find the equilibrium solution.

Day 3

Section 1.3, 2.1

CLASSIFICATION OF DIFFERENTIAL EQUATIONS
Definitions.

- If y is a function of just one variable, $y(t)$, then the derivatives are ordinary derivatives and the differential equation is called an **ordinary differential equation** (we study these in this class).
- If y is a function of more than one variable, $y(t, x, v)$, the the derivatives are partial derivatives and the differential equation is called a **partial differential equation**.
- The **order** of a differential equation is the order of the highest derivative that appears (if y' appears it is first-order, if y'' appears it is second-order, and so on).
- A differential equation is **linear** if it is a linear function of the variables y, y', y'', and so on. (*That is, we cannot have y raised to a power, or multiplied by a derivative, etc.*). Otherwise it is **nonlinear**.

In this class, we will only deal with ordinary diffy Qs.

INTEGRATING FACTORS

Example 1: Take the derivative: $t^4 \sin t$

RECALL PRODUCT RULE!!

Example 2: Solve $(4 + t^2)\frac{dy}{dt} + 2ty = 4t$

Integrate with respect to t!

NOTE: The left side has a term with dy/dt, and a term with y -- it looks like it could be the result of a product rule. Any guess as to the two functions? Try it!

SOLN: $y = \frac{2t^2+c}{4+t^2}$

NOTE: For this to work, the function in front of the y (2t) has to be equal to the derivative of the function in front of dy/dt (4+t^2).

Example 3: $\frac{dy}{dt} + \frac{1}{2}y = \frac{1}{2}e^{t/3}$

ANS: $y = \frac{3}{5}e^{t/3} + ce^{-t/2}$

PROBLEM: This one also has a term with dy/dt, and a term with y. BUT it isn't a nice product rule - the other parts don't match up properly.

LEIBNIZ: If we multiply the equation by a certain function $\mu(t)$, then it will make the 'other parts' work correctly. How do we find this mysterious function?

Multiply by $\mu(t)$, observe that $\frac{1}{2}\mu(t)$ had better equal the derivative of $\mu(t)$ -- so we have $\frac{d\mu(t)}{dt} = \frac{1}{2}\mu(t)$.

Divide by mu, integrate, solve for mu to get: $\mu(t) = ce^{t/2}$ - DON"T NEED CONSTANT c!!

Now multiply both sides by this function $e^{t/2}$, the integral of the left side should be $e^{t/2}y$.

RULE: *For equations of the form* $\frac{dy}{dt} + ay = g(t)$, *the integrating factor is* e^{at}.

Example 4: $\frac{dy}{dt} - 2y = 4 - t$

NOTE: need to use integration by parts on the last term!

Example 5: $ty' + 2y = 4t^2$

Day 4

Section 2.2 - Separable Equations

NOTE: in this section, we will use x as our independent variable, instead of t.

Def. A differential equation is said to be **separable** if it can be written in the form $M(x) + N(y)\frac{dy}{dx} = 0$.
We solve such an equation by integrating the two terms $M(x)$ and $N(y)\frac{dy}{dx}$ with respect to x.
That is, if we can collect the y's together (multiplied by y'), and separately collect the x's together.

Example 1. Show that the equation $\frac{dy}{dx} = \frac{x^2}{1-y^2}$ is separable, and then find the solution.

ANS: $-x^3 + 3y - y^3 = c$
OBSERVE: We have gotten rid of y'. Have we gotten y by itself? Can we?
What does this equation look like for different values of c? Try c=1, c=0, c=-3.
Defn: Graphs of solutions of differential equations are called **integral curves**.

- A general solution consists of a family of such curves.
- Finding a particular solution means identifying a single curve ("finding the value of C"), based on a given initial condition.

QUES: If I wanted you to narrow your answer down to just one of the integral curves, what additional information would you need?

QUES: Is each one a function? NO.
FACT. The equation $-x^3 + 3y - y^3 = c$ defines the solutions **implicitly**. Any function $y = \phi(x)$ which satisfies this equation is a solution to the differential equation. Sometimes we are able to solve for y, and produce an **explicit** solution $y = \phi(x)$, sometimes not.

Example 2: Find an explicit solution to the initial value problem $\frac{dy}{dx} = \frac{3x^2+4x+2}{2(y-1)}$, $y(0) = -1$, and determine the interval in which the solution exists.

ANS: $y^2 - 2y = x^3 + 2x^2 + 2x + 3$.
$y = 1 \pm \sqrt{x^3 + 2x^2 + 2x + 4}$. This is TWO solutions - which one satisfies the initial condition?

Now, for which x is this function defined? (what do we need to look for? When is the expression under the radical positive? You can use a graphing calculator to help you.

Example 3: Solve the equation $\frac{dy}{dx} = \frac{4x-x^2}{4+y^3}$ and find the solution passing through the point (0,1). Determine its interval of validity.
ANS: $y^4 + 16y + x^4 - 8x^2 = 17$, valid on interval from -3.3488 to 3.3488 (To find: note vertical tangent -- from original eqn, this means the denominator $4 + y^3 = 0$. Solve this to get y, plug back in to get x.

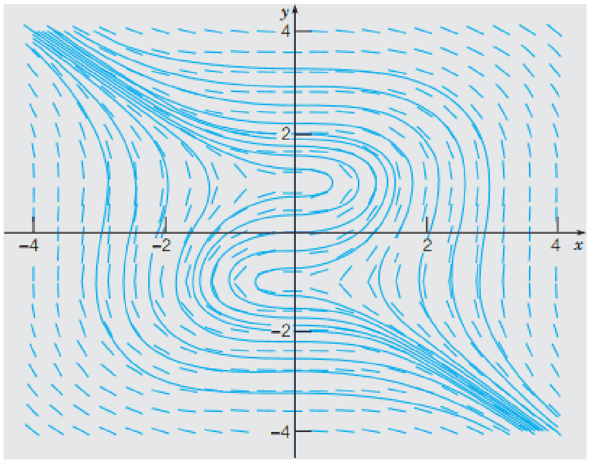


FIGURE 2.2.1 Direction field and integral curves of $y' = x^2/(1 - y^2)$.

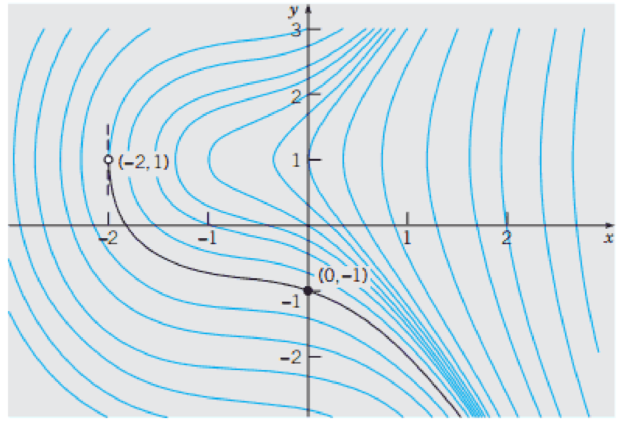


FIGURE 2.2.2 Integral curves of $y' = (3x^2 + 4x + 2)/2(y - 1)$; the solution satisfying $y(0) = -1$ is shown in black and is valid for $x > -2$.

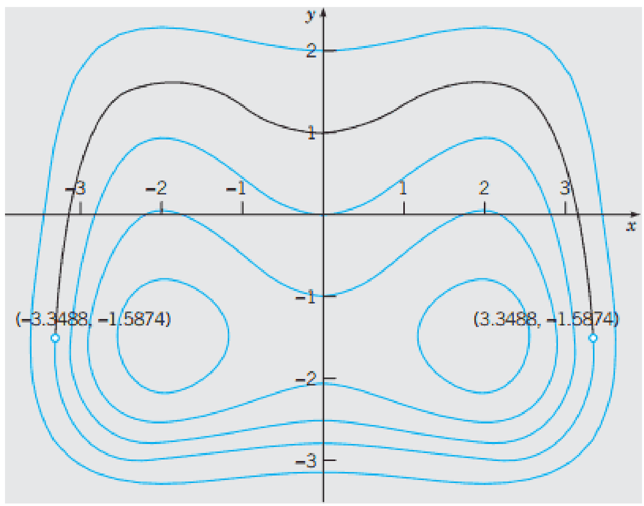


FIGURE 2.2.3 Integral curves of $y' = (4x - x^3)/(4 + y^3)$. The solution passing through $(0, 1)$ is shown by the black curve.

Day 5

Sec 2.2 - homogeneous equations

NOTE: Mention the "rule/shortcut" for integrating factors!

SHORTCUT FOR SOLVING INTEGRATING FACTORS PROBLEMS:

Step 1: Rewrite the differential equation in the standard form: $\frac{dy}{dt} + p(t)y = g(t)$

In practice, this usually just means getting the y and $\frac{dy}{dt}$ on the same side, and dividing to get rid of anything in front of the $\frac{dy}{dt}$.

Step 2: Find μ, by plugging in: $\mu = e^{\int p(t)dt}$

That is, integrate the function in front of y, and then raise e to the power of the result. This gives μ

Step 3: Find y, by plugging in: $y = \frac{1}{\mu(t)} \int \mu(t)g(t)dt + C$

That is, multiply μ by the function $g(t)$ from the right hand side of the differential equation, integrate, and multiply the result by $\frac{1}{\mu}$.

NOTE: The standard form mentioned in Step 1 shows up a lot – in fact, even if you are not using the shortcut formulas above, it is considered "pretty standard" to rewrite your equation in standard form before solving the problem.

WARM-UP: $\int \frac{3x}{x^2+4} dx$ *HINT: RECALL u-substitution.*

REMINDER: $\frac{d}{dx}\tan^{-1}\frac{x}{a} = \frac{a}{a^2+x^2}$, and $\int \frac{1}{a^2+x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a}$

HOMOGENEOUS EQUATIONS

Defn. If the right side of the equation $\frac{dy}{dx} = f(x, y)$ can be expressed as a function of the ratio $\frac{y}{x}$, we call the equation **homogeneous**.

FACT: We can use a change of variables to make any homogeneous equation **separable**.

Example 1. Consider $\frac{dy}{dx} = \frac{x^2+xy+y^2}{x^2}$.

a. Rewrite the right side as a function of $\frac{y}{x}$.

b. Consider the substitution $v = \frac{y}{x}$. Use this substitution to replace all occurrences of "y" with "v", "x", and "$\frac{dv}{dx}$"

c. Show that the resulting equation is separable, and use this to find an explicit solution for y.

HINT: After integrating, don't forget to substitute back $v = \frac{y}{x}$ *to obtain an answer in terms of y.*

ANS: $y = \frac{1}{x}\tan(\ln|x| + C)$

GROUPS:

Example 2. Find a general solution to $\frac{dy}{dx} = \frac{x^2+3y^2}{2xy}$. Give an explicit solution (that is, solve for y).

HINT: You may need to use u-substitution.
ANS: $y = \pm\, x\sqrt{cx - 1}$

Day 6

Sec 2.4

BIG PICTURE: An initial value problem consists of a differential equation, together with an initial value or condition.

Ex: $\frac{dy}{dt} + ty = t^3, y(1) = 3.6$

BASIC QUESTION: Given an initial value problem, does a solution exist? Is the solution unique, or is there more than one solution that works?

ANSWER: It depends on the equation. Today we will look at two Theorems that guarantee the existence and uniqueness of solutions for certain kinds of differential equations.

Example 1. For each differential equation below,
1. is it first order or second order?
2. is it linear or nonlinear?
3. Without actually solving it, decide which is the right solution strategy: integrating factor, or separable, or homogeneous?

a. $\frac{dy}{dt} = \frac{t+3y}{t-y}$
b. $2y' - ye^{t/3}$
c. $y' = (1 - 2t)y^2$
d. $\frac{dy}{dt}t^2y = xt\sqrt{1 - t^2}$
e. $\frac{dy}{dt} = t^2e^{2t} - 3y$
f. $\frac{dy}{dt} = \frac{2\cos 2t}{3+2y}$
g. $y' - 3\sqrt{y't + y} = t^2$

FACT: For first order, linear equations, a solution always exists, and it is unique.

THEOREM 1 (Theorem 2.4.1 in book): Existence and Uniqueness for First Order Linear Equations.
If the functions p and g are continuous on an open interval I: $\alpha < t < \beta$ containing the point $t = t_0$, then there exists a unique function $y = \phi(t)$ that satisfies the differential equation:

$y' + p(t)y = g(t)$

for each t in I, and that also satisfies the initial condition

$y(t_0) = y_0$

where y_0 is an arbitrary prescribed initial value.

PROOF: The "SHORTCUT" for solving these problems always gives a solution.

$\mu = e^{\int p(t)dt}$, $y = \frac{1}{\mu(t)} \int \mu(t)g(t)dt + C$

Focus on what the theorem says, how to use it!

Example 2. Use Theorem 1 to find an interval in which the initial value problem $ty' + 2y = 4t^2$, $y(1) = 2$ has a unique solution. Do NOT find the actual solution.
HINT: rewrite in standard form. Where is p(t) continuous?

Day 7

Sec 2.4 - First order nonlinear, Bernoulli eqns

RECAP: If the diffy q is first order, and is linear, then it has a solution *under what condition??*
(p(t), g(t) must be *continuous*)

Now, what if an equation is first order, but is not linear? We are going to focus on the following type of equation - the derivative y' appears only once, and we can write it in the form:

$$\frac{dy}{dt} = f(y, t)$$

To understand this theorem, we will need to know something about derivatives of functions like $f(y, t)$ -- functions of not one, but two variables.

PARTIAL DERIVATIVES. KEY IDEA: When taking a partial derivative of a function of two or more variables, we consider the other variables to be **constant**.
NOTE: The partial derivative symbol uses a "curly d" - different from the standard derivative which uses regular d.

Example 1. If $f(y, t) = t^3 + t^2y^2 + y^4$, find: a. $\frac{\partial f}{\partial t}$ b. $\frac{\partial f}{\partial y}$

THEOREM 2 (Theorem 2.4.2 in book): Existence and Uniqueness for First Order Nonlinear Equations.
Let the functions f and $\frac{\partial f}{\partial y}$ be continuous in some rectangle $\alpha < t < \beta,\ \gamma < y < \delta$containing the point (t_0, y_0). Then, in some interval $t_0 - h < t < t_0 + h$ contained in $\alpha < t < \beta$, there is a unique solution $y = \phi(t)$ of the initial value problem:

$$\frac{dy}{dt} = f(y, t),\ y(t_0) = y_0$$

NEXT TIME: Look at 1-2 examples? Then on to Bernoulli

Example 2. Consider the initial value problems: a. $\frac{dy}{dx} = \frac{3x^2+4x+2}{2(y-1)}$, $y(0) =- 1$ b. $y' = y^{1/3}$, $y(0) = 0$

NOTE: The latter doesn't satisfy continuity condition - sol'n is not unique!
$y =\pm (\frac{2}{3}t)^{3/2}$, $t \geq 0$ are two solutions, as is y=0.

BERNOULLI EQUATIONS.
Sometimes we can solve a nonlinear equation by changing into a linear equation through a change of variables. The most important such equations are:

Defn. A **Bernoulli Equation** is a differential equation of the form $y' + p(t)y = q(t)y^n$ for some real number *n*.

SOLVING A BERNOULLI EQUATION: A Bernoulli equation can be transformed into a linear equation by a) dividing by y^n and b) using the substitution $v = y^{1-n}$. The resulting linear equation can be solved using either Integrating Factors or the corresponding shortcut.

Example 3: $t^2y' + 2ty - y^3 = 0,\ t > 0$

ANS: $y = \pm\sqrt{\frac{5t}{2+5t^2}}$

Day 8

Sec 2.6 - Exact Equations
NOTE: UPDATE HANDOUT TO MATCH EXAMPLES (1,2 are based on new prob)

One more type of first order equation - neither linear nor separable, nor homogenous, nor Bernoulli, so existing techniques won't work. Called **exact**.
NOTE: Of all first order differential equations, the ones that can actually be solved using techniques or tricks like those we've studied so far are rather special - most canNOT be solved in this way.

Example 1. Solve: $6x^5 + 2xy^2 + (2yx^2 + 5y^4)\frac{dy}{dx} = 0$ (give the solution implicitly)

Verify that this equation is an implicit solution: $x^6 + x^2y^2 + y^5 + c = 0$
How do we check? Take the derivative, and see if we get the original differential equation.
NOTE: Is this a regular derivative (treating y as a function of x), or a partial derivative (treating y as a constant)? CHECK THE DERIVATIVE NOTATION IN THE EXAMPLE!
Note: The left side of the implicit solution will be important - it's a function of x and y, let's call it 'psi':
$\psi(x, y) = x^6 + x^2y^2 + y^5 + c$

THE BIG QUESTION: How do we find the function $\psi(x, y) = x^6 + x^2y^2 + y^5 + c$?

Example 2. Find the partial derivatives of the function $\psi(x, y) = x^6 + x^2y^2 + y^5 + c$
RECALL: How many partial derivatives are there? When we take a partial derivative wrt one variable, how do we treat the other variable?

NOTE: The differential equation $6x^5 + 2xy^2 + (2yx^2 + 5y^4)\frac{dy}{dx} = 0$ can be broken into two parts:
$M(x, y) + N(x, y)y' = 0$
M(x,y) is the result of taking the partial derivative of ψ with respect to x: $\frac{\partial}{\partial x}\psi(x, y) = M(x, y)$ *(We also write ψ_x)*
N(x,y) is the result of taking the partial derivative of ψ with respect to y: $\frac{\partial}{\partial y}\psi(x, y) = N(x, y)$ *(We also write ψ_y)*

Starting with M, N, how do we find ψ? Integrate M with respect to x to find ψ
BEWARE: When we take the partial derivative of ψ with respect to x, what "disappears"? *Anything that is a pure function of y.* So when we integrate, we have to put back "a function of y" -- this plays the role that a constant usually plays in standard integration.

$\int 6x^5 + 2x^2dx = x^6 + x^2y^2 + h(y)$.

This is our $\psi(x, y) = x^6 + x^2y^2 + h(y)$. BUT we have to figure out what h(y) is. How do we do it?
Take the partial derivative of ψ with respect to y -- this should equal M.
$\psi_y = 2x^2y + h'(y)$. This had better equal N, so $2x^2y + h'(y) = 2yx^2 + 5y^4$, so in this case $h'(y) = 5y^4$.
Integrate to find $h(y) = y^5 + c$.

QUESTION:
Given a differential equation of the form: $M(x, y) + N(x, y)y' = 0$, how do we know that there is a function

$\psi(x, y)$ that will work as in the example above? We have to verify the following condition:
CONDITION: If such a $\psi(x, y)$ exists, then $\psi_{xy} = \psi_{yx}$, and vice versa. Therefore, to check if such a ψ exists, we simply have to check whether $M_y = N_x$.

Let's put all of this together.

SOLVING EXACT EQUATIONS:
Given a differential equation of the form: $M(x, y) + N(x, y)y' = 0$
1. Verify that the equation is **exact** by checking that $M_y = N_x$.
2. Integrate *M* with respect to x to obtain $\psi(x, y)$. *Treat y as a constant. Don't forget to add a "constant" term* $h(y)$.
3. Take the partial derivative ψ_y and set it equal to *N*, solve for $h'(y)$.
4. Integrate $h'(y)$ to find $h(y)$
5. The general solution to the differential equation is given implicitly by: $\psi(x, y) = c$.

Example 3. Solve $(y \cos x + 2xe^y) + (\sin x + x^2e^y - 1)y' = 0$
ANS: $y \sin x + x^2e^y - y = c$

Example 4. Solve $(3xy + y^2) + (x^2 + xy)y' = 0$
NOTE: $M_y \neq N_x$, *and so this equation is not exact - this method will not work.*

Day 11/12

Sec 2.7 - Numerical Approximations and Euler's Method

IDEA: TAKE AN EXTRA DAY TO INTRODUCE NUMERICAL APPROX.
CLASS CHALLENGE WARM-UP: Given a slope field / differential equation and initial point, find value of solution at terminal point (make it one they can solve).
CLASS CHALLENGE: given a (picture of) a slope field, and an initial point, find best approximation possible at another point. DONT give formula for y' -- instead, offer to give them value of derivative at any points they wish.
RESOURCES:
this worksheet starts with some good problems re: Euler's Method -- #1,2:
http://www.math.wisc.edu/~yliu/Spring%202014%20math222/Euler's%20Method%20And%20Applications%20I.pdf

Example 1. Consider the initial value problem $\frac{dy}{dt} = 3 - 2t - 0.5y, y(0) = 1$. Find y(1).
Example 2. Consider $y' = y \sin t - 1$. Suppose $y(0) = 2$. What is $y(3)$?

WRITE: How do we go about answering these questions?
- Solve for y. Use the initial condition to find C. Plug in t=1 to get y(1).
What's the problem here?

In general, it can be very difficult or even impossible to solve for y. However we "should be able to use computers to get an approximate answer". How do we do this mysterious thing?

Example 1. Consider the initial value problem $\frac{dy}{dt} = 3 - 2t - 0.5y, y(0) = 1$. Find y(1).
Consider the solution y(t).
- What is one point on the graph of the solution y(t)?
- Can we determine the slope of the solution y(t) at that point?
- Can we determine the slope of a solution at *any* point?
- What is the slope at (2,3)? At (4, 1.5)?
- If we do this everywhere, we get what is called a **direction field**.

?First, solve this exactly - solve for y, then plug in to find y(1). Sketch!

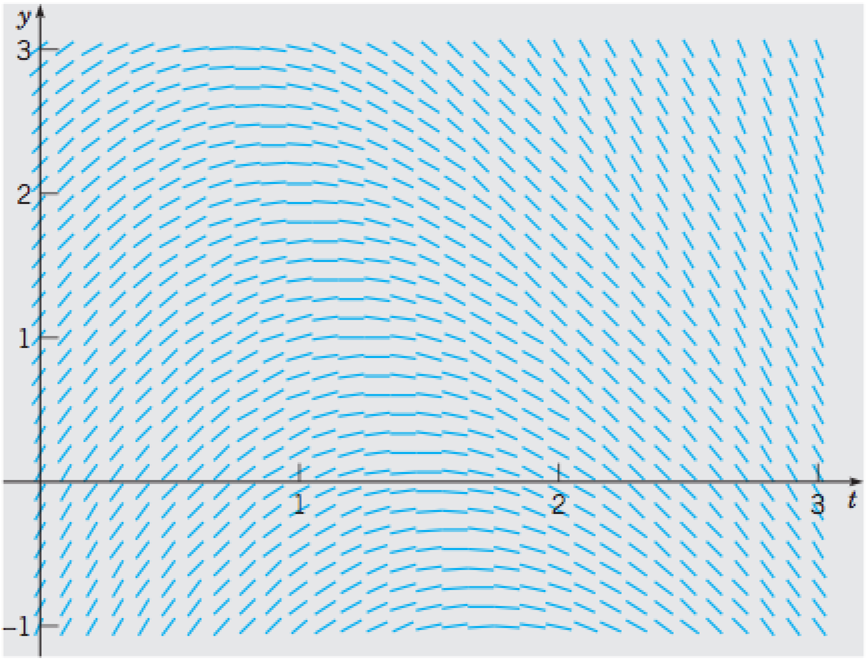


FIGURE 2.7.1 A direction field for Eq. (2): $dy/dt = 3 - 2t - 0.5y$.

From the picture, it seems like we should be able to just "glue together the slopes" to make a curve! That's the basic idea behind Euler's method.

NOTE: The differential equation $\frac{dy}{dt} = 3 - 2t - 0.5y$ has the form $y' = f(t, y)$, where the function f gives the slope.

1. Write the equation of the tangent line to the curve at the point (0,1).

   RECALL: Point-slope form of a line $y - y_0 = m(t - t_0)$

   ANS: $y = 1 + 2.5t$, $y = y_0 + f(0, 1)(t - t_0)$ where the point is (0,1), and the slope is $y' = f(0, 1)$.

   SKETCH. Does this follow the curve we want? NO - we need to break it up into steps, each of the same width.

2. Let the step size h=0.2, and let the first point be $(t_0, y_0) = (0, 1)$. How do we find the next point, (t_1, y_1)?

   QUES: What is t_1*?* $t_1 = t_0 + h$. *In this example,* $t_1 = 0.2$.

   What is y_1*? Plug t1 into the equation* $y = 1 + 2.5(0.2) = 1.5$

   Write the equation of the tangent line to the curve at the point (0.2, 1.5).

3. Find the point (t_2, t_3).

4. Continue until you reach t=1.

EULER'S METHOD

For the differential equation $y' = f(t, y)$ with initial condition (t_0, y_0), to generate an approximate solution with step size h:

- we can calculate all the values of t_n immediately using $t_n = t_0 + hn$
- to calculate the values of y_n you must proceed one at a time, since each one depends on the previous one:

$y_{n+1} = y_n + f(t_n, y_n)(t_{n+1} - t_n)$, or simply $y_{n+1} = y_n + f(t_n, y_n)h$.

QUESTION: How close was our answer to the "actual" answer in Example 1? Since we can actually solve this equation, we can find out.

SOLUTION: $y = 14 - 4t - 13e^{-t/2}$

TABLE 2.7.1 Results of Euler's Method with $h = 0.2$ for $y' = 3 - 2t - 0.5y,\ y(0) = 1$

t	Exact	Euler with $h = 0.2$	Tangent line
0.0	1.00000	1.00000	$y = 1 + 2.5t$
0.2	1.43711	1.50000	$y = 1.13 + 1.85t$
0.4	1.75650	1.87000	$y = 1.364 + 1.265t$
0.6	1.96936	2.12300	$y = 1.6799 + 0.7385t$
0.8	2.08584	2.27070	$y = 2.05898 + 0.26465t$
1.0	2.11510	2.32363	

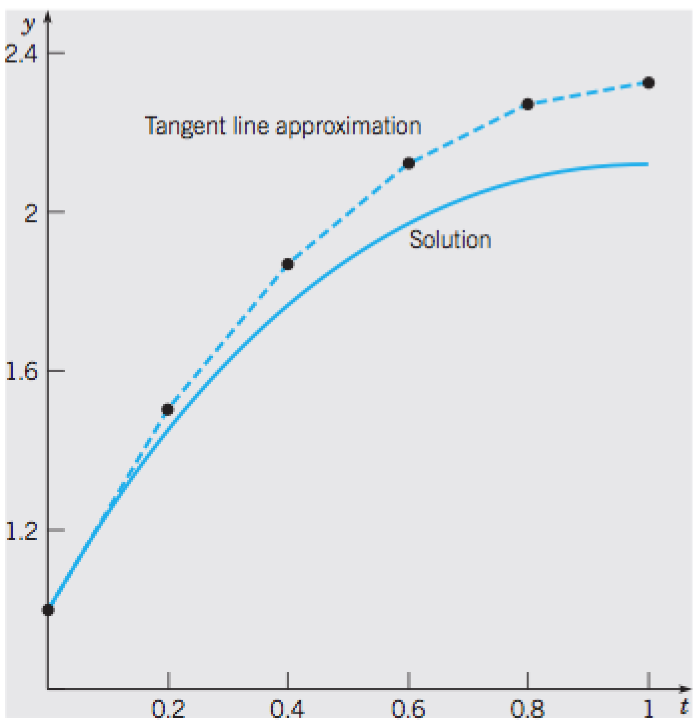


FIGURE 2.7.3 Plots of the solution and a tangent line approximation with $h = 0.2$ for the initial value problem (11): $dy/dt = 3 - 2t - 0.5y,\ y(0) = 1$.

QUESTION: How can we get a **better approximation**?
Use more steps - use a smaller h.

Finally, complete Example 2 using a step size of h=0.5.

DAY 12:
Start: 5 minute writing assignment, think/pair/share -
1. What's the main idea of Euler's Method?
2. If we want to use Euler's Method to approximate a solution to a differential equation, what should we have to start?

EULER'S METHOD OVERVIEW:
See notes taken on "observation" paperwork 3/12/15

BASIC PROBLEM:
Given the initial value problem $\frac{dy}{dt} = f(t, y)$ and $y(a) = b$, find the value of the solution at the point where $t = c$, that is, find $y(c)$.

EULER'S METHOD:
Approximate the solution by generating a series of points:
$(t_0, y_0) = (a, b)$ = the initial condition
(t_1, y_1)
(t_2, y_2)
…
$(t_n, y_n) = (c, y_n)$, where y_n is the approximate value of $y(c)$

n = number of steps
h = step size = distance between successive t-values = $t_{i+1} - t_i$
To find *h,* divide the distance from *a* to *c* by the number of steps n: $h = \frac{c-a}{n}$
Starting from a given point (t_i, y_i), how do we find the next point (t_{i+1}, y_{i+1}):
a find the slope dy/dt at the given point (t_i, y_i), dy/dt=f(ti,yi)
b. move along the **tangent line** from (ti,yi) to (ti+1,yi+t)
c. ti+1 = ti + h
d. yi+1 = yi + slope*h
NOTE: finish typing up the steps above (tidy up/fix notation)

Day 13

Chapter 3: Second Order Differential Equations
Sec 3.1 - Homogeneous Equations with Constant Coefficients

Defn. A second order **ordinary** differential equation has the form $\frac{d^2y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right)$, where *f* is some given function.
NOTE: As before, we will often denote the independent variable by t or x, and the dependent variable by y, or v, or some other letter.

Defn. Such an equation is **linear** if it has the form $\frac{d^2y}{dt^2} = g(t) - p(t)\frac{dy}{dt} - q(t)y$, that is, if *f* is a linear function of y and y', and we often write it in the form: $y'' + p(t)y' + q(t)y = g(t)$.

NOTE: If the equation is NOT in the form above, then it is **nonlinear**. Nonlinear second order equations are generally *hard* - usually numerical or geometrical methods work best - and we will no say much more about them until Chapters 8 and 9.
NOTE: Even **linear** second-order equations are *often hard* - so we will *start simple*.

Defn. A second order linear equation is said to be **homogeneous** if $g(t) = 0$ (otherwise it's **nonhomogeneous**).
Crazy fact: What does this definition have to do with the **homogeneous** equations we studied in Chapter 2?
Nothing!

Defn. A second order linear equation has **constant coefficients** if the functions $p(t)$, $q(t)$ and $g(t)$ are constant functions.

Today, we will be looking at homogeneous linear second order differential equations with constant coefficients -- that is, equations of the form: $y'' + by' + cy = 0$

Example 1. Solve $y'' - y = 0$. Find the solution that satisfies the initial conditions $y(0) = 2$, $y'(0) = -1$

NOTE: Rearrange: $y'' = y$. We are looking for a function whose second derivative is equal to the function itself.
QUES: Can you think of such a function?
TWO ANS: $y = e^t$ and $y = e^{-t}$. *Check 'em. What about multiplying by a constant?*
GENERAL SOLUTION: $y = c_1e^t + c_2e^{-t}$

For the specific solution, first use y(0)=2 to obtain $c_1 + c_2 = 2$, then differentiate and use y'(0)=1 to get $c_1 - c_2 = 1$. Now solve.

Observations from previous example:
1. Solutions were exponential functions.
2. Linear combinations of solutions were also solutions.

Let's apply these same ideas to $y'' + by' + cy = 0$. First, let's see if we can find a solution of the form $y = e^{rt}$ (an exponential function) - the only question is, ***what is the constant r?***

To find r: differentiate twice, plug in, to obtain: $(r^2 + br + c)e^{rt} = 0$. For this to equal zero, we must have the part in parentheses equal zero: $r^2 + br + c = 0$

Defn. The **characteristic equation** of the differential equation $y'' + by' + cy = 0$ is the equation $r^2 + br + c = 0$.

- If we can solve the characteristic equation for *r*, then $y = e^{rt}$ is a solution of the differential equation.
- The general solution of the differential equation is given by $y = c_1e^{r_1t} + c_2e^{r_2t}$ (where r_1, r_2 are the roots of the characteristic equation)

Example 2. Find the general solution of $y'' + 5y' + 6 = 0$. Then find the solution satisfying initial conditions $y(0) = 2,\ y'(0) = 3$.

ANS: general solution: $y = c_1e^{-2t} + c_2e^{-3t}$. particular solution: $c_1 = 9,\ c_2 =- 7,\ y = 9e^{-2t} - 7e^{-3t}$

Sec 3.4 - Repeated Roots

Example 3. Find the general solution $y'' + 4y' + 4y = 0$.
NOTE: Just complete the above example through finding the roots, and demonstrate the alternative form of the general solution at the end of class!! (skip the rest of this lecture)

NOTE: The two roots of the characteristic equation are r1=r2=-2. One solution must be $y = e^{-2t}$. *What's the "other" solution?*

OPTIONAL.

GUESS: that the "other" solution looks like "something times $y = e^{-2t}$".

That is, suppose there is a solution of the form $y = v(t)e^{-2t}$ for some function v(t). Let's try to find v(t).

Differentiate y twice, plug into diffy q (it's a mess - but everything cancels!), obtain: $v''(t) = 0$.
Integrate twice to find $v(t) = c_1t + c_2$.

Thus $y = c_1te^{-2t} + c_2e^{-2t}$. *The second term corresponds to the original solution, and the term* $y = te^{-2t}$ *is the "other solution" we were seeking. Altogether, this gives the general solution.*

CLAIM: $y = te^{-2t}$ is also a solution. *Check it!*
GENERAL SOLUTION: $y = c_1te^{-2t} + c_2e^{-2t}$

RULE: If $y'' + by' + c' = 0$ has a characteristic equation with a repeated root *r*, then the general solution is $y = c_1te^{rt} + c_2e^{rt}$.

GROUPS:
Example 4. Find the solution of the initial value problem
$y'' - y' + 0.25y = 0,\quad y(0) = 2,\quad y'(0) = \frac{1}{3}$.

ANS: roots are r1=r2=½. c1=2, c2=-⅔. Solution: $y = 2te^{-t/2} - \frac{2}{3}e^{-t/2}$

Day 14

Sec 3.3 - Complex Roots

Example 1: Solve $y'' - 4y' + 13y = 0$.
NOTE: $r = 2 \pm 3i$. ?? ANS: $y = c_1 e^{(2+3i)t} + c_2 e^{(2-3i)t}$??

To proceed, we need to know how to deal with raising e to the power of a complex number.
?? Derive Euler's formula??

EULER'S FORMULA: $e^{it} = \cos t + i \sin t$

QUES: How to rewrite $y = e^{(2+3i)t}$?
ANS: $e^{(2+3i)t} = e^{2t}(\cos 3t + i \sin 3t)$

QUES: For example 1, what are the TWO basic solutions? What is the general solution?
NOTE: Need identities $\cos(-x) = \cos x$, and $\sin(-x) =- \sin x$
FACT: All solutions obtainable through the general solution above can also be obtained through the simpler version:
$y = e^{2t}(c_1 \cos 3t + c_2 \sin 3t)$

GENERAL FACT: If the characteristic equation for $y'' + by' + cy = 0$ has complex roots $r = \lambda + i\mu$, then the general solution for the differential equation is given by $y = e^{\lambda t}(c_1 \cos \mu t + c_2 \sin \mu t)$.

Example 2: Find the solution of the initial value problem $16y'' - 8y' + 145y = 0,\ y(0) =- 2,\ y'(0) = 1$.

ANS: $y =- 2e^{t/4} \cos 3t + \frac{1}{2}e^{t/4} \sin 3t$

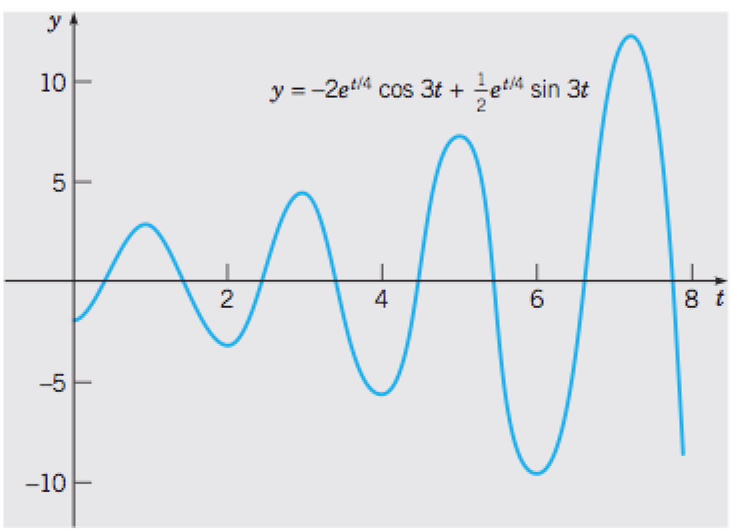


FIGURE 3.3.2 Solution of the initial value problem (25): $16y'' - 8y' + 145y = 0,\ y(0) = -2,\ y'(0) = 1$.

Day 15

Sec 3.5 - Nonhomogeneous Equations, the Method of Undetermined Coefficients

Example 1: $y'' - 3y' - 4y = 3e^{2t}$.

QUES: What makes this equation different from those we've been studying? ANS: *not homogeneous!*

CONNECTION BETWEEN HOMOGENEOUS AND NONHOMOGENOUS DIFFERENTIAL EQUATIONS
Defn. Given the nonhomogeneous second-order differential equation $y'' + p(t)y' + q(t)y = g(t)$, where *p, q,* and *g* are given continuous function on an open interval *I*, the corresponding homogeneous equation is
$y'' + p(t)y' + q(t)y = 0$.

THEOREM. *"The difference of two solutions of the nonhomogeneous equation give a solution to the homogeneous equation."*
If Y_1 and Y_2 are two solutions of the nonhomogeneous equation above, then their difference $Y_1 - Y_2$ is a solution of the corresponding homogeneous equation. If, in addition, y_1 and y_2 are a fundamental set of solutions of the nonhomogeneous equation, then $Y_1 - Y_2 = c_1y_1(t) + c_2y_2(t)$ for some constants c_1, c_2.

THEOREM. *"The general solution of the nonhomogeneous equation combines the general solution to the homogeneous equation and a single solution to the nonhomogeneous equation."*
The general solution of the homogeneous equation can be written in the form
$y = \phi(t) = c_1y_1(t) + c_2y_2(t) + Y(t)$, where:

- $c_1y_1(t) + c_2y_2(t)$ is the general solution to the homogeneous equation, and
- $Y(t)$ is any particular solution of the nonhomogeneous equation.

PROBLEM: We know how to find the general solution to the homogeneous equation. How do we find the specific solution to the nonhomogeneous equation?

THE METHOD OF UNDETERMINED COEFFICIENTS
We make a guess, based on g(t), of the *form* that a solution $Y(t)$ will take, without knowing the coefficients. We substitute this expression into the differential equation, and attempt to determine the coefficients. If we can't do it, we modify our guess about the form of Y(t) and try again.

Example 1: $y'' - 3y' - 4y = 3e^{2t}$.

- Find a particular solution $Y(t)$ to this differential equation.
- Find the general solution to the corresponding homogeneous equation.
- Find the general solution.

Based on the right side $3e^{2t}$, we guess that the solution will have the form $Y(t) = Ae^{2t}$. Can we figure out A? Find Y' and Y'', plug into the equation, and solve.

ANS: $A =- 1/2,\ y(t) = c_1e^{4t} + c_2e^{-t} - \frac{1}{2}e^{2t}$

Example 2: Find a particular solution to $y'' - 3y' - 4y = 2 \sin t$

QUES: What form should Y(t) have? Asint ?
QUES: When we take the derivative, what do we get? cos, then sin. To make sure that we have enough flexibility to cancel things out, let's guess that $Y(t) = A \sin t + B \cos t$.

Example 3: Find a particular solution to $y'' - 3y' - 4y = 4t^2 - 1$
NOTE: Y(t) should be a polynomial of same degree as right side: $Y(t) = At^2 + Bt + C$.

Day 16

Sec 3.7 - Mechanical Vibrations

Today we're going to focus on a particular application - springs.

Setup: We start with a spring. We hang a mass on it.
m = mass
L = elongation of the spring caused by the mass (at equilibrium)

A spring is an object in motion, whose motion is changing because of various forces acting on it. What basic relationship describes the way that force interacts with motion?
NEWTON: f=ma

Definitions: $u(t)$ = the position, or displacement from equilibrium, at time *t*.
Note: u is positive in the downward direction.
Note: the total displacement (including that caused by the mass) is given by $L + u$

Considering Newton's equation, which part can we express in terms of u? Also, force changes over time. So we have: $mu''(t) = f(t)$

What are the forces acting on it?

- weight (gravity): $w = mg$, constant, acting in downward direction.
- the spring force F_s is (approximately) proportional to the total elongation of the spring, $F_s = -k(L + u)$
  - L+u is the total displacement
  - k is the ***constant of proportionality***.
  - TO FIND k: Consider the object at rest, note that gravity is exactly balanced by spring force, so $mg - kL = 0$. If we know m, g and L, we can find k.

The following two additional forces can help to increase the accuracy of the model in many real-world situations (though we will ignore them for now).

- A damping or resistive force F_d. Always acts to slow the mass (acts in the direction opposite motion).
- An external force $F(t)$ applied to the mass. (including this allows us to account for many external factors that might be acting on the system - for example, motion of the mount to which the spring is attached, etc).

Combining things, we get: $mu''(t) = mg + F_s(t) + F_d(t) + F(t)$. We will ignore the last two terms, which gives us:
$mu''(t) = mg - k(L + u(t))$. Simplifying, we obtain:
$mu'' + ku = 0$ (NOTE: recall that the weight exactly counterbalances the spring force at equilibrium so mg-kL=0)

GROUPS(?):
Example 1. Suppose that a mass weighing 10lb stretches a spring 2 in. If the mass is displaced an additional 2 in and is then set in motion with an initial upward velocity of 1 ft/s, determine the position of the mass at any later time. Express your answer in terms of a single cosine function.
Hint: Use pounds and feet in this problem. Note that acceleration due to gravity is $g = 32ft/s^2$

Hint: Are pounds a measurement of mass?

FIND m:
Does m=10lb?? NO: pounds are not a measurement of mass, they are a measurement of weight: w=10 lb. Recall w=mg. Do we know gravity g? g=9.8m/s^2. BUT are we using meters? What is it in feet? g=32 ft/s^2.
Thus:

$$m = \frac{10lb}{32ft/s^2} = \frac{5}{16} \cdot \frac{lb \cdot s^2}{ft}$$

FIND k: note: L is given in inches, need to convert to ft. L=2/12 =⅙ feet.

$$k = \frac{mg}{L} = \frac{5/16 \cdot 32}{(2/12)} = 60\ lb/ft$$

Thus we have: $\frac{5}{16}u'' + 60u = 0.$
Find general solution, then find specific solution satisfying $u(0) = 1/6\,ft,\ u'(0) = -1ft/s$

ANS: $u = \frac{1}{6}\cos(8\sqrt{3}t) - \frac{1}{8\sqrt{3}}\sin(8\sqrt{3}t)$

Final step - how to combine these two trig functions into a single trig function?
TRIGONOMETRIC IDENTITY:
$A\cos ct + B\sin ct = R\cos(ct - \delta)$
where $R = \sqrt{A^2 + B^2}$, $\tan\delta = \frac{R\sin\delta}{R\cos\delta} = \frac{B}{A}$
Hints: To find δ, first determine the quadrant by checking the signs of sine and cosine (A and B) in the fraction. Use inverse tangent to find delta, then adjust accordingly to account for the quadrant.

Hints: There is a very thorough explanation of this trig identity here:
http://www.mathcentre.ac.uk/resources/uploaded/mc-ty-rcostheta-alpha-2009-1.pdf

ANS: $R \approx 0.18162\,ft,\ \delta \approx -0.40864rad,\ u = 0.18162\cos(8\sqrt{3}t + 0.40864)$

Sec 3.7 Electrical Circuits

ELECTRICITY BASICS:
We use electricity to move energy from one place to another.

Think of energy as gold, and electrons are the mine carts we use to move gold around - in a circuit, electrons are loaded up with gold in the battery, travel around the circuit till they reach, say, a light bulb, then unload the gold (which is used to light the light) - the empty mine carts continue around the circuit.

Energy is measured in Joules (J) , but we usually approach the energy more indirectly - for example:
Power P is a measurement of energy per second, measure in Watts (one Watt = 1 Joule per second)
The amount of power moving through a circuit is determined by two factors:

1. how many electrons are flowing through the circuit (current)
2. how much energy each electron is carrying (voltage)

Current (I) is measured in amps (A) - one amp equals 6,241,507,648,655,549,400 (6.24... x 10^18) "electrons per second" (six quintillion electrons per second - this number is a Coulomb of electrons)

NOTE: Charge (Q) is simply a measure of how many electrons. Current equals charge per second, $I(t)=Q'(t)$

Voltage is measured in volts (V) - one volt equals one Joule per Coulomb of electrons "energy per electron"

How do we measure power? power = # of electrons * energy per electron, or:
W=A*V

Orienting example: Things like light bulbs give their power requirements in Watts - like a 100W bulb. Your wall outlet in your home tells you how much energy each electron is carrying - in the US, usually get 110V current. Now we can answer the question - when you plug in the light bulb, how many electrons are flowing through it each second ("what is the current through the bulb?"). There must be enough current to deliver 100 W of power. 100W = A*110V, solve for A. (A = 10/11 Amps)

Components of electrical circuits:
Power source (like a battery, or a generator, or solar cell, etc.) -- these supply energy to electrons moving around a circuit, **E(t) = volts at time t**. If the power source is a battery, it always gives each electron the same amount of energy (e.g. a "9 Volt battery"), but depending on the other stuff in the circuit can send out more, or less, electrons.
Resistor (like a light bulb) -- *resistance* measure how difficult it is to force electrons to more through a material. The **resistance R=V/I** is measured in Ohms (Ω, or Volts per Amp). For a given resistor (like a light bulb), the resistance is **constant**.
Voltage drop across a resistor: V=RI, where I(t) is the current and R is constant

For a fixed current, the resistance measures how much energy is stolen from each electron passing through the component (how much gold is taken from each mine cart). BUT since current can vary and R is fixed, it says that when you increase the current (more electrons per second), it actually steals MORE energy from each electron - so forcing more current through a resistor end up in a loss of efficiency (the resistor takes more energy from each one).
Alternatively, resistance measures how much voltage is required for each amp of current that you want to flow across the resistor. To double the current, you must double the voltage (since R remains constant).
Does this refer to voltage in the circuit, or only voltage passing through that point? If two resistors follow each other, is the voltage experienced by the second one less than the voltage experienced by the first one?

Conceptual problem: When we talk about the voltage V in a simple circuit, are we talking about

- *"how much energy the battery imparts to each electron", e.g. a 9V battery in a circuit means V always equals 9, or are we talking about (V is associated with the entire circuit) - NOTE: this is called the* ***impressed voltage***
- *"voltage drop across elements", or differences in energy around the the circuit, e.g. a resistor has a voltage drop of some number of volts, so electrons entering the resistor have more energy than electrons leaving the resistor. V will be different depending on what part of the circuit we are measuring.*

ANSWER?: Kirchhoff's Second Law: In a closed circuit, the ***impressed voltage*** *will equal the* ***sum of the voltage drops*** *in the rest of the circuit*

Capacitor - stores up energy (model: two plates separated by an insulator, so no electrons jump across). When Voltage is applied, charge Q (# of electrons) builds up on one of the plates, and pushes electrons away from the other plate. The **capacitance C** is given by **C=Q/V**, charge per volt (or "# of electrons per (energy per electron)"), or Farads -- tells us how many electrons build up on one side of the capacitor for every volt. If we want twice as many electrons to build up (twice the charge), we need to double the voltage.
Voltage drop across a capacitor: V = Q/C, where Q(t) is charge and C is constant.

Inductor - (model: coil of wire) - resists change in flow of current, stores energy in a magnetic field. Think of it as a heavy waterwheel in a pipe - when water starts flowing, it takes some time for the waterwheel to get up to speed (during which there is resistance), after which water flows normally. When the water source is shut off, the wheel keeps spinning, pushing the water along until the wheel slows. The **inductance L = V/(dI/dt)** "Volts per change in current" (measured in Henries H). As the current changes, how much energy is stored up (or released) in the inductor?
Voltage drop across an inductor: V = L*dI/dt

Kirchhoff's Second Law: In a closed circuit, the ***impressed voltage*** *will equal the* ***sum of the voltage drops*** *in the rest of the circuit.*
Thus, for a circuit with an inductor, a resistor, and a capacitor, along with a battery delivering constant voltage V, we have:
V = L*dI/dt + RI + Q/C
where V, L, R and C are constant, I(t) is the current flowing through the circuit at time t, and Q(t) is the charge at time t. Recall that I(t) = Q'(t)
To obtain a differential equation for the charge Q, take the derivative with respect to time t.
0 = L*I" + R*I'+ Q'/C

QUESTION FOR CLASS:
How many people understand this diagram?

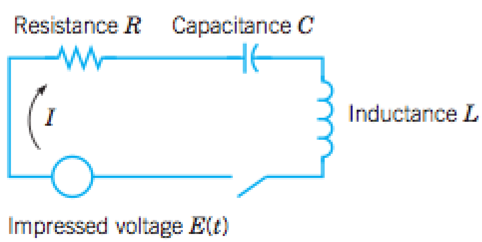


FIGURE 3.7.9 A simple electric circuit.

From Ohbong (applications/problems): http://www.intmath.com/differential-equations/6-rc-circuits.php

Electrical circuits - check out this great (ppoint) introduction to circuits (first result in google search):
https://www.google.com/webhp?sourceid=chrome-instant&ion=1&espv=2&ie=UTF-8#q=introduction%20to%20electricity%20and%20electrical%20circuits

Hydraulic analogy: http://en.m.wikipedia.org/wiki/Hydraulic_analogy

Good for basic definitions/units:
http://physicsnet.co.uk/a-level-physics-as-a2/current-electricity/charge-current-potential-difference/

Day 17

Finish up spring example from last time. Review questions for final?

Day 18

Exam #2

Day 19

Review of Taylor Series, Section 5.2 - Series Solutions around an ordinary point
SUGGESTED EXERCISES: Sec 5.2, p264 #1, 3, 5 (a,b,d only)

QUES: Is cos(x) a polynomial? Can we make a polynomial that "looks like" cos(x)?

https://www.desmos.com/calculator

Plot some partial MacLaurin Series:
Example: $\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \frac{1}{40320}x^8 - \frac{1}{3628800}x^{10} + ...$
Graph it! See what it looks like.
Where does this come from?

One of the MOST powerful ideas to come out of calculus is the ability to take (almost) ANY function, and write it as a polynomial. (possibly an infinite polynomial, called a Taylor series).

Definition. If a function *f* has a Taylor series expansion about x=0 (a MacLaurin Series), then it is given by:
$$f(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

PROJECT 1: Calculate the cosine of 1 radian, that is, find $\cos 1$**, using only basic arithmetic (add/subtract/multiply/divide).**

1. Find the first few terms (up to x^8) of the Taylor series about $x = 0$ (the MacLaurin Series) for $y = \cos x$.
2. Plug in $x = 1$ to the resulting Taylor polynomial, and record the result to 8 decimal places.

You did it! Now, how good is your answer? Check it against the true value of cos 1 (be sure to use radians!). How many decimals are correct?

------------- ONLY GOT THIS FAR ON DAY 19 -----------

Example: Use the formula to find the first few terms in the Taylor series for cos(x) at x=0.
Example: Write this in Sigma notation.

CLASS:

Example 1: Find the first few terms of the Taylor series expansion of e^x(about x=0). Write in Sigma notation.

Example 2: Find the first few terms of the Taylor series expansion of lnx about x=0.
Example 3: Now try it about x=1. Write in Sigma notation.

Definition. If a function *f* has a Taylor series expansion about $x = x_0$, then it is given by:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

COMMON TAYLOR SERIES:

$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}x^{2k}$ $\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}x^{2k+1}$ $e^x = \sum_{k=0}^{\infty} \frac{1}{k!}x^k$ $\ln x = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}(x - 1)^k$

Section 5.2 - Series Solutions of Differential Equations around an ordinary point.

USING SERIES TO SOLVE DIFFERENTIAL EQUATIONS

Defn. Given a homogeneous linear second order differential equation $P(x)y'' + Q(x)y' + R(x)y = 0$, we can divide by P(x) to obtain the form: $y'' + p(x)y' + q(x)y = 0$ -- provided $P(x)$ is not zero.
Any point x_0 for which $P(x_0) \neq 0$is called an **ordinary point** (otherwise it is a **singular point**).

Even if it is impossible to find a "nice" solution for such a differential equation, it is still often possible to express the solution in terms of a series.

Example 4. Find the series solution of the initial value problem $y'' + y = 0, \; -\infty < x < \infty$.

a. Write y as a power series about about $x = 0$ $y = \sum_{n=0}^{\infty} a_n x^n$

b. Take derivatives of y and substitute into the differential equation.

c. Find the first few even terms a_2, a_4, a_6, ... given in terms of a_0, and the first few odd terms given in terms of a_1

d. Find a formula relating a_n and a_{n+2}, called a ***recurrence relation****.*

e. Find a formula for the even terms, and a formula for the odd terms.

f. Are these series familiar? What are they?

g. Now consider the initial value problem $y'' + y = 0$, $y(0) = 3$, $y'(0) =- 3$. Estimate the value of $y(2)$ using the first eight terms of the series solution. Recall for a Taylor Series about $x = 0$ (a Maclaurin Series) that $a_0 = \frac{y(0)}{0!}$, $a_1 = \frac{y'(0)}{1!}$

ON BOARD: Write out y=a0+a1x+a2x^2+... (up to a5).
QUES: How many constants? (infinite)!
- Find y', y''
- substitute y'', y into the differential equation
- collect like terms (terms with same power of x). Ask them to generate the next couple of terms (up to x^5) by

pattern matching.
- equate left and right sides -- set each coefficient expression equal to 0.
- Take a0, a1 as given. our goal is to use these equations to EXPRESS LATER TERMS IN TERMS OF EARLIER ONES!
- Make a list of a0, a1, a2, … up to a7, all IN TERMS OF a0 and a1
- put these back into the original power series to obtain an expression for y in terms of a0, a1

TWO DIRECTIONS WE CAN GO FROM HERE:
1. Solve IVP by finding values for an up to some point - get an approximate solution!
2. Simplify this expression, try to write it in sigma notation, relate it to other (known) series

NOTE: To combine series together, need to shift the index so they agree - e.g. replace n by n+2 and start the sum at 0 rather than 2...

Example 5. Find a series solution of Airy's equation: $y'' - xy = 0, \ -\infty < x < \infty$.
a. Write y as a power series about $x_0 = 0$
b. Take derivatives of y and substitute into the differential equation.
c. Find the recurrence relation.
d. Find the first 12 terms of the series.
e. Find a formula for the a_n (break them into two series).
f. Give the general series solution for Airy's equation.

Day 20

(Power Series solutions continued)
(NOTE: last time, I only covered intro to Taylor Series, cosine example. Gave definition of Taylor Series, but didn't cover anything with Sigma notation…)

RECALL: A Taylor Series begins with a function $f(x)$ and a point $x = x_0$. We want to write down the polynomial that *best matches the function f(x) at the point* $x = x_0$.

DO Example 1 (Day 19) ON BOARD: Find the first few terms of the Taylor Series $f(x) = e^x$ *about the point* $x = 0$ *(the Taylor Series at* $x = 0$*)*

We want: $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + ...$
How do we find the coefficients $a_0, a_1, a_2, ...$?

FACT: $a_n = \frac{f^{(n)}(0)}{n!}$ *What do these symbols mean?*
Make table: n, $f^{(n)}(x)$, $f^{(n)}(0)$, a_n

Have class find first few. Now, what's a general formula for a_n?
Discuss Sigma notation. Write formula using Sigma notation.

DO Example 2 and Example 3 - GROUPS?:
Example 2: Find the first few terms of the Taylor series expansion of lnx about x=0.
Example 3: Now try it about x=1. Write in Sigma notation.

NOTE: Look at common Power Series LIST.

DO Example 4 Day 19 sheet

Day 21

6.1 Definition of Laplace Transform
GOALS: Defn of Laplace Transform, Computing basic Laplace Transforms, linearity of Laplace Transforms
NOTE: UPDATE FORMULA SHEETS TO USE F(s) instead of L{f(t)} (Final Exam Review has an updated version)
Overview: What is the Laplace transform all about? How do we use it/why do we study it?

- The Laplace transform of a function $f(t)$ is another function called $\mathcal{L}\{f(t)\}$, or $F(s)$.
  NOTE: This is similar to the way that the derivative of a function $f(t)$ is another function $f'(t)$ or $\frac{df}{dt}$.
  NOTE: The variable changes when we compute the Laplace transform - if the original *f(t)* is a function of *t*, then the Laplace transform is a function of another variable *s*.
- We use it to make solving differential equations easier, following this outline:
  1. Start with a differential equation.
  2. Take the Laplace transform of both sides.
     This replaces the differential equation with a much simpler (algebraic) equation.
  3. Solve the algebraic equation.
  4. Simplify the solution.*
     ** requires partial fraction decomposition*
  5. Take the inverse Laplace transform of the solution.
     This gives the solution to the original differential equation.
- Yes, but WHAT is the Laplace transform? *Ask me about it sometime (go on…)*

TRICKY PARTS: #2, #4, #5.
Today: #2 - how to take the Laplace transform.
- Finding the Laplace transform using the definition
- some basic Laplace transforms
- linearity of the Laplace transform
- Finding the Laplace transform using a table

Defn. If $f(t)$ is a function** defined for $t \geq 0$, then the Laplace transform of *f(t)* is:

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)dt.$$

*** NOTE: In fact, we also require the function $f(t)$ to satisfy two additional conditions:*
1. $f(t)$ must be piecewise continuous on any interval $[0, A]$ (for any positive A)
*2. $f(t)$ must be of **exponential order**: $|f(t)| \leq Ke^{at}$ when $t \geq M$ (for some constants K, a, and M, with K and M positive). This says that it cannot grow too quickly.*

Example 1: Find the Laplace Transform of the constant function $f(t) = 1$.
ANS: $F(s) = \frac{1}{s}$, $s > 0$

-------------------- Spr 2015 - today I stopped here ------------------

TABLE OF LAPLACE TRANSFORMS:

function	**Laplace transform**
1	$\frac{1}{s}$, $s > 0$
e^{at}	$\frac{1}{s-a}$, $s > a$
t^n, n=positive integer	$\frac{n!}{s^{n+1}}$, $s > 0$
$\sin at$	$\frac{a}{s^2+a^2}$, $s > 0$
$\cos at$	$\frac{s}{s^2+a^2}$, $s > 0$
$e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}$, $s > a$
$e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}$, $s > a$
$t^n e^{at}$, n=positive integer	$\frac{n!}{(s-a)^{n+1}}$, $s > a$
$af(t) + bg(t)$	$a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$
$f'(t)$	$s\mathcal{L}\{f(t)\} - f(0)$
$f''(t)$	$s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0$

LINEARITY RULE:

For function f,g and constants a,b, $\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$

OpenLab #3: Flipping the classroom / Laplace Transforms
- videos: using tables to compute Laplace Transforms
- table: http://3.bp.blogspot.com/_3J4lrjHF-IY/StIpYnZ883I/AAAAAAAAATw/9WjkOgtQww4/s1600/Table.12.2.jpg
- videos: Laplace Transforms
- video: partial fractions decomposition

Day 22

Laplace Transform / Inverse Laplace Transform (cont'd)

TODAY: taking Laplace transform, inverse Laplace transform

RECALL DEFN of Laplace Transform (& two addl conditions: piecewise continuous, exponential order)

Laplace transform of:
$\mathcal{L}\{1\}$
$\mathcal{L}\{e^{at}\}$
$\mathcal{L}\{t^{n}\}$
LINEARITY RULE: For function *f,g* and constants *a,b*, $\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$

Ex: $\mathcal{L}\{5 + 6t^{5} + 2e^{3t}\}$

GROUP WORKSHEET
(Submit answers on a separate sheet)
Part I. Find the Laplace Transform of each function, and determine the interval on which it is defined.

5. $t^{2} + 4t^{3} - 7t^{6}$
6. $\sin t + \cos 3t$
7. $5e^{2t} - 4\sin 3t$
8. $5t^{2} + 3\sin 5t - 2e^{6t}\cos 2t$

Part II. Find the inverse Laplace Transform of each function.

6. $\frac{3}{s} + \frac{4!}{s^{5}} + \frac{1}{s-7}, \ s > 7$
7. $\frac{1}{s^{2}+25} + \frac{s}{s^{2}+25}, \ s > 0$
8. $\frac{1}{(s-4)^{2}+9}$, s>2
9. $\frac{1}{(s-6)^{7}} + \frac{5}{2s-7}, s > 6$
10. $\frac{5}{s^{2}-8s+41}, s > 4$

Day 23

6.2 Solution of Initial Value Problems using the Laplace Transform

Ex: Find the Inverse Laplace Transform:

a) $\frac{1}{s-1} + \frac{4}{s+4}$

b) $\frac{5s}{s^2+3s-4}$

(HINT: They are equal!)
Starting from b), how do we get to a)?

PARTIAL FRACTION DECOMPOSITION (discuss, example).

SOLVING IVPs USING LAPLACE TRANSFORM
We will need to be able to take the Laplace transform of the derivative of a function.

Laplace transform of a derivative

Theorem. Suppose *f* is continuous and of exponential order, and *f'* is piecewise continuous.

Then $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$.

Similarly, $\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$.

Example 1. Use the Laplace transform to solve the differential equation $y'' - y' - 2y = 0$ with initial conditions $y(0) = 1,\ y'(0) = 0$.

ANS: $Y(s) = \frac{1/3}{s-2} + \frac{2/3}{s+1},\ y = 1/3\, e^{2t} + 2/3\, e^{-t}$

Example 2. Find the solution of the differential equation $y'' + y = \sin 2t$ satisfying initial conditions $y(0) = 2,\ y'(0) = 1$.

ANS: $Y(s) = \frac{2s}{s^2+1} + \frac{5/3}{s^2+1} - \frac{2/3}{s^2+4},\ y = 2\cos t + 5/3 \sin t - 1/3\ \sin 2t$

Day 23

SUGGESTED EXERCISES: Sec 8.1, p460 #1, 3, 5 (a,c only)
8.1 - Backwards Euler Method

USE THIS: SHOWS EULERS APPROXIMATIONS FOR ARBITRARY h:
http://slopefield.nathangrigg.net/

How do we solve differential equations for which none of our current techniques work? How do we use computers to solve differential equations?

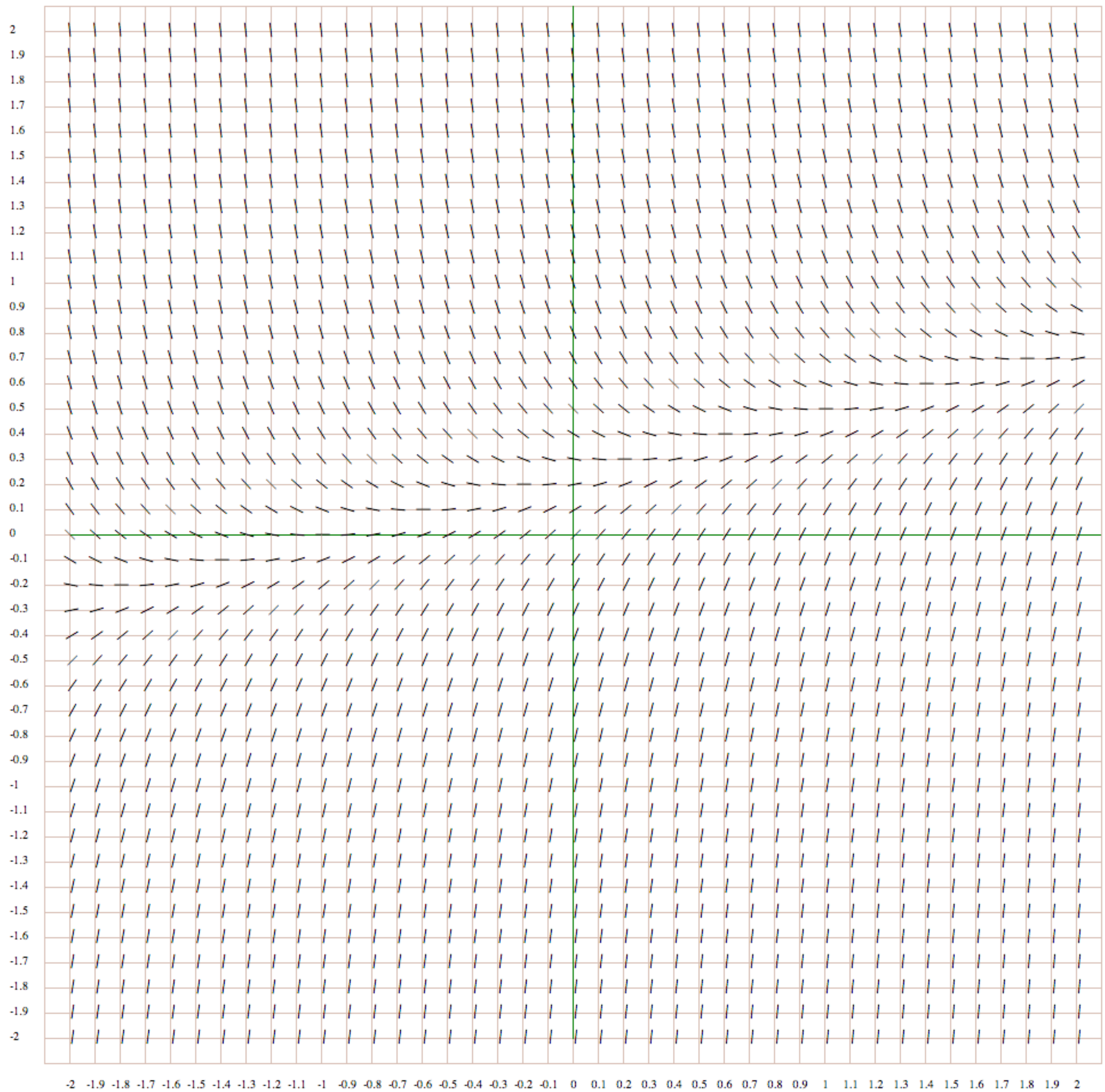
BASIC SETUP: First order linear differential equation $\frac{dy}{dt} = f(t, y)$ with initial condition $y(t_0) = y_0$.
GOAL: Find approximate value of y for a given t.
STRATEGY: Find a series of points that approximate the correct values of y: $(t_0, y_0), (t_1, y_1), (t_2, y_2), \ldots, (t_k, y_k)$, where the final point gives the approximation we desire.
NOTE: In general, we know the t-values of the points - we just space them equally apart, distance h from one to the next. Our goal is to find the y-values for each point.

RECALL EULER'S METHOD:
Given a point, how do we find the next point?
1. Start with a point (t_n, y_n).
2. Find the correct slope at that point, $f(t_n, y_n)$.
3. Determine the equation of the tangent line at the point (t_n, y_n)
4. Use the tangent line to determine the y-value of the next point (t_{n+1}, y_{n+1}).

NOTE: In general, steps 3 and 4 can be consolidated into: $y_{n+1} = y_n + hf(t_n, y_n)$

INSERT PICTURE HERE
- show an initial value problem, and slope field, with goal point.
- show "actual solution"
- discuss Euler's method, show how the solution looks (maybe very bad - oscillating etc). Error?
- show that with greatly increased step size, get a good approximation.
- show reverse Euler's method, show how the solution looks. Error?

Example 1. Given the initial value problem $\frac{dy}{dt} = 1 + t - 4y,\ y(0) = 1$, estimate the value of y at $t = 2$.
NOTE: The exact solution is $y = \frac{1}{16}(4t + 13e^{-4t} + 3)$, *which gives the actual answer y=0.6877725634*
a. Try Euler's method with *h=1*. How close are we?
b. Try again with *h=0.5*, and with *h=0.2*.



NOTE: Euler's method with h=1 gives lousy answer, y=8.
With h=0.5 answer is closer, y=1.5
With h=0.2 answer is y=0.6875

BACKWARDS EULER'S METHOD
IDEA: Start with a point (t_n, y_n). Consider the next point (t_{n+1}, y_{n+1}). In Euler's method, we write the equation of the line that passes through BOTH of these points, but has the correct slope at the FIRST POINT.
In the backwards Euler's method, we want to choose our second point so that the line passing through them has the correct slope at the SECOND POINT.

Given a point, how do we find the next point?

1. Start with a point (t_n, y_n).
2. We want to find the next point (t_{n+1}, y_{n+1}) so that (t_n, y_n) lies on the tangent line at (t_{n+1}, y_{n+1}):

   $y_{n+1} = y_n + hf(t_{n+1}, y_{n+1})$

   NOTE: Same as in Euler's method, but uses the slope at the new point instead of the old point.
3. Plug in h, y_n, t_{n+1} and solve for y_{n+1}. This gives the point (t_{n+1}, y_{n+1}).

Example 1 cont'd. Use Backwards Euler's method with *h=1* to estimate the value of y at *t=2*.

	Eulers: y1=y+h(1+t-4y)	**Backwards Euler:** y1=(y+h+t1h)/(1+4h)	**Exact:** y=1/16 (4t + 13e^-4t + 3)
t	**y**	**y**	**y**
0	1	1	1
1	-2	0.6	0.4523814566
2	8	0.72	0.6877725634

NOTE: Backwards Euler's, with h=1, gives answer already of y=0.72
With h=0.5, get y=0.6975
With h=0.2, get y=0.6897756

Example 2. Given the initial value problem $y' = t^2 - y,\ y(0) = 2$, use the Backward Euler's method with *h=0.2* to estimate the value of *y* at *t=0.6*.

EXACT ANS: $y = t^2 - 2t + 2,\ y(0.6) = 1.16$
Backwards Euler: y=1.244259259

	Eulers: y1=y+h(t^2-y)	**Backwards Euler:** y1=(y+h t1^2)/(1+h)	**Exact:** y=t^2-2t+2
t	**y**	**y**	**y**
0	2	2	2
0.2	1.6	1.673333333	1.64
0.4	1.288	1.421111111	1.36
0.6	1.0624	1.244259259	1.16

Day 24

SUGGESTED EXERCISES: Sec 8.2, p466 #1, 3, 5 (a,c only)
8.2 - Improved Euler Method

TODAY: another improvement on Euler's method.

RECALL TWO THINGS FROM CALCULUS:
1. Definite integral formula:
$\int_a^b f(x)dx = F(b) - F(a)$, where $F(x)$ is the antiderivative (integral) of $f(x)$, $F'(x) = f(x)$

Now, we can rearrange this equation as follows to get something extremely useful: $F(b) = F(a) + \int_a^b f(x)dx$

Why is this useful? Suppose I am studying a function F(x) and I know $F(a)$ (the value of the function at a point $x = a$), and I know the derivative $F'(x) = f(x)$. Using these, I can find the value of the function at another point $F(b)$.

2. How to approximate an integral $\int_a^b f(x)dx$:

Method 1. *Sketch f(x).* Simply take the width (b-a) and multiply by the height at the left endpoint $f(a)$:
$\int_a^b f(x)dx \approx (b - a) \cdot f(a)$
This gives an approximation equivalent to finding area of rectangle with height f(a).
Method 2. Better approximation. Instead of using f(a), use the average value from the two endpoints:
$\int_a^b f(x)dx \approx (b - a) \cdot \frac{f(a)+f(b)}{2}$.
This gives a better approximation - equivalent to finding area of trapezoid.

IMPROVED EULER METHOD
Let's combine these ideas to improve on Euler's method.
RECALL BASIC SETUP: First order linear differential equation $\frac{dy}{dt} = f(t, y)$ with initial condition $y(t_0) = y_0$.
GOAL: Find approximate value of y for a given t.
Need to be able to generate a sequence of points -
Suppose we know one point (t_n, y_n), need to find the next point (t_{n+1}, y_{n+1}).

First, we apply the definite integral formula -- suppose our function is $y(t)$, our derivative is $y' = f(t, y)$, our first point is (t_n, y_n) and our second point is (t_{n+1}, y_{n+1}). Substituting, we get $y_{n+1} = y_n + \int_{t_n}^{t_{n+1}} f(t, y)dt$

QUESTION: Suppose we can't carry out the integral. How do we go about approximating it? Using Method 1, we have $\int_{t_n}^{t_{n+1}} f(t, y)dt \approx (t_{n+1} - t_n) \cdot f(t_n, y_n)$. Since $t_{n+1} - t_n = h$, we have:

$$y_{n+1} = y_n + hf(t_n, y_n)$$

This is exactly our formula for Euler's method.

How can we improve this? We can use method 2 to approximate the integral instead:

$$y_{n+1} = y_n + h \cdot \frac{f(t_n, y_n) + f(t_{n+1}, y_{n+1})}{2}$$

PROBLEM: In Euler's method, we know (t_n, y_n) and we even know t_{n+1}, but we don't know y_{n+1} -- that's what we're trying to find. How to solve this?

IDEA: First, use Euler's method to get an initial approximation of y_{n+1}: " $y_n + hf(t_n, y_n)$ ". Then, use this value to calculate the slope at the right endpoint $f(t_n + h, y_n + hf(t_n, y_n))$ in the formula above.

$$y_{n+1} = y_n + h \cdot \frac{f(t_n, y_n) + f(t_{n+1}, y_n + hf(t_n, y_n))}{2}$$

IMPROVED EULER FORMULA: $y_{n+1} = y_n + \frac{f(t_n, y_n) + f(t_n + h, y_n + hf_n)}{2} h$

NOTE: This is a "two step" method -- first we calculate $y_n + hf_n$, then we use that value to plug in and calculate y_{n+1}.

IMPROVED EULER METHOD

Given a point, how do we find the next point?
Start with $(t_n,\ y_n)$ and h.
Calculate:

$$k_1 = f(t_n, y_n)$$
$$k_2 = f(t_n + h, y_n + hk_1)$$
$$y_{n+1} = y_n + h \cdot \frac{k_1 + k_2}{2}$$

Now we have (t_{n+1}, y_{n+1})

*NOTE: wolfram alpha computes the improved Euler method (Heun's method) with a weighted average, counting the slope at the initial point once and the slope at the (estimated) terminal point 3 times, something like: (fn + 3f(...))/4 *h. Other sources (e.g. wikipedia?) describe Heun's method as also referring to a variation which minimizes truncation error - they give the butcher table (which I don't follow, but which tellingly contains in the final line the fractions ¼ and ¾).*

Example 1: Given the initial value problem $\frac{dy}{dt} = \frac{(\sin t)^2}{y}$, $y(0) = 1$, use the improved Euler's method to estimate the value of y at $t = 2$ with a step size of h=1.
Example 2: Try Example 1 again with a step size of h=0.5 .

				Improved Euler's ***y1 = y + (k1+k2))/2 * h***

t	h	k1	k2	y
0				1
1	1	0	0.7080734183	1.354036709
2	1	0.5229351712	0.440508363	1.835758476

Day 25

SUGGESTED EXERCISES: Sec 8.2, p466 #1, 3, 5 (a,c only)
8.3 - Runge-Kutta Method

Like the other numerical methods, the Runge-Kutta method involves finding an approximate solution to an initial value problem by finding a sequence of points. As usual, the main idea has to do with how we move from one point to the next. This method is now called the “classic fourth order four-stage Runge-Kutta method”, but it is often referred to simply as *the* Runge-Kutta method.

NOTE: This is based on “Simpson’s Rule”, which uses a quadratic function (instead of constant or linear) to approximate the integral.

Recall that an EXACT solution to “the next point” from a given point can be given by $y_{n+1} = y_n + \int_{t_n}^{t_{n+1}} f(t, y)dt$, but the integral is generally impossible to evaluate directly. Instead, we use an approximation of the integral. In Runge-Kutta, the integral is approximated by using a *weighted average* of various values of $f(t, y)$ (recall that $y' = f(t, y)$):

$$y_{n+1} = y_n + h\left(\frac{k_1+2k_2+2k_3+k_4}{6}\right)$$

where:

$k_1 = f(t_n, y_n)$

$k_2 = f(t_n + \frac{1}{2}h,\ y_n + \frac{1}{2}hk_1)$

$k_3 = f(t_n + \frac{1}{2}h,\ y_n + \frac{1}{2}hk_2)$

$k_4 = f(t_n + h,\ y_n + hk_3)$

Example 1: Given the initial value problem $\frac{dy}{dt} = t^2 - y,\ y(0) = 1$, use the Runge-Kutta method to estimate the value of y at $t = 1.5$ with a step size of h=0.75.

Day 28

SUGGESTED EXERCISES: Sec 5.1 p253 #19-27 odd
Taylor Series - working with Sigma notation

Example 1. Find the first five terms ($n = 0 \ldots 4$) of the Taylor Series solution to the Initial Value Problem $y' - y = 0, y(0) = 1$.

Some facts about series:

1. Two series can be added if they have the same indices (even if the index variable is different):
   $\sum_{n=0}^{\infty} a_n + \sum_{m=0}^{\infty} b_m = \sum_{k=0}^{\infty} (a_k + b_k)$
2. We can use substitution to change the indices of a series:
   $\sum_{n=0}^{\infty} a_n = \sum_{m=1}^{\infty} a_{(m-1)}$ (here, the substitution used was $m = n + 1$)

Example 2. a. Combine $\sum_{n=0}^{\infty} (2n + 6)x^n + \sum_{n=0}^{\infty} (n + 5)x^n$

b. Rewrite this series with an index starting at 0: $\sum_{n=2}^{\infty} \frac{1}{n}$

c. Combine: $\sum_{n=0}^{\infty} \frac{1}{n+1} - \sum_{n=1}^{\infty} n$

Example 2. Complete the first Example again using Sigma notation.

- Based on this, what do you think the next term will be?
- What about the term with $n = 8$? or $n = 20$?
- Find a general formula for a_n.
- Find the complete Taylor series for $y(x)$

NOTE: write down a recurrence relation between terms!

Example 3. Find the Taylor series solution for $y'' - y = 0,\ y(0) = 1,\ y'(0) = 0$.

Backwards Euler's method:
Good explanation, but contains applets that won't run @ citytech:
https://ece.uwaterloo.ca/~dwharder/NumericalAnalysis/14IVPs/stiff/complete.html

Plotting slope fields in Sage:
http://www.sagemath.org/doc/reference/plotting/sage/plot/plot_field.html

LAPLACE TRANSFORMS.

- I like this guy's style - here's his playlist on Laplace Transform:
  https://www.youtube.com/playlist?list=PL5750E3CE53DB625A

  video: The Laplace Transform - The Basic Idea of How We Use It (1:32)
  https://www.youtube.com/watch?v=Z_wQvCyKjwE
  Short, sweet, concise - lays out basic process of solving ODEs using Laplace transform

- videos: Finding Laplace Transform using a table: http://youtu.be/ES2Lwzrw_UE
  Table of Laplace Transforms: http://youtu.be/DXK0kbrUujM

- Finding Inverse Laplace Transforms: https://www.youtube.com/watch?v=Y8GXpS31CGI
  (NOTE: examples start after 90 seconds of discussion)

- Two part example showing all basic steps of solving an ODE using Laplace transform
  Ex: $y'' - 2y' + y = 3e^{t}$, $y(0) = 1$, $y'(0) = 1$
  part 1: http://youtu.be/kbp9qWS-Bsk
  part 2: http://youtu.be/8Pgd-D00H0U

- video: Khan Academy - Using the Laplace Transform to solve a nonhomogenous eq
  Ex: $y'' + y = \sin 2t$, $y(0) = 2$, $y'(0) = 1$
  https://www.youtube.com/watch?v=nlUG4OKN1IY

- video: Khan Academy - Using the Laplace Transform to solve a homogeneous differential equation
  $y'' + 5y' + 6y = 0$, $y(0) = 2$, $y'(0) = 3$
  Part 1: https://www.youtube.com/watch?v=3uYb-RhM7lU
  Part 2: http://youtu.be/EdQ7Q9VoF44

- video: Lec 19 | MIT 18.03 Differential Equations, Spring 2006 (47 min)
  https://www.youtube.com/watch?v=sZ2qulI6GEk
  Introduces Laplace transform, gives general idea (continuous analogue of Power Series), shows basic rules, calculates 1-2 simple Laplace transforms. Assumes familiarity w/ Power Series & some other topics
  NOTE: accompanying video from recitation session, asks you to give defn, compute some basic Laplace transforms, etc --
  https://www.youtube.com/watch?v=BniJM-ireXQ
- video: Lec 20 | MIT 18.03 Differential Equations, Spring 2006 (51 min)
  https://www.youtube.com/watch?v=qZHseRxAWZ8
  "exponential type". Using Laplace Transforms to solve linear ODEs

PARTIAL FRACTIONS

- basic example w/ some intro material: https://www.youtube.com/watch?v=HZTv4zCgEnA
- partial fractions - includes a breakdown of steps, as well as forms for various types of factors in denom, with examples: https://www.youtube.com/watch?v=pZ9FfGy3Cfw

WHAT HAVE I LEARNED ABOUT LAPLACE TRANSFORM:

- It's the continuous analogue of Power Series
- Only really apply it to exponential functions, trig functions, polynomials
- It converts a differential equation to an algebraic equation. After solving, use inverse Laplace transform to convert back & obtain solution.
- Inverse Laplace NEEDS PARTIAL FRACTIONS - use it to simplify expression so inverse Laplace can be applied

RESOURCES:
Ezra's 2680 OpenLab site (Fall 2013):
http://openlab.citytech.cuny.edu/halleckmat2680fa2013/exams/

Vika's 2680 site:
http://websupport1.citytech.cuny.edu/faculty/vgitman/diffeqs.html

Suggestions of intro/overview material
http://math.stackexchange.com/questions/107264/essay-about-the-art-and-applications-of-differential-equations

Sage math resources:
PDF text on teaching Intro Diffy Qs using Sage (freely available online):
sage.math.washington.edu/home/wdj/teaching/DiffyQ/des-book.pdf

Sage reference page for ODEs:
http://www.sagemath.org/doc/reference/calculus/sage/calculus/desolvers.html

NOTES FROM EZRA ON TEACHING 2680 (from Dec 2013):
Hi Henry, Nadia, Jonas and Neil,

I just wanted to give a small report on my experience teaching 2680 this semester. This is the first time I have used Boyce and DiPrima to teach, although I did use a much earlier version of it when I took a Junior-level ODE course almost 30 years ago.

I started with 31 students and finished with 12 passing, with the grade distribution uniform among A-D.

I followed the syllabus more or less. I added a few topics, particularly forced vibrations 3.8 and step functions and discontinuous forcing functions 6.3-4 and **subtracted a few, particularly convolution 6.6 and the last 2 numerical topics runge-kutta and multipstep 8.3-4.**

I had planned to get to Runge-Kutta 8.3, but found the students struggling enough with improved Euler. And actually, this is better than most instructors are doing. Both Smith and Parker say they did very little with the numerical beyond Euler. I am not sure, but when I asked a student taking the course in the evening with an adjunct, they were quite behind and I doubt they did much of the numerical.

My approach was to give a quiz at the beginning of class, based on the homework from the class before, but found I had to spend quite a bit of time reviewing before the quiz to have any chance of success among the students. For presentation, I relied on slides which I prepared by modifying the ones provided by the publisher.
For my slides as well as other aspects of the class, go to my openlab site:
http://openlab.citytech.cuny.edu/halleckmat2680fa2013/

I think the slides benefited the better students in the class. I imagine that **the weaker students, since they struggled with the calculus and algebra,** would have benefited more from chalk talks, with less theory and more problems.

I asked students to do the webwork problem sets that Smith and Parker put together as well as to submit written homework, including printouts of all the problems which asked for technology, which are considerable. **The students from CET and ETET knew MATLAB. Some of the other students were familiar with MAPLE.** I also tried to make use of graphing calculators and had some success with Euler using sequence mode, although it was not straightforward. For improved Euler, I am not sure if sequence mode works, although the calculators can be programmed, which is what I did as student.

I want to point out that I have a definite bias, which is to get students to engage with qualitative aspects. I emphasized these on exams and if I were to create a syllabus, would downplay the analytic techniques, especially the myriad of first order solution techniques. I guess for students who have taken-are taking-will take Calculus III, we should leave in Exact and of course, you shouldn't pass a first course on ODE's without mastering 1st order linear and separable, but Bernoulli and homogeneous, how important are they? I think students would benefit more if we did a bit with systems. Of course having a prerequisite of linear algebra would be great, but if we could at least do parts of the first few sections of chapter 7 or something similar to the first 20 or so pages of
http://www.math.uconn.edu/~mckenna/notes.pdf

I will look at the free text being made available over the next few weeks to see if would be as good a fit as to our needs as Boyce-DiPrima. How many of the students were unable to finish because they did not purchase the text? Who knows, maybe a couple, but the main problem is the students preparation. Many of the students who did not finish could not remember their calculus. Some even had basic algebra issues. What Andrew Parker told me was that his classroom in spring 13 did not have another class after his, so he offered to work with students after class, in particular on calculus. This way, he was able to keep several students in the class who otherwise would have dropped out.

Best,
Ezra