Mathematics Framework
Chapter 2: Teaching for Equity and Engagement

First Field Review Draft

Note to reader: The use of the non-binary, singular pronouns *they*, *them*, *their*, *theirs*, *themself*, and *themselves* in this framework is intentional.

Equity and Engagement: An Introduction

In California, all teachers strive to ensure every child has an equal opportunity to succeed. Teachers of mathematics can provide equitable education by making sure all students receive the attention, respect, and resources they need to achieve their potential. California classrooms combine diverse communities and students who bring a rich variety of cultural and linguistic resources that teachers can draw on to create culturally-relevant lessons (Ladson-Billings, 2009; Hammond, 2020; Milner, 2011). Cultural relevance is important for learning and also for expanding a collective sense of what mathematical communities look and sound like to reflect California's diverse history. A focus on equity recognizes that mathematics, over the years, has developed in a way that has excluded many students (see Chapter 1). Because of these inequities, teachers need to work consciously to counter racialized or gendered ideas about mathematics achievement (Larnell, Bullock, & Jett, 2016). It is common for people to claim that avoiding aspects of race, culture, gender, or other characteristics as they teach mathematics, means they are being equitable; but the evolution of mathematics in educational settings has resulted in dramatic inequities for students of color, girls, and students from low income homes (Joseph, Hailu & Boston, 2017; Milner & Laughter, 2015). These inequalities include not only access to high-quality curriculum and resources, but also to instruction that appropriately leverages students’ diverse knowledge bases, identities, and experiences for both learning and developing a sense of belonging to mathematics (Langer-Osuna & Esmonde, 2017). A “color-blind” approach allows such systemic inequities to continue (Battey, 2013; Martin, 2007). The examples that follow are provided to help educators utilize and value students’ identities, assets, and cultural resources to support learning and ensure access to high achievement for all students in California—particularly English learners, who are linguistically and culturally diverse, and those who have been disenfranchised by systemic inequities. (See also https://www.cde.ca.gov/pd/ee/assetbasedpedagogies.asp.)

> “Creating, supporting, and sustaining a culture of access and equity require being responsive to students' backgrounds, experiences, cultural perspectives, traditions, and knowledge when designing and implementing a mathematics program and assessing its effectiveness. Acknowledging and addressing factors that contribute to differential outcomes among groups of students are critical to ensuring that all students routinely have opportunities to experience high-quality mathematics instruction, learn challenging mathematics content, and receive the support necessary to be successful. Addressing equity and access includes both ensuring that all students attain mathematics proficiency and increasing the numbers of students from racial, ethnic, linguistic, gender, and socioeconomic groups who attain the highest levels of mathematics achievement.”
>
> –NCTM Position Statement, Access and Equity in Mathematics Education

Students learn best through active engagement with mathematics and one another (Freeman, et al, 2014). As such, this framework highlights the importance of students’ active engagement in classrooms. When teachers launch investigations into relevant content with the Drivers of Investigation (DI) identified in this framework, they elicit students' curiosity and provide motivation for them to engage deeply with authentic mathematics. The framework suggests the following Drivers of Investigation:

1) Make sense of the world
2) Predict what could happen
3) Impact the future

Research conducting in preceding decades has produced a wealth of information showing that the highest mathematics achievement, understanding, and enjoyment comes when students are actively engaged—when they are developing mathematical curiosity, asking their own questions, reasoning with others, and encountering mathematical ideas in multi-dimensional ways. This can occur through numbers, but also through visuals, words, movement, and objects, considering the connections between them (Boaler, 2016, 2019; Cabana, Shreve & Woodbury, 2014; Louie, 2017; Hand, 2014; Schoenfeld, 2002). The principles of the Universal Design for Learning (UDL) guidelines outline a multi-dimensional guide that benefits all students, and can be particularly useful when applied to mathematics. The San Francisco Unified School District illustrates its approach to UDL for teachers, providing advice and guidance (http://www.sfusdmath.org/universal-design-for-learning.html). When students are engaged in all these kinds of experiences, they can come to view mathematics, and their own relationship to mathematics far more positively. The contrasting approach—of students sitting in rows watching a teacher demonstrate methods before reproducing them in short exercise questions through passive student engagement—has led to widespread mathematical disinterest, perpetuating the students perspective that mathematics is merely a sterile set of rules.

Students must view mathematics as a vibrant, inter-connected, beautiful, relevant, and creative set of ideas. As educators increase access for students to engage with and thrive in mathematics and value the different ways questions and problems can be approached and learned, many more students view themselves as belonging to the mathematics community (Boaler & Staples, 2008; Boaler, 2016; Langer-Osuna, 2015;

Walton et al, 2012). This community serves a diversity of learners more equitably, prepares students to think mathematically in their everyday lives, and helps our society develop many more students interested and excited by Science, Technology, Engineering, Arts, and Mathematics (STEAM) pathways. In this expanded view of mathematics, teachers have space to analyze lessons through an equity lens, considering ways to draw upon the rich cultural and linguistic resources of their students (Nasir, et al, 2014; Fernandez et al, 2017; Louie, 2017). The following sections draw from California's English Language Development Standards (ELD Standards) (https://www.cde.ca.gov/sp/el/er/documents/eldstndspublication14.pdf), the California Department of Education advice for integrating the ELD Standards into mathematics teaching (https://www.cde.ca.gov/re/cc/eldresources.asp) and the principles of UDL (http://www.sfusdmath.org/universal-design-for-learning.html).

California's diverse student population bring with it linguistic diversity. Various supports exist to ensure that the state's large population of language learners and multilingual students can learn and thrive. These supports reflect important recommendations for supporting students learning English—for example Moscovitch, (2014), Lagunoff et al. (2015), and Turner et al. (2013). These recommendations, that focus on different ways of giving students access to meaningful mathematics, are helpful for all students. A framework outlined by Darling (2019) seems particularly important in encouraging linguistically and culturally diverse language learners, as well as other students:

1. **Take an asset approach and recognize multilingualism as a power**
2. **Include group work** (strategically grouping for language development)
3. **Make work visual** (include graphic organizers, visual examples, encourage visual communication)
4. **Build on students' lived experiences and cultures** (allow native and home language use)
5. **Scaffold learning and language development** (scaffold with sentence frames and/or sentence starters)

6. **Give opportunities for pre-learning** (opportunities to learn prerequisite material ahead of time)

Darling (2019). See also https://ell.stanford.edu/publication/mathematics-common-core-and-language.

A recent framework proposed for equitable teaching in California, that addresses barriers to mathematics equity, can be accessed at https://equitablemath.org/.

Five Components of Equitable and Engaging Teaching

How does a teacher create an equitable and engaging mathematics environment that supports all students? The following sections describe five important components that based on research and supported by practice.

1. Plan Teaching Around Big Ideas

Mathematics is a subject made up of important ideas and connections. Curriculum standards tend to divide the subject into smaller topics, but it is important for teachers and students to think about the big ideas that characterize mathematics at their grade level and the connections between them. Instead of planning teaching around the small topics or methods set out in the standards, or the chapters of textbooks, teachers can plan to teach the "big ideas" of mathematics (Nasir et al, 2014). Lessons designed around big ideas facilitate the linking of one or more Content Connections with Standards for Mathematical Practice, and with one of the Drivers of Investigation, as described in Chapter 1. (See for example big ideas across grades K–8: https://www.youcubed.org/wp-content/uploads/2017/11/Big-Ideas-paper-12.17.pdf.)

It is helpful if mathematics teachers are given release time in which they can sit with colleagues and work out the big ideas in their grade level or course, then choose rich, deep tasks that invite students to explore and grapple with those big ideas (c.f., Arbaugh, & Brown, 2005). The cluster headings that organize the standards in the Common Core State Standards for Mathematics (CA CCSSM) also give a broader view of mathematical ideas than the detail of individual standards and can usefully form a guide for such discussions and choosing of tasks. These tasks can then form the basis of a course and, if the tasks are rich enough, they likely include many of the smaller methods and ideas set out in the standards. Rather than preparing a set of many problems to work through in a lesson, one rich task may be planned as the basis for an entire lesson, or at times worked on through several days of exploration, sense-making, and discussion. More detail is given on these kinds of tasks in section two.

There are times when teachers will share ways to approach a mathematics problem or discuss with students new methods to learn important mathematical concepts. Important research studies have considered the best times to do such direct instruction (Schwartz & Bransford, 1998; Deslauriers et al., 2019). The studies contrasted the approach used in many classrooms—a practice of teaching students the methods and then providing opportunities to practice those methods—with a different approach, one where teachers introduced questions first, then allowed students time to use intuition in considering ways they may solve the questions. In these studies, teachers taught new methods to students when they needed them to solve problems. The students who learned through this approach achieved at significantly higher levels, leading the researchers to conclude that their understanding came because they were primed to learn the methods—methods they knew were required to solve the problems—and so they were engaged and interested. Students also engaged in struggle, which is the most productive part of learning (see also Boaler, 2019). When students learn methods before they use them, they might ask the legitimate question, *When will I ever need this?* The vignette below describes a high school classroom in which the teacher taught mathematical methods when students needed them to solve the problem. In working on this task, the students received opportunities to learn.

Vignette–36 Fences

Lori, a high school geometry teacher, introduces a problem to students. Lori explains that a farmer has 36 individual fences, each measuring one meter in length, and that the farmer wants to put them together to make the biggest possible area. Lori takes time to ask her students about their knowledge of farming, making reference to California’s role in the production of fruit, vegetables, and livestock. The students engage in an animated discussion about farms and the reasons a farmer may want a fenced area. While some of Lori’s long-term English learners show fluency with social/conversational English, she knows some will be challenged by forthcoming disciplinary literacy tasks. To support meaningful engagement in increasingly rigorous course work, she ensures images of all regular and irregular shapes are posted and labeled on the board, along with an optional sentence frame, “*The fence should be arranged in a* [blank] *shape because* [blank]*.”* These support instruction when Lori asks students what shapes they think the fences could be arranged to form. Students suggest a rectangle, triangle, or square. With each response, Lori reinforces the word with the shape by pointing at the image of the shapes. When she asks, “How about a pentagon?” she reminds students of the optional sentence frame as they craft their response. Lori asks, the students think about this and talk about it as mathematicians. Lori asked them whether they want to make irregular shapes allowable or not.

After some discussion, Lori asks the students to think about the biggest possible area that the fences can make. Some students begin by investigating different sizes of rectangles and squares, some plot graphs to investigate how areas change with different side lengths.

Susan works alone, investigating hexagons—she works out the area of a regular hexagon by dividing it into six triangles and she has drawn one of the triangles separately. She tells Lori that she knew that the angle at the top of each triangle must be 60 degrees, so she could draw the triangles exactly to scale using compasses and find the area by measuring the height.

Niko has found that the biggest area for a rectangle with perimeter 36 is a 9 x 9 square—which gave him the idea that shapes with equal sides may give bigger areas and he started to think about equilateral triangles. Niko was about to draw an equilateral triangle when he was distracted by Jaden who told him to forget triangles, he had found that the shape with the largest area made of 36 fences was a 36-sided shape. Jaden suggested to Niko that he find the area of a 36-sided shape too and he leant across the table excitedly, explaining how to do this. He explained that you divide the 36-sided shape into triangles and all of the triangles must have a one-meter base, Niko joined in saying, “Yes, and their angles must be 10 degrees!” Jaden said, “Yes, and to work it out we need tangent ratios which Lori has just explained to me.”

Jaden and Niko move closer together, incorporating ideas from trigonometry, to calculate the area.

As the class progressed many students started using trigonometry, some students were shown the ideas by Lori, some by other students. The students were excited to learn about trig ratios as they enabled them to go further in their investigations, they made sense to them in the context of a real problem, and the methods were useful to them. In later activities the students revisited their knowledge of trigonometry and used them to solve other problems.

Opportunities for learning – California Mathematics Standards

G-SRT.B.4: Prove theorems about triangles.

G-SRT.B.5: Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.

G-CO.D.12: Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.).

G-CO.D.13: Construct an equilateral triangle, a square, and a regular hexagon inscribed in a circle.

G-SRT.B.5: Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.

G-SRT.C.6: Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles.

G-SRT.C.8: Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems.

G-MG.A.3: Apply geometric methods to solve design problems

G-SRT.C.8.1: Derive and use trig ratios for special triangles

This, and similar tasks, can be found at www.youcubed.org/tasks/.

2. Use Open, Engaging Tasks

What are Open Tasks?

Open tasks are those that enable students to take ideas to different levels (Vale, et al, 2012). When tasks have a low floor and a high ceiling, it means that any student can access the task but the task extends to high levels (Boaler, 2016; Krainer, 1993). When questions are narrow and focused, only some students are cognitively challenged at an appropriate level, and the questions are often not very interesting. When tasks are open, they allow all students to work at levels that are appropriately challenging for them, within the content in their grade. Two examples of such tasks are given below; one based in a real-world context, and one that encourages exploration of mathematical ideas through numerical patterns. A list of websites that provides additional tasks can be found at the end of the chapter, before the references.

Why Use Open Tasks?

Open tasks invite students to engage in multi-dimensional ways (Vale et al., 2012). They provide an ideal opportunity to integrate the important principles of UDL which recognize the strengths of learner variability and student diversity (see http://udlguidelines.cast.org/?utm_source=castsite&lutm_medium=web&utm_campaign=none&utm_content=aboutudl). When students work on open tasks they take advantage of opportunities to engage in different ways, using multiple ways of representing mathematical ideas, and expressing understanding (see also Lambert, 2020). Open tasks provide teachers with opportunity to listen carefully, to make sense of student thinking, and to assess formatively as the lesson progresses. See Chapter 11 for further discussion of how the use of open tasks enables teachers to gather important information about students' learning.

Real-world tasks can offer students opportunities to mathematize contexts that connect to their lived experiences. When teachers get to know their students—not only how they think about mathematics, but also their curiosities, interests, and experiences—they are better able to choose, craft, and launch tasks that engage students with big ideas in meaningful and relevant ways. This is particularly effective when teachers find out about and bring into mathematics the culture of their students, engaging in culturally relevant pedagogy (Ladson-Billings, 2009; Hammond, 2020). Data Science tasks, such as those we highlight in the Data Science chapter and in the chapter for grades 6–8, are naturally open, and provide many opportunities for students to connect mathematics to their lives. Students can, for example, design wheelchair ramps, plan a new school garden, or survey peers to find out how they have been impacted by distance learning, drawing from their own knowledge and interests as they learn new mathematics. These tasks that draw from students' lives are very different from the imagined contexts that often fill textbooks and present mathematics in ways that students may find irrelevant and "other worldly" (Boaler, 2016). With carefully chosen projects students can learn to address the inequities they experience, learning mathematical tools that allow them to highlight inequities and plan new ways forward (see also section 3, Teach Toward Justice, below, and Gutstein [2003, 2006]; Berry et al. [2020]).

Neuroscience research has shown that the highest achieving people have more interconnected brains—with different brain pathways communicating with each other (Menon, 2015; Kalb, 2017). This develops in mathematics students when they encounter mathematical ideas in different ways. In one example, Park and Brannon

(2013) found that when students worked with numbers and also saw the numbers as visual objects, brain communication was enhanced and student achievement increased. A range of different research studies throughout K–12 have shown the importance of visual thinking in mathematics (West, 2004; Alibali, & Nathan, 2012; Boaler, et al., 2016; Boaler, 2019). Researchers even found that after four 15-minute sessions of playing a game with a number line, differences in knowledge between students from low-income backgrounds and those from middle-income backgrounds were eliminated (Siegler & Ramani, 2008). All mathematical ideas can be considered in different ways—visually, through touch or movement, through building, modeling, writing and words, through apps, games and other digital interfaces, as well as through numbers and algorithms. Fingers have been shown to be particularly important as a visual and physical representation for students, enabling the development of important brain areas (Boaler et al., 2016). The tasks we use in classrooms should encourage multi-dimensional forms of engagement. Tasks that offer multiple ways to engage with and represent mathematical ideas also support students with learning differences (Lambert & Sugita, 2016). The UDL guidelines can support students with learning differences because they are designed to support learning for all (see, for example, http://udlguidelines.cast.org/?utm_source=castsite&lutm_medium=web&utm_campaign=none&utm_content=aboutudl).

When students have gaps in understanding or unfinished learning from previous grades, teachers must provide support without making premature determinations about the student's abilities—that they are a low achiever, require intervention, or need to be placed in a group learning different grade-level standards. Learning develops at different times and at different rates, and what educators perceive as an apparent lack of understanding in a student may not indicate an actual absence of knowledge. Too often, students are segregated according to perceived ability in hopes of providing environments where teachers can accommodate students' talents and struggles through differentiation. This approach exacerbates inequities both within the classroom through ability grouping (students sitting in math or reading groups for example [Sparks, 2018] and across classrooms through tracking [Cohen & Lotan,1997; Schmidt, 2009; Boaler, 2016]). Ability grouping has been shown as unnecessary, particularly when math instruction is designed to offer open tasks students can engage with at different levels and feel supported in appropriate ways as they work (see, for example, Boaler, 2019). These environments create discussions enriched by the range of strategies and perspectives that students offer. For example, students benefit from discussing connections between direct modeling and more abstract reasoning strategies. By focusing on inclusive approaches to teaching, progress, not perfection, is the goal. One such type of approach is aligned with the principles of UDL, a framework for inclusive teaching. The principles, and their associated guidelines, are presented at the end of this chapter before the References section.

Open, multi-dimensional tasks upend the conventional arguments for tracking; rich classroom discussions at both the whole-class and small-group levels rely on the different strategies students bring and the ensuing approaches they take to articulate their thinking. This approach not only supports learning, it serves to position students across a range of backgrounds as mathematical thinkers. Open, multi-dimensional tasks offer authentic opportunities for all students to contribute their unique perspectives. This start can engage all students and draw them into mathematical conversations on an equal footing (Boaler, 2019b). When students begin appropriately-structured group work or engage in classroom discussions of a mathematics problem or situation, they feel supported by an activity designed to use the different ways they see or think about the issue. And with appropriate language development scaffolds and strategies, all students can contribute in shared participation (see, for example, https://www.youcubed.org/evidence/our-teaching-approach/). Group work and mathematical discussions are productive when students share the intellectual work (Langer-Osuna, 2016; Langer-Osuna et al., 2020).

Using open tasks represents a shift in instructional practice, away from a traditional "lecture" approach. Professional learning that provides teachers the opportunity to investigate important mathematics through rich, authentic, culturally relevant tasks themselves will support their continued growth and ability to enrich their students' learning experiences. Such professional learning, combining pedagogical knowledge with mathematics content, is an essential element in eliminating the inequities inherent in tracking students, and should be an ongoing part of a district's plan.

Teaching with Open Tasks

Open, multi-dimensional tasks invite student discussion (Stein & Smith, 2011). Students are likely to see the problems differently, and therefore respond with different assets in discussing problem approaches and solutions. This diversity in approach and thinking is generative because students are given the opportunity to make sense of mathematical ideas from multiple perspectives, which supports conceptual understanding and strategic reasoning (National Research Council, 2001; Stein & Smith, 2011).

Classrooms that use open tasks and encourage mathematical reasoning often have a similar structure:

- The teacher launches a problem (or problem context).
- Students work through the problem in peer partnerships or small groups.
- The class gathers for whole-class discussion based on students' solutions and reflection (Smith & Stein, 2011).

Planning to teach in this way means educators must attend to the ways they can support—rather than control—student thinking. Smith and Stein's text, *5 Practices for Orchestrating Productive Mathematical Discussions* (2011), offers a useful approach to planning and implementing such tasks effectively. Chapin, O'Connor, and Anderson's book, *Classroom Discussions* (2013), also offers useful tools for facilitating productive classroom discourse. Useful classroom activities are also illustrated in the following vignettes about productive partnerships and peer re-voicing.

Anticipating the strategies students might use and the challenges and confusions students are likely to encounter allows the teacher to strategically plan questions before lessons begin (Smith & Stein, 2011). During planning, teachers should understand not only the needs of English learners, but also their myriad assets, such as their linguistical and cultural diversity, and design instruction that is universal and accessible to all. In the recommendations below, developed for the teaching of addition and reasoning (Lagunoff et al., 2015), the language specialists suggest paying careful attention to the terminology.

When students participate as audience members for classmates' presentations and explanations of the models and strategies that they used—when they observe others describing their reasoning—students ultimately determine whether or not the explanations clearly describe the learning and respond with ways the explanations could have been improved. Also, through limited prompting and strategic support from the teacher, students determine whether their peers have used correct terminology (e.g., add, subtract, one-digit, two-digit) when describing their processes. To support students at the **Emerging** level of English proficiency, the teacher provides more substantial support. For example, she ensures that students understand the specific term under discussion (e.g., one-digit, two-digit) and asks a direct question such as, "Mary said this is a two-digit number" as she points to a number. "Is this a two-digit number?" (Lagunoff et al., 2015).

Teachers are encouraged to align instruction with the outcomes of the California ELD Standards, which state that linguistically and culturally diverse English learners receive instruction that values their home cultures. This instruction recognizes students' primary language as an *asset* and draws on them to build new learning. It views language as a resource rather than a deficit, and treats students' every-day and home languages as linguistic resources to engage students in mathematics (Moschkovich, 2013). To do so, Moschkovich (1999) suggests that teachers listen for the mathematical ideas being expressed by students, noticing how students might draw on multiple language bases (i.e., translanguaging), extra-linguistic communication such as gesturing and using representation. Teachers could then re-voice students' ideas (for example, a teacher might say, “So I hear you say that this shape is not a triangle because it has four sides and triangles only have three sides. Is that right?”) to both check their understanding of students’ expressed ideas and to also offer the idea back to the student with potentially clearer mathematical language.

The work of anticipation starts with working through the day’s task as part of planning, as well as thinking about one’s individual students and what is known about them. If teachers listen closely to students’ thinking during the lesson by using classroom discourse as formative assessment (Cirillo & Langer-Osuna, 2018), they can make use of the questions they have prepared strategically and responsively, and support the English learners as they learn the content. This formative assessment—the in-the-moment work of teaching—is some of the most important (Munson, 2018). Despite careful pre-planning, however, surprises still arise during lessons. Teachers need to be flexible, improvising additional questions and prompts that might support emerging understanding and enable students to communicate the mathematics more coherently. To do this kind of work effectively, teachers need to feel supported when feeling vulnerable or uncertain; they need resources and time to do their best, then reflect on what worked and what did not in order determine next steps. Through this, they grow professionally as student-centered, responsive educators. Expert guidance from a mathematics coach or involvement in a long-term professional learning program can support teachers as they develop their capacity to leverage students’ responses to maximize learning.

Classrooms should include, at times, the use of real-world data. It should be rooted in contexts students can engage with as a way to understand mathematics as an important tool for participating meaningfully in their community. Mathematics is a quantitative lens through which to view the patterns that exist throughout the world. When grappling with the data, students can pose questions about issues that matter to them, drawing upon content from relevant issues like cyber bullying, neighborhood resources, or water quality. Data related to these and other issues can draw from not only a range of mathematical ideas and curiosities from students, but from a range of feelings about relevant, complex social issues. Trauma-informed pedagogy in mathematics highlights the importance of allowing students to identify and express their feelings as part of mathematics sense-making, and to allow students to address what they learn about their world by suggesting recommendations and taking action (Kokka, 2019). However, not all mathematics problems need to be related to the world—students can be fully engaged exploring pure number patterns, for example. In the following examples, two problems are highlighted; one is purely numerical, and one draws from real-world data.

Number talks are a pedagogical practice that involve discussing numbers in ways that “open” these kinds of problems and expand the ways students encounter them. For example, a student can work on a question such as 18 x 5 in a textbook question, or in response to a teacher question, with the expectation that one answer is the goal. Alternatively, teachers can “open” the problem by presenting it as a number talk. In a number talk, teachers ask the class of students to work out the answer to 18 x 5, mentally and then share with the teacher when they have a solution, using a “quiet thumb.” Teachers then ask the class for the different answers that students may have found, and write them on the board. After the different answers are collected teachers can ask if anyone would like to defend their answer. Ideally, different students will share different ways of thinking about the problem, with visual, as well as numerical solutions.

For example:

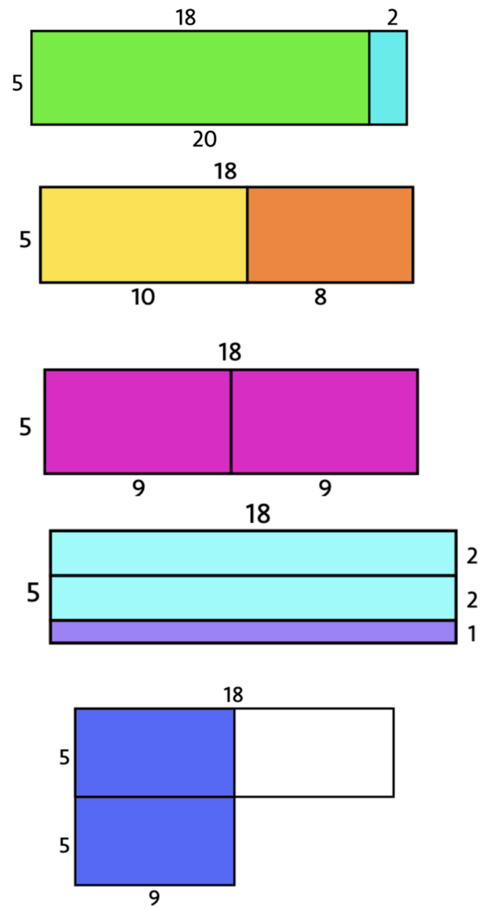
20 x 5 = 100
2 x 5 = 10
100-10 = 90

10 x 5 = 50
8 x 5 = 40
50 +40 = 90

9 x 5 = 45
9 x 5 = 45
45 + 45 = 90

18 x 2 = 36
18 x 2 = 36
18 x 1 = 18
36 + 36 + 18 = 90

18 x 5 = 9 x 10



Number talks were created by Ruth Parker and Kathy Richardson, and have been developed in several books and video and online resources, given below. Any number problem can be used with students across K–12. When students become familiar with different mathematical strategies, visuals, and approaches, they feel more prepared to engage in open tasks.

Resources for Teaching “Number Talks:”

Humphreys, C & Parker, R. (2015) Making Number Talks Matter: Developing Mathematical Practices and Deepening Understanding, Grades 3–10. Stenhouse.

Humphreys, C & Parker, R. (2018) Digging Deeper: Making Number Talks Matter Even More, Grades 3–10. Stenhouse.

Parish, S. (2014). Number Talks: Helping Children Build Mental Math and Computation Strategies, Grades K–5, Updated with Common Core Connections. Math Solutions.

Films and other resources: https://www.youcubed.org/resource/number-sense/.

Two Examples of Open Tasks

Example 1 Four 4s. Can you find every number between 1 and 20 using exactly four 4s and any operation?

Source: https://www.youcubed.org/tasks/the-four-4s/

Opportunities for Mathematics Content Learning	Opportunities for Mathematics Practices Learning	Opportunities for Language Development and Teacher Actions
Order of operations: 3. NBT Number and operations in Base Ten • Use place value understanding and properties of operations to perform multi-digit arithmetic. 6. EE Expressions and Equations • Apply and extend previous understandings of arithmetic to algebraic expressions o Write and evaluate numerical expressions involving whole-number exponents.	MP 1) Make sense of problems & persevere in solving them MP 2) Reason abstractly and quantitatively MP 3) Construct viable arguments & critique the reasoning of others	ELD PI.A - allow time for struggle; ask: • How could you get started on this problem? • What does it mean that “any operation” is allowed? • What does this symbol (parentheses, equal sign, fraction bar) mean to you?

(continued) 3. NF Number and Operations – Fractions • Develop understanding of fractions as numbers 4. NF Number and Operations – Fractions • Build fractions from unit fractions by applying and extending previous understandings of operations on whole numbers. 5. NF Number and Operations – Fractions • Use equivalent fractions as a strategy to add and subtract fractions. • Apply and extend previous understandings of multiplication and division to multiply and divide fractions. 1. OA Operations and Algebraic Thinking • Addition and Subtraction 3. OA Operations and Algebraic Thinking • Represent and solve problems involving multiplication and division.	(continued)	(continued) • What math language will help you prove your answer? ELD P2.C - allow time for rehearsal of response; ask: • How are those two examples connected? • How could that be written as one equation? ELD P1.A (above) ELD P1.C - encourage practicing language with partner; ask: • Which words on the word wall could help express this? • How else could we explain this answer?

(continued) • Understand properties of multiplication and the relationship between multiplication and division. 8. EE Expressions and Equations • Work with radicals and integer exponents. 3. OA Operations and Algebraic Thinking Number sense • Solve problems involving the four operations, and identify and explain patterns in arithmetic.	(continued)	(continued)

Example 2

Who attends your school? Which racial and gender groups are represented? And how does your school data compare to state or national data?

Opportunities for Mathematics Content Learning	Opportunities for Mathematics Practice Learning	Opportunities for Language Development
Construct a survey 7.SP Statistics and Probability Use random sampling to draw inferences about a population **Measures of spread** 6.SP Statistics and Probability Develop understanding of statistical variability **Collect and analyze data** 8.SP Statistics and Probability	1) Make sense of problems and persevere 4) Model with mathematics 5) Use appropriate tools strategically	ELD PI.A ELD PI.B ELD P2.B

Investigate patterns of association in bivariate data **Use a spreadsheet:** SMP.5 Alg. II/ Mathematics III: Statistics and Probability: S-ID: Interpreting Categorical and Quantitative Data **Ratios** 6.RP Ratios and Proportional Relationships Understand ratio concepts and use ratio reasoning to solve problems. **Number sense** 6.NS Number System Apply and extend previous understandings of numbers to the system of rational numbers		

Launching Open Tasks

To successfully launch tasks, teachers should discuss key contextual features and mathematical ideas, soliciting ideas from students to create shared language for anything that might be unfamiliar or confusing without reducing the cognitive demand of the task (see Lagunoff et al., example above). Whole-class discussions during the launch are also important opportunities to support students in learning how to effectively and inclusively share ideas during small group work. The vignette below describes an example of such a discussion in a fourth-grade classroom:

Vignette – Productive Partnerships (Langer-Osuna, Trinkle, & Kwon, 2019)

Tracy, a fourth-grade teacher, joins her students at the carpet in the front of the room to launch the day's lesson on place value. One of the first lessons of the year, she introduces the idea of "productive partnerships" with students before releasing them into small group work. When productive partnerships are the norm in a classroom, students engage in and strengthen their capacity for several mathematical practices, particularly SMPs 1, 3, 5, and 6, all of which involve reasoning, representing mathematical ideas, and communicating. Tracy has planned her lesson carefully, making it accessible for her students with learning differences, as well as language learners. Tracy knows that all of her students can better engage if she aligns her expectations with the principles of UDL, particularly encouraging students to represent their ideas in multiple ways—visually, numerically, and physically (See https://www.udll.com/media-room/articles/the-seven-principles-of-universal-design/).

Tracy begins by asking students what it means to be productive. Students talk with a partner and offer different perspectives and ideas to the whole class. She then calls on a student volunteer to pretend to be her partner and act out what the class suggests they try to work "productively" as a partnership.

T: How can we show that we are ready to work with our partners?

S: Sit!

T: We should sit? Ok, let's sit. How should we sit?

Students offer different ideas—sit facing each other, sit side-by-side to share the materials—which Tracy and her student partner model for the class. Tracy solicits suggestions for how they might attend to each other, decide on turns, or what to do if they reach a disagreement. After discussion, she tells the class that they will try out these ideas in their partnerships today, then moves on to launch the day's mathematics problem: Four 4s.

Tracy is confident that all her students will be able to engage in this open task utilizing their unique strengths. Her linguistically and culturally diverse students, especially the English learners, will experience important learning opportunities through communicating their reasoning to their partners and contributing to the class discussion.

Tracy relies on the CA ELD Standards which, in grade four, specify that Tracy's English learners will "develop an understanding of how language is a complex, dynamic, and social resource for making meaning." She wants to use the informal nature of this portion of the lesson to illuminate how math "is organized in different text types and across disciplines using text structure, language features, and vocabulary depending on purpose and audience." The students will make use of several mathematical practices (e.g., SMP 1, 2, 3, 6, 8), and will build skills as they invent and solve calculation problems using the four arithmetic operations (4.OA.4; 4.NBT 4, 5, 6). Tracy posts the problem statement on the whiteboard; she asks the students to read it silently first and then leads a choral reading: "Can we find every number between 1 and 20 using exactly four 4s and any operation?"

She signals for quiet thinking time, and after a few seconds, says, "When I first read this problem, I was not sure what it meant for us to do. Which words in this problem might have caused me confusion?" She uses a think aloud strategy, repeating, "BLANK confused me because…. BLANK confused me because…." After another pause, she asks the students to turn to a partner and ask, "What confused me?" The chatter provided formative feedback, and Tracy continued by prompting them to discuss what they think it means—which mathematical operations can they think of to use? "Try to be ready to explain what we should do, or perhaps share an example of a number you were able to find between 1 and 20 using exactly four 4s. In a few minutes, we will share our ideas with the whole class."

Partners turn toward each other to begin their discussion of the task. Partner discussions are based on an integrated ELD strategy called Three Reads constructive conversations (https://achieve.lausd.net/page/11500), where students first read to understand, then read to identify and understand the math, then read to make a plan. Their discussion is framed by cues on the board: "1.) Understand; 2.) Understand the math; 3.) Make a plan." She observes that many students are stuck between the second and third stage; they are not entirely sure of how to proceed, especially with regard to using all the operations. Many of the students have limited themselves to addition and are ready to suggest one way to get 16.

For example, one pair describes what they think the problem asks them to do:

Partner 1: Well, we can add all the fours together, and that makes 16.

Partner 2: Yeah, that works, but aren't we supposed to get *all* the numbers from 1 to 20 as our answers. How are we supposed to do that?

Partner 1: Oh. What else can we do with the fours?

Tracy brings the class together to thank the students for their successful productive partnerships and to begin discussing what the problem asks and what solutions students have discovered.

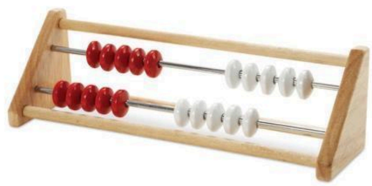
Supporting Student Partnerships and Small-Group Work

Students can explore the mathematics inside open, multi-dimensional tasks in collaboration with peers. In order to realize the many benefits of student-led work, students must learn to share and discuss ideas inclusively. Issues of status, stereotypes, and peer relationships can get in the way of mathematical sense-making by biasing who participates and in what ways to the mathematical work at hand (Cohen & Lotan,1997; Esmonde & Langer-Osuna, 2011; Shah, 2017; LaMar, Leshin, & Boaler, 2020). Established classroom norms and routines can support students in attending to and making sense of their peers' mathematical ideas in ways that position one another's thinking as worthy of taking into consideration (see also Cabana, Shreve, & Woodbury, 2014). Chapin, O'Connor, & Anderson (2013) provide further support for teachers in supporting productive classroom discussions, considering the mathematics to talk about, and the moves that encourage productive discussions.

The following vignette highlights a particular routine—peer revoicing—that helped first graders take turns sharing, listening, and making sense of one another's math ideas.

Vignette – Peer Revoicing (Langer-Osuna, Trinkle, & Kwon, 2019)

Hope, a grade one teacher, introduces peer revoicing during a whole-class carpet discussion. She wants her young learners to practice a way of interacting that supports mutual attention and making sense of one another's mathematical thinking (SMP.3, 5, 6). Using a large rekenrek, she models revoicing with a student partner. The student partner first said how many beads she sees on the Rekenrek and how she knows (DI 1, CC 2; 1.OA.3, 6).



S: I see eight beads because there are five on the top and three on the bottom and that's five, six, seven, eight.

T: So, I hear you say that you see eight beads because there are five beads on the top and three beads on the bottom and you counted up from five, six, seven, eight. and that's how you knew there were eight. Is that right?

S: [nods head] Yup.

Hope then modeled the language used for the revoicing. "Let's practice that" she said to her class. "I hear you say 'mmmmm,' is that right?"

The class repeated as a chorus, "I hear you say 'mmmmm,' is that right?"

Students then practiced at the carpet with their partners and a class set of rekenreks before taking their rekenreks back to their tables for partner work.

At their table, students took turns representing numbers. Ana represented the number 10 and turned it toward her partner Sam. Sam counted the beads one by one and then stated:

Sam: “I see a 10 because there are 1, 2, 3, 4, 5 on the top and 5 on the bottom.”

Ana: “So I hear you say, wait. Can you repeat?”

Sam: [giggles] I said I see a 10 because there are 5 on the top and 5 on the bottom and that makes 10.

Ana: “So I hear you saying that you see a 10 because there are 5 on the top and 5 on the bottom, is that right?”

Sam: “and that makes 10”

Ana: “and that makes 10. Is that right?”

Sam: Yes

Ana: Ok, my turn. You do a number now.

Revoicing is a talk move between two people where the contribution of the speaker is restated by the listener, who checks with the speaker to confirm understanding. It often includes a statement such as, “So I hear you say…” followed by a restatement of the speaker’s words and then a check for understanding, such as “Is that right?” Peer revoicing is a powerful routine for promoting both shared understanding of mathematics and mutual recognition as young mathematicians. Peer revoicing structures the dialogue between the speaker and the listener, ensuring that the contributions build meaningfully upon each other. Teachers can also intervene on status issues as they confer with groups of students. Complex instruction offers a status intervention technique where teachers strategically find opportunities to elevate the mathematical contributions of a student perceived as low-status by pointing out the student’s idea, strategy, or drawing as useful to the group and worthy of consideration by peers (Cohen & Lotan, 1997; Cabana, Shreve, & Woodbury, 2014; LaMar, Leshin & Boaler, 2020).

Orchestrating Reflective Whole-Class Discussions

Whole-class discussions are good opportunities for teachers to listen closely to and facilitate students' ideas as students work to articulate their thinking. To implement the SMPs, it is necessary that teachers give careful attention to the types of questions they use; high quality, probing questions empower students to deepen their understanding. But all too commonly, questions that demand only simple recall or superficial explanation dominate classroom conversation (Simpson, et. al., 2014).

The Mathematics Assessment Project (MAP) offers a series of professional development modules (see https://www.map.mathshell.org/index.php). One of these modules, *Improving Learning through Questioning,* includes guidance on how and why to use open-ended questions, and provides examples such as, "What patterns can you see in this data? or Which method might be best to use here? Why?" Questions of this type take students beyond simple recall of known facts; instead, they call for original thought and connections of concepts. MAP research has found that in order to be effective, questions must be designed to include all students and to elicit thinking and reasoning. Teachers should provide think time, avoid judging student responses, and pose follow-up questions that encourage continued thinking. https://www.map.mathshell.org/pd/modules/4_Questioning/pdf/4_Questioning_Guide.pdf.

Whole-class discussions at the close of a lesson provide additional opportunities to reflect on the impact of student partnerships and small-group work so that students increasingly internalize the expectations and learn the tools of inclusive, productive, shared mathematical work. Teachers might ask, "What went well in your partnerships today that we can learn from? What was difficult? What might we try tomorrow to be better partners?" Responses not only allow students an opportunity to express their thoughts like a mathematician, but the responses can provide invaluable formative feedback for teachers to use when defining the next steps in the learning progression(s).

Approaches described in this chapter can benefit all students, but they may be particularly useful for vulnerable students, including students learning English and students with disabilities. A positive learning environment relies on foundational supports that are broadly available and incorporated into the classroom norms. Creating an inclusive mathematics classroom means incorporating strategies that support the participation of all students, with particular attention to under-served students. One approach to support participation of linguistically and culturally diverse English learners in mathematical discussions is outlined by Moschkovich (1999):

> Teachers should attend to the mathematical ideas being expressed rather than focusing on correcting vocabulary. By instead revoicing and rephrasing students' statements, the teacher allows the student the right to evaluate the correctness of the teachers' interpretation. Second, revoicing helps keep the discussion mathematical by reformulating the statement in ways closer to the standard mathematics discourse. (See also: https://ell.stanford.edu/publication/mathematics-common-core-and-language)

Studies show that students with learning differences are supported in mathematics classrooms when they include multi-dimensional tasks that incorporate multiple representations, multiple ways to engage, and multiple forms of expression (Foote & Lambert, 2011; Lambert & Sugita, 2016; Moschkovich, 1999; Boaler & LaMar, 2019). In addition to the approaches discussed throughout the chapter, the following strategies can also support the participation of students with learning differences in mathematical discussions:

- Including paraprofessionals in the instruction allows students opportunities to rehearse and share in preparation for whole-class discussion (Baxter et al., 2005). This functions similarly to a think-pair-share completed prior to whole-class discussion.

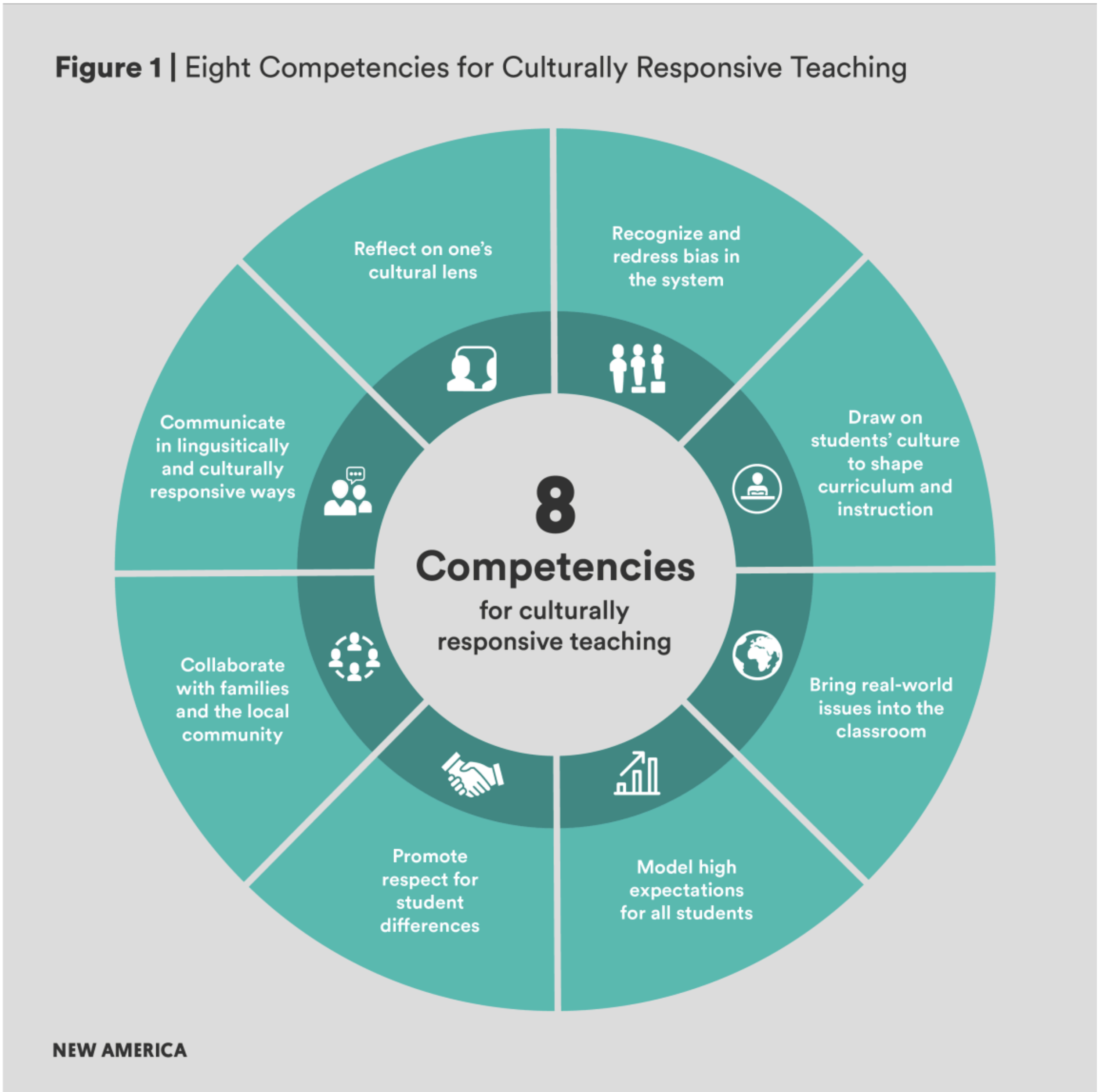
- Teachers can help create a classroom culture where all students can and *do* readily access resources—like math notebooks, media apps and websites, and manipulatives—whenever they need them. Students with learning differences may use these resources more often or for longer amounts of time than other students during whole class discussions and benefit from being able to draw on them as necessary (Foote & Lambert, 2011).
- Asking follow-up questions sets up the expectation and the support for students to be accountable to explaining their strategies. (Lambert & Sugita, 2016).

Strategies that support students with learning differences ultimately create a positive learning environment for all students. Incorporating gestures (including facial and corporal), artifacts (such as props and images), and multiple styles of language as part of classroom discourse (include formal and informal dialects, Spanglish, etc.), attending to and revoicing students' mathematical ideas, allowing students time to rehearse and prepare for whole-class discussions, not limiting the use of resources, and using follow-up questions to help students complete or extend their explanations support the participation of all learners. Such strategies are particularly helpful for language learners.

3. Teach Toward Social Justice

Mathematics educators have an imperative to impart upon their students the argument that mathematics is a tool that can be used to both understand and change the world. Mathematics has traditionally been viewed as a neutral discipline, which has occluded possibilities for students to develop more personal and powerful relationships to mathematics and has led too many students to believe mathematics is not for them. A different perspective enables teachers to not only help their students see themselves inside mathematics but develop knowledge and understanding that allows them to use mathematics toward betterment in their worlds. Teachers can take a justice-oriented perspective at any grade level, K–12, helping students feel belonging (Brady et al, 2020), and empowering them with tools to address important issues in their lives and communities. In a special issue (Summer, 2016, https://www.todos-math.org/assets/documents/TEEM/teem7_final1.pdf), TODOS, an affiliate organization of the National Council of Teachers of Mathematics, presents six articles written by educators who are involved in teaching and learning mathematics from a social justice perspective. Each author describes "the promises, tensions, and struggles of engaging themselves and others...in fundamentally changing the experience of learning and teaching mathematics."

A framework for addressing culturally responsive teaching that gathered information from fifty states outlines eight competencies:



Source: https://newamerica.org/education-policy/reports/culturally-responsive-teaching/, page 12.

Students come from many different backgrounds, experiences, and cultural identities. Multicultural education draws on students' experiences through their family, community, and cultural and linguistic forms of knowing, referred to as funds of knowledge (Gonzalez, Moll, & Amanti, 2006), in ways that go far beyond food, music, and folklore. A multicultural approach is foundational to participating in the global economy.

Multicultural education can be implemented into mathematics by exploring students' lives and histories, centering contributions that historically marginalized people have made to mathematics in the design and implementation of curricula, creating opportunities for teachers and students to share their own autobiographies as mathematics doers and learners, and creating spaces for students to participate as authors of their mathematical learning experiences. Esmonde and Caswell (2010) describe a fifth-grade mathematics project focused on access to water as a human right, integrating topics of volume, capacity, operations, and proportional reasoning to explore their families' usage of water and access to water in developing countries. In the Number Book Project (Esmonde & Caswell, 2010), kindergarteners and their families shared number stories, songs, and games that parents or others knew as children. They then designed classroom activities that drew on these number stories, songs, or games.

In the example that follows (from Diez-Palomar & Lopez Leiva, 2018), a group of students explored their family's immigration experiences through a measurement lesson on the topic of unit conversion, specifically between the US system and the metric system. Many of the students had experienced immigrating with their families to the US or knew relatives who had, as well as had family members living on the other side of the Mexican border. Through map explorations and a series of discussions, students used and expanded their math skills, as you see in the vignette below:

Vignette: Exploring measurements and family stories (from Diez-Palomar & Lopez Leiva, 2018, p. 49)

On a map, [two] students located the different places where their relatives lived or that they had heard mentioned. They selected the starting and ending points of immigration and figured out the distances. The discussion continued:

> Mary Jo: Yeah so right here to here. Like right here to right here is a mile.
>
> Jocelyn: I think it's more than a mile.
>
> Mary Jo: Eight miles?

Jocelyn: There’s a scale on the map somewhere, let’s look. Let’s measure this, how long is this? Okay, first of all what are these numbers here, what do those represent?

Mary Jo: Inches, one inch.

Jocelyn: Then what are these numbers?

Mary Jo: Millimeters.

Jocelyn: What’s millimeters?

Mary Jo: Millimeters are more than, no.

Jocelyn: Do you see them mm? Where’s the mm?

Mary Jo: Oh, these are millimeters, these are inches. …”

Multicultural children’s literature can also be used as contexts to connect learning mathematics with students’ cultural experiences (Esmonde & Caswell, 2010; Leonard, Moore, & Brooks, 2013). For example, in *The Great Migration: An American Story* (Lawrence and Myers 1995), young children explore quantity in terms of population shifts. In *First Day in Grapes* (Perez 2002), a boy from a family of migrant workers uses his knowledge of mathematics to earn the respect of his peers. Drawing on *The Black Snowman* (Mendez, 1989), students can explore money problems through contexts linked to the African Diaspora. *One Grain of Rice* (Demi, 1997) offers students a context for exploring exponents and the importance of sharing food through the story of a peasant girl who tricks a raja into giving her the royal storehouse’s entire supply of rice. *Multicultural Mathematics Materials* (NCTM) by Marina Krause also includes several games and activities that draw on Hopi and Navajo materials.

Empowering students with mathematics also includes removing the high stakes of errors and sending the message that learning is always unfinished and that it is safe to take mathematical risks. This mindset creates the conditions for students to develop a sense of ownership over their mathematical thinking and their right to belong to the discipline of mathematics. For example, in the vignette below, the teacher, Ms. Wong, offers students an open-ended math task, which she then implements in a way that positions all learners as belonging to their mathematics sense-making community (SMP.3, 7, 8.).

Vignette: This vignette comes from research based in a California high school committed to social justice and a mathematics classroom designed to foster positive mathematics identities (Gargroetzi, 2020). The following is a transcript from the lesson that occurred.

Function 1

Directions: Copy the table below onto your binder paper:

Input	3	5	1	2	11	-3
Output	13	21	5			

1. Describe in words the function that converts the input to the output.
2. What would the output be for the following inputs? Show your work.
    a. 2?
    b. 11?
    c. -3?

Ms. Wong: Raise your hand if you tried a rule that didn't work.

Gabriel raised his hand and was called on, shared an idea he had that didn't work out.

Ms. Wong: I love that you just shared something that you tried that didn't work! Raise your hand if you found a rule that does work.

Scott is called on. He explains he put the numbers in order least to greatest, then noticed that from 1 to 3 the outputs went up by eight, and from 3 to 5 the outputs went up by eight. So, from 1 to 2 the outputs should go up by four. So, he says, the pattern is plus four. Ms. Wong documents Scott's thinking on the board using arrows to show the "plus four." She nearly bounces with excitement.

Bay shares the rule he found: Add the number to itself, then multiply by two, then add one. (He demonstrates with 3 and then 5)

Maple: I have the same rule [as Bay] but in a different format. Y=4x+1. (She goes up to the white board to demonstrate her thinking and puts up multiple examples. Ms. Wong asks about x and y. Maple explains that y is the output and x is the input.

Jill: Mine is the same as Maple's—times 4, add one.

Ms. Wong I'm up here amazed! But also raise your hand if you are confused? Talk with your group—which rule makes the most sense to you?

Ok. Raise your hand if you liked Scott's rule (no hands). It's okay, sometimes genius is misunderstood! Bay's rule? (Bay's table all raises their hands, but nobody else), Maple and Jill's rule (everyone else raises their hands). Okay, since Maple and Jill's was the most popular—I'm not saying that means it's correct—let's use it to check. Ms. Wong began the discussion about this task by intentionally asking students to share if they tried a rule that "didn't work." In doing so, she sent the message that ideas were valued for reasons beyond being correct, that doing mathematics sometimes involved errors or confusion, and broadened possible ways for students to participate, lowering the risk of contributing to discussions. These moves positioned all students in the classroom as mathematical thinkers, learners, and community members (Gargroetzi, 2020).

Learning is not just a matter of gaining new knowledge—it is also about a change in identity. As teachers introduce mathematics to students, they are helping them shape their identity as people (Langer-Osuna & Esmonde, 2017; Boaler & Greeno, 2000).

Teaching mathematics through discussions and activities that broaden participation, lower the risks associated with contributing, and position students as thinkers and members of the classroom community, are powerful ways to support students in seeing themselves as young mathematicians. However, even within a classroom that utilizes these approaches, inequities can manifest through patterns about whose ideas are attended to and become influential, and ultimately who is seen as relatively more mathematical (Langer-Osuna, 2011; Shah, 2017). Students from social groups that have been historically affiliated with the discipline of mathematics can come to have undue influence over those from social groups not stereotypically associated with mathematics (Esmonde & Langer-Osuna, 2013). These processes are often unconscious, based on stereotypes and implicit bias, and get in the way of creating robust, productive, and inclusive sense-making mathematics classroom communities (Shah, 2017). Teachers can support discussions that center mathematical reasoning rather than issues of status and bias by intentionally defining what it means to do and learn mathematics together in ways that include and highlight the languages, identities, and practices of historically marginalized communities. One way in which they can do this is by emphasizing and welcoming students' families into classroom discussion (González, Moll, & Amanti, 2006; Turner & Celedón-Pattichis [2011], Moschkovich, J. [2013]; Battey et al., 2016).

Vignette (from Turner et al., 2013)

In the vignette below, the teacher emphasizes the importance of communicating mathematical ideas, and attending and responding to the mathematical ideas of others across languages (DI 1, CC 3, SMP.3, 6; 4.OA.4, 5).

This vignette comes out of classroom research on the participation of linguistically and culturally diverse English learners in mathematical discussions (Turner, et al, 2013). It documents an actual classroom experience. The teacher and students (grades 4–5) are discussing multiplicative relations using a paper-folding task where students folded a piece of paper to make 24 equal parts. Note how the teacher and class members engage with Ernesto's thinking about the mathematics in this task. Ernesto is an English learner; by focusing attention on his reasoning, the teacher is validating his status as a contributor to the mathematical discourse within the class.

Teacher: Ernesto, nos dices cómo lo hiciste? (Ernesto, would you tell us how you solved it?)

Ernesto: Lo doblé cinco veces, a la misma (I folded it five times, the same way—) [Stands up to come to the front of the room]

Teacher: [Hands Ernesto a piece of paper to show his folds] A ver, escúchenlo. (Let's see. Let's listen to him.)

Ernesto: Lo doblé. cinco veces, igual. Así. (I folded it five times, equally. Like this.) [Folds paper five times in the same direction, using an accordion-like fold] [Unfolds paper] Y me da seis partes. (And it gives me six parts.)

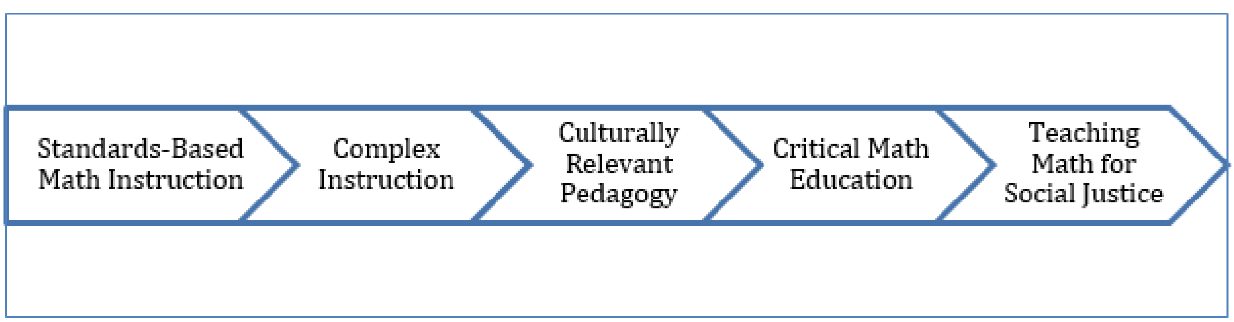
Teacher: His idea is to fold it five times, five times, and you get six parts. Does anyone have something to say to Ernesto? What do you think of how he did that? Anybody agree? [pause] Anybody else do it that way?

Corinne: It's different from ours, because he folded it five times to make six parts, and we—all three of us [the students who shared previously]—folded it in half, and [then] three times to make six parts.

Teacher: So, you noticed some way that Ernesto's strategy is a little bit different.

Reflection: The classroom community could be relied on to translate for others, and the emphasis remained on positioning all learners as thinkers and as members of the same community. In doing so, students who historically are marginalized from mathematical discussions—in this case, English Learners, were positioned as contributors and thinkers alongside their English-speaking peers. Further, students from dominant cultures—in this case monolingual English speakers—had the opportunity to engage with the mathematical ideas of typically silent students, to take their ideas into consideration, and to build on and make connections to their mathematical thinking.

Mathematics educators committed to social justice also work to both raise awareness of the ways textbook examples exclude and stereotype certain students (Bright, 2016; Yeh & Otise, 2019) and to provide curricular examples that equip students with a tool kit and mindset to combat inequities with mathematics (Aguirra & Frankenstein, 1995; Gutstein, 2006, Gutstein & Peterson, 2005; Moses & Cobb, 2001). The tasks have been developed to help students read and write the world with mathematics. First learning to use mathematics to highlight inequities—*reading the world* with mathematics—and then learning to change the world with mathematics—*writing the world* with mathematics (Gutstein, 2003; 2006). Note that these tasks correspond neatly to Drivers of Investigation (DI), DI 1(making sense of the world) DI 2 (predicting what could happen) and D3 (Impacting the future). Examples of such tasks include making data visualizations to show food availability for different communities, analyzing the ethnicities of different math tracks in high school (Berry et al, 2020), looking at the ways gun violence affects children (http://www.creatingbalanceconference.org/resources/), modelling border policies (Berry et al, 2020), considering the nature of safe water to drink (Aguirre & Civil, 2016), and celebrating Black mathematicians. Berry's approach builds upon four other bodies of work related to equitable teaching:



Source: Berry et al., 2020, p.19

A social justice approach to mathematics enables the *humanizing* of mathematics (Goffney, I. & Gutiérrez, 2018; Su, 2020). Students start to see mathematics as something that relates to their lives and that can work to empower individuals and communities. In Ms. Wong's classroom, for example, tasks are not only deliberately designed to engage students in meaningful mathematics, but are also, at times, designed to support students in noticing that they are already important members of the mathematics classroom community, supporting belonging.

Vignette (from Wei & Gargroetzi, 2019)

Math Identity Rainbows

Purpose: To reflect on and share the strengths that you and your teammates bring to the group

Each person will get six different colored strings. Each color represents a different math practice

Your task is to arrange the cords according to your relative strengths and weaknesses.

Math Identity Rainbow Cords and Identification

- Pink is persevering: "I try my best and don't give up, even when I face challenges."
- Orange is numerical reasoning: "I have good number sense and use numbers flexibly."
- Yellow is communicating: "I can explain my reasoning clearly to others."
- Blue is modeling: "I can use methods and tools to arrive to solutions."
- Purple is pattern recognizing: "I can generalize patterns and see connections between concepts."
- White is reflecting: "I know what I've learned and what I still need to learn."

Directions: Arrange the cords in the order of your strengths (strongest practices on top).

Through this task, Ms. Wong offered a definition of mathematical competence as multi-faceted. Ms. Wong emphasized, “All of these are extremely important to being mathematicians and everyone has these qualities but you have different strengths, right? So, the idea is you are going to order these cords on your desk so that the top strand is what you think your biggest strength is” (Gargroetzi, 2020). Students reflected individually and then shared their top strength with their partner. Students then discussed the strengths each group member brought to their mathematical work. In doing so, students had the opportunity to notice that together they were part of a mathematical community in which each member offered different, important strengths.

In the following vignette, a Ms. Ross leads students into a discussion of textbook questions, to consider the ways textbook examples may exclude and stereotype certain students (Bright, 2016; Yeh & Otise, 2019). As the students consider the questions they learn the mathematics inside the questions and work to reformulate the questions to better reflect the students in the classroom (SMP.2, 3, 5, 6).

Vignette

Ms. Ross teaches fifth grade at the Jackie Robinson Academy. She has been focusing on developing her students’ sociopolitical consciousness through language arts and wants to bring mathematics into their thinking (SMP.1, 2). To begin the process, the class is led in an analysis of word problems from their fifth-grade mathematics textbook (NF.1, 2, 4, 5, 6). Ms. Ross selects three word problems to connect with the class’s current read-aloud of George, a novel by Alex Gino that shares the story of a 10-year-old transgender fourth grader and her struggles with acceptance among friends and family. In doing so, the teacher is reflecting the recommendations of California’s *Health Framework*, which suggests that sensitive discussions of gender are important for students (https://www.cde.ca.gov/ci/he/cf/). Ms. Ross reads the questions aloud to the class:

Amie used 7/9 yard of ribbon in her dress. Jasmine used 5/6 yard of ribbon in her dress. Which girl used more ribbon? How much more did she use?

A fifth grade class is made up of 12 boys and 24 girls. How many times as many girls as boys are in the class?

Ms. Hernandez knitted a scarf for her grandson. The scarf is 5/6 of one yard long and 2/9 of one yard wide. What is the area of the scarf?

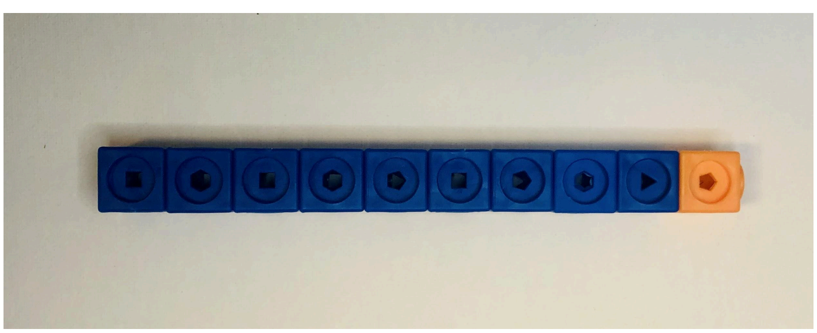
Ms. Ross uses a Say, Mean, Matter graphic organizer based on the following questions:

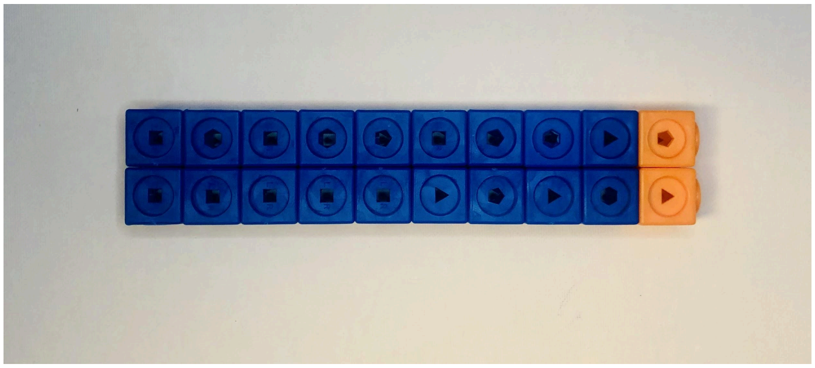
Say: What does the text say?

Mean: What does this mean? How do I integrate this? Read between the lines.

Matter: Why does this matter? Why does this matter to me or others? What are the implications?

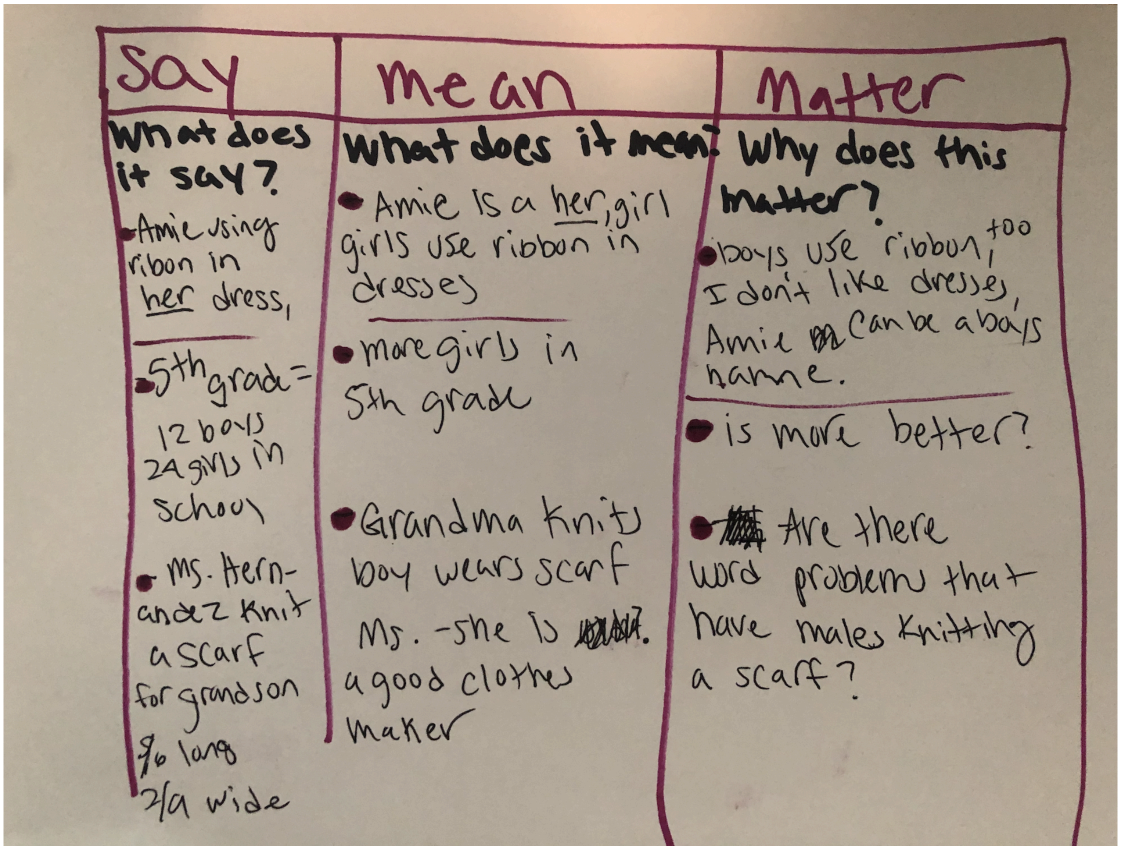
Ms. Ross asked the first question, “What does this text say?” to engage her students in an analysis of the word problems with the question. Across the room, students sit forward in their chairs, some tracing the words on their papers with their fingers in attempts to examine the text more closely. After some silence, Ms. Ross repeats her question: “What does it say?” and students slowly begin to share out—explaining the word problems through words, drawings, and numbers—a process that transforms the class conversation into one of fractions and measurements.





Several students remark on the knitted scarf problem, leading to a group discussion comparing the characteristics and units of measure between area and volume. The excitement builds in the classroom as students traverse mathematical concepts, discussing different methods of problem solving based on what makes sense for each student and the methods of operation to use in relation to the problem context.

The conversation shifts as Ms. Ross then asks the students, “What does this mean?” In this second level of analysis, students are now asked to consider, for themselves, what the text means. Ms. Ross gives students time to think about the meaning on their own, and then, students engage in sharing with their partners. After a few minutes of talking in pairs, the whole class comes together for conversation, having had multiple opportunities to see the problem from differing perspectives (see Figure 2). The open question, “What does this mean?” could potentially go in many possible directions, allowing students to bring their own noticings, curiosities, and concerns to bear on the problem. As depicted in the image below, the students in this particular classroom came to notice the role of gender in the mathematics problem.

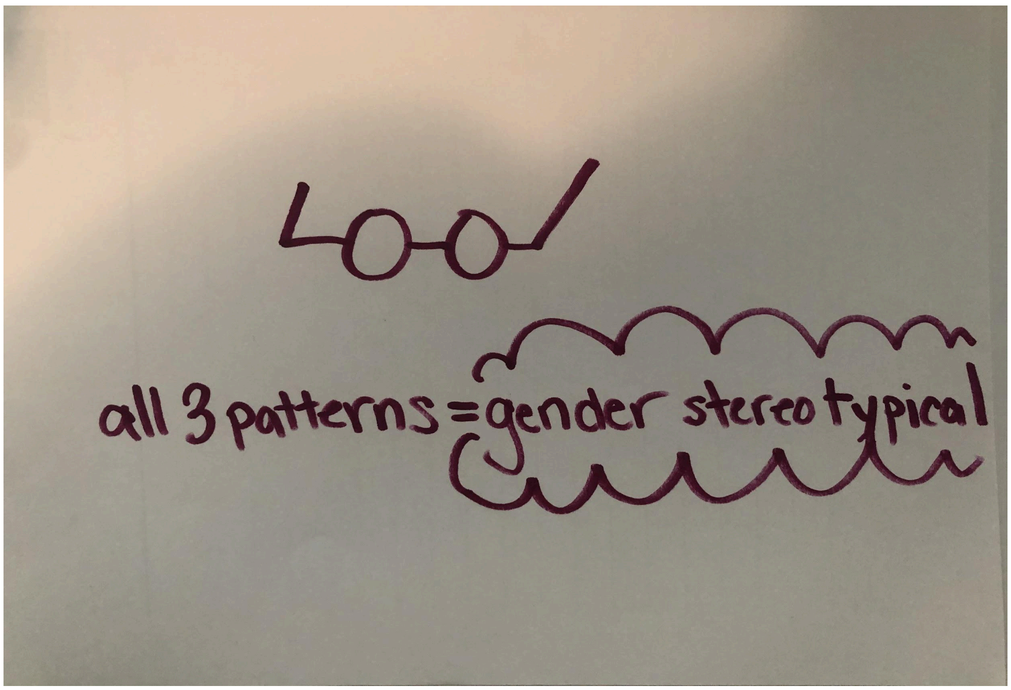


As the third and final prompt within the exercise, Ms. Ross asks students, “Why does this matter?” This time, she is asking students to identify perspectives and points of view in the text, and encouraging them to look for the “silences” in texts. Students are able to take what they noticed and named – in this case, how gender played out in the problem – and consider its implications, enabling critical thinking. In doing so, Ms. Ross’ question asks students to grapple with: What prior knowledge and experiences aside from mathematics is needed? Whose lived experiences are not included?

During classroom discussion, several questions and concerns arise as a result from their conversations: (see Figure 3):

- What constitutes boys’ things and girls’ things?
- Problems with girls’ names provide context related to looking pretty, being helpful, and being a homemaker.
- Problems with boys’ names focus on sports and competition.

- Playing sports is seen as a boy’s thing while playing house is a girl’s thing.
- Are certain things—toys, games, activities, etc.—the sole and primary preserve of either girls or boys?
- Are there word problems about ribbons, cooking, or knitting that use a boy’s name?
- Do these word problems really *matter* in real life? Do they represent mathematical calculations needed to engage in daily life?



Students’ Mathematical Investigations

The word problem analysis serves as a springboard for students to investigate their own questions. One student asks, “Are there word problems that have a male knitting a scarf, cooking, cleaning?”; and another ponders, “Does the textbook always use girl names for girl stuff and boy names for boy stuff?” Lastly, another student asks, “Are there word problems that challenge gender stereotypes?” When examining the entire textbook, the students noted that there were a few instances of gender-fluid problems (e.g., David’s dad baked a dozen cookies to share with him, his sister, and his mom); however, the problem continued to conflate gender with a heterosexual identity. The class could not find problems involving non-nuclear families (e.g., two moms, a single dad) or gender nonconforming characters (e.g., John cutting ribbon). Ms. Ross has students notice these patterns, but also asked students to question why certain items (e.g., toys, activities, careers) are perceived as being “for” only girls or boys, and the implications for these assumptions. She continues to engage her students by asking, “*Why does this matter? Who does this privilege? Who is silenced?”*

Reframing Mathematics

At the last stage, student groups collaboratively rework and reframe word problems into new scenarios that disrupt the taken-for-granted and expand possibilities, in this case, for gender nonconforming and LGBTQ representations that students chose to make visible: Juan is cutting ribbon to make a pink bow or Molly’s dad knits a scarf for his husband. Here are web links to view three groups’ recreated word problems and their justifications for changes:
https://www.youtube.com/watch?v=fNMnQs-b6Wg&feature=youtu.be.

This vignette is adapted from: Yeh, C., & Otis, B. M. (2019). Mathematics for Whom: Reframing and Humanizing Mathematics. Occasional Paper Series, 2019 (41). Retrieved from https://educate.bankstreet.edu/occasional-paper-series/vol2019/iss41/8/
Other resources for teaching mathematics with a social justice perspective include:

- Mathematical Modeling with Cultural and Community Contexts: https://sites.google.com/qc.cuny.edu/m2c3/home
- The five strides of Equitable Math.org: https://equitablemath.org/

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4. Invite Student Questions and Conjectures

Open tasks about big ideas in mathematics foster curiosity. Teachers can invite students to follow their curiosity by making space for their questions and conjectures. One of the most important yet neglected mathematical acts in classrooms is that of students asking or posing mathematical questions. These are not questions to help students move through a problem; they are questions that are sparked by wonder and intrigue (Duckworth, 2006). Examples of questions a student may ask include, “What is half of infinity?” “Is zero even or odd?” or “Does the pattern that describes the border of a square work if the shape is a pentagon?” Questions sparked by curiosity might also sound like pushing back on the ideas at play in the classroom, whether introduced by the teacher or peers. Students begin questions with, “But what about…?” or “But didn’t you just say…?” Such questions should be valued and students given time to explore them. They are important in the service of creating active, curious mathematical thinkers.

Mathematically curious students who explore big ideas through open tasks are well primed to engage in another important mathematical act—that of making a conjecture. Most students in science classrooms know that a hypothesis is an idea that needs to be tested and proven. The mathematical equivalent of a hypothesis is a conjecture. When students are encouraged to come up with conjectures about mathematical ideas, and the conjectures are discussed and investigated by the class, students come to realize that mathematics is a subject that can be explored deeply and logically. It is through conjectures that curiosity and sense-making are nurtured. The Drivers of Investigation which are centered in this framework are intended to create opportunities for students to be curious and develop conjectures, as they work on investigations with the goal of “making sense of the world,” “predicting what could happen,” and/or “impacting the future.”

Snapshot

A teacher presented their fourth-grade students with a list of eight equations, noting that not all of them were true statements of equality. The students worked with partners to decide which were true, which were false, and to explain how they knew.

2 x (3 x 4) = 8 x 3
4 x (10 + 2) = 40 + 2
5 x 8 = 10 x 4
6 x 8 = 12 x 4

9 + 6 = 10 + 5

9 – 6 = 10 – 5

9 x 6 = 10 x 5

Ryan and Anen worked together, and after a few minutes, the teacher could see that they were very excited! The teacher stopped by their workplace and after listening to their explanation, and posing a few challenges, invited them to describe their “magic” trick with multiplication to the class. At the front of the class; Anen wrote equation c, 5 x 8 = 10 x 4, on the board, and asked everyone to use a hand signal to show true or false. Almost all students indicated it is a true equation. Ryan asked the class about example d, 6 x 8 = 12 x 4. Again, the class agreed that it is true.

Anen and Ryan continued, saying that something special was going on, and they had a conjecture they think *probably* works all the time, but they want to be sure. They explained that in 5 x 8 = 10 x 4, they noticed “5” on the left side of the equation is half of the “10” on the right side, and the “8” on the left side is two times the “4” on the right side. So, they concluded, trying to use proper mathematical language, and pointing at the numbers as they spoke, “If you have factors like that where one first factor is half of the other first factor, and the second factor is twice as big as the other second factor, they’ll always be equal!”

The teacher called for the class to explore this conjecture and to see whether they could find a way to prove whether it is always true or not. Now the whole class was interested and trying to prove or disprove the Ryan/Anen conjecture.

The teacher supported the discussion in several ways. They:

- brought the class together to listen according to class norms
- encouraged the speakers to pause occasionally so that their classmates would have time to think and try out ideas
- asked students to repeat, revoice, or add on to each other’s statements

- re-stating Ryan's and Anen's explanations using precise mathematical terms
- checking with students who are learning English to ensure that they are both communicating with and supported by their partners during the student-led presentation
- called for others in the class to express their own conjectures and challenges
- focused students' attention to Anen and Ryan's explanations and questions
- posed questions to both the presenters and the other class members as the discussion progressed, such as:
    - why is this true?
    - will this always work?
    - does this work for other operations, or only for multiplication?
    - how can we know?
    - how are these numbers related?

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5. Center Reasoning and Justification

Reasoning is at the heart of doing and learning mathematics. Chapter 4 includes a description of how students come to conjecture, reason, and justify along with other important related acts together. All students can reason deeply with and about mathematics.

Open tasks invite students to reason about mathematics and, through discussion, justify their thinking. Reasoning with and about mathematics supports and enhances everyday life. A student who learns to reason about their ideas is learning to be a good communicator of mathematics, a skill that is essential in 21st Century employment. Employers used to value highly the people who could calculate and come up with correct answers, but now computers perform calculations and employees are needed to program computers, make sense of solutions, reason about mathematical pathways and communicate their thinking so that other team members connect with them (Wolfram, 2020). Flexible and creative thinking is more highly valued in today's workplace than fast calculating (Mlodinow, 2018; Wolfram, 2020).

Reasoning is fostered when students have the opportunity to talk about mathematics with each other through whole class discussions and small group work on open tasks. Students can be given open tasks and asked to discuss ideas with each other and reason about them. An example of a group discussion of a mathematical task, with students engaging in sense making, reasoning and justification, can be seen at (https://www.youcubed.org/mathematical-mindset-teaching-guide-teaching-video-and-additional-resources/).

This framework for teaching with open tasks is consistent with the recommendations from the National Council of Mathematics Teachers (NCTM), in Catalyzing Change:

Mathematics Teaching Practices: Supporting Equitable Teaching Practices

Mathematics Teaching Practices	Equitable Teaching
Establish mathematics goals to focus learning. Effective teaching of mathematics establishes clear goals for the mathematics that students are learning, situates goals within learning progressions, and uses the goals go guide instructional decisions.	• Establish learning progressions that builds students' mathematical understanding, increases their confidence, and support their mathematical identities as doers of mathematics. • Establish high expectations to ensure that each and every student has the opportunity to meet the mathematical goals. • Establish classroom norms for participation that position each and every student as a competent mathematics thinker.

	Establish classroom environments that promote learning mathematics as just, equitable, and inclusive.
Implement tasks that promote reasoning and problem solving. Effective teaching of mathematics engages students in solving and discussing tasks that promote mathematical reasoning and problem solving and allow multiple entry points and varied solution strategies.	- Engage students in tasks that provide multiple pathways for success and that require reasoning, problem solving, and modeling, thus enhancing each students' mathematical identity and sense of agency. - Engage students in tasks that are culturally relevant. - Engage students in tasks that allow them to draw on their funds of knowledge (i.e., the resources that students bring to the classroom, including their home, cultural, and language experiences).
Use and connect mathematical representations. Effective teaching of mathematics engages students in making connections among mathematical representations to deepen understanding of mathematics concepts and procedures and to use as tools for problem solving.	- Use multiple representations so that students draw on multiple resources of knowledge to position them, as competent. - Use multiple representations to draw on knowledge and experience related to the resources that students bring to

	mathematics (culture, contexts, and experiences). ● Use multiple representations to promote the creation and discussion of unique mathematical representations to position students as mathematically competent.
Facilitate meaningful mathematical discourse. Effective teaching of mathematics facilitates discourse among students to build shared understanding of mathematical ideas by analyzing and comparing approaches to arguments.	● Use discourse to elicit students' ideas and strategies and create space for students to interact with peers to value multiple contributions and diminish hierarchal status among students (i.e., perceptions of differences in smartness and ability to participate). ● Use discourse to attend to ways in which students position one another as capable or not capable of doing mathematics. ● Make discourse an expected and natural part of mathematical thinking and reasoning, providing students with the space and confidence to ask questions that enhance their own mathematical learning.

	● Use discourse as a means to disrupt structures and language that marginalize students.
Pose purposeful questions. Effective teaching of mathematics uses purposeful questions to assess and advance students' reasoning and sense making about important mathematical ideas and relationships.	● Pose purposeful questions and then listen to and understand students' thinking to signal to students that their thinking is valued and makes sense. ● Pose purposeful questions to assign competence to students. Verbally mark students' ideas as interesting or identify an important aspect of students' strategies to position them as competent. ● Be mindful of the fact that the questions that a teacher asks a student and how the teacher follows up on the student's response can support the student's development of a positive mathematical identity and sense of agency as a thinker and doer of mathematics.
Build procedural fluency from conceptual understanding. Effective teaching of mathematics builds fluency with procedures on a foundation of	● Connect conceptual understanding with procedural fluency to help students make sense of the mathematics and develop a

conceptual understanding so that students, over time, become skillful in using procedures flexibly as they solve contextual and mathematical problems.	positive disposition toward mathematics. ● Connect conceptual understanding with procedural fluency to reduce mathematical anxiety and position students as the mathematical knowers and doers. ● Connect conceptual understanding with procedural fluency to provide students with a wider range of options for entering a task and building mathematical meaning.
Support productive struggle in learning mathematics. Effective teaching of mathematics consistently provides students, individually and collectively, with opportunities and supports to engage in productive struggle as they grapple with mathematical ideas and relationships.	● Allow time for students to engage with mathematical ideas to support perseverance and identify development. ● Hold high expectations, while offering just enough support and scaffolding to facilitate student progress on challenging work, to communicate caring and confidence in students.
Elicit and use evidence of student thinking. Effective teaching of mathematics uses evidence of student thinking to assess progress toward mathematical understanding and to adjust	● Elicit student thinking and make use of it during lesson to send positive messages about students' mathematical identities.

instruction continually in ways that support and extend learning.	- Make student thinking public, and then choose to elevate a student to a more prominent position in the discussion by identifying his or her idea as worth exploring, to cultivate a positive mathematical identity. - Promote a class culture in which mistakes and errors are viewed as important reasoning opportunities, to encourage a wider range of students to engage in mathematical discussions with their peers and the teacher.

Conclusion

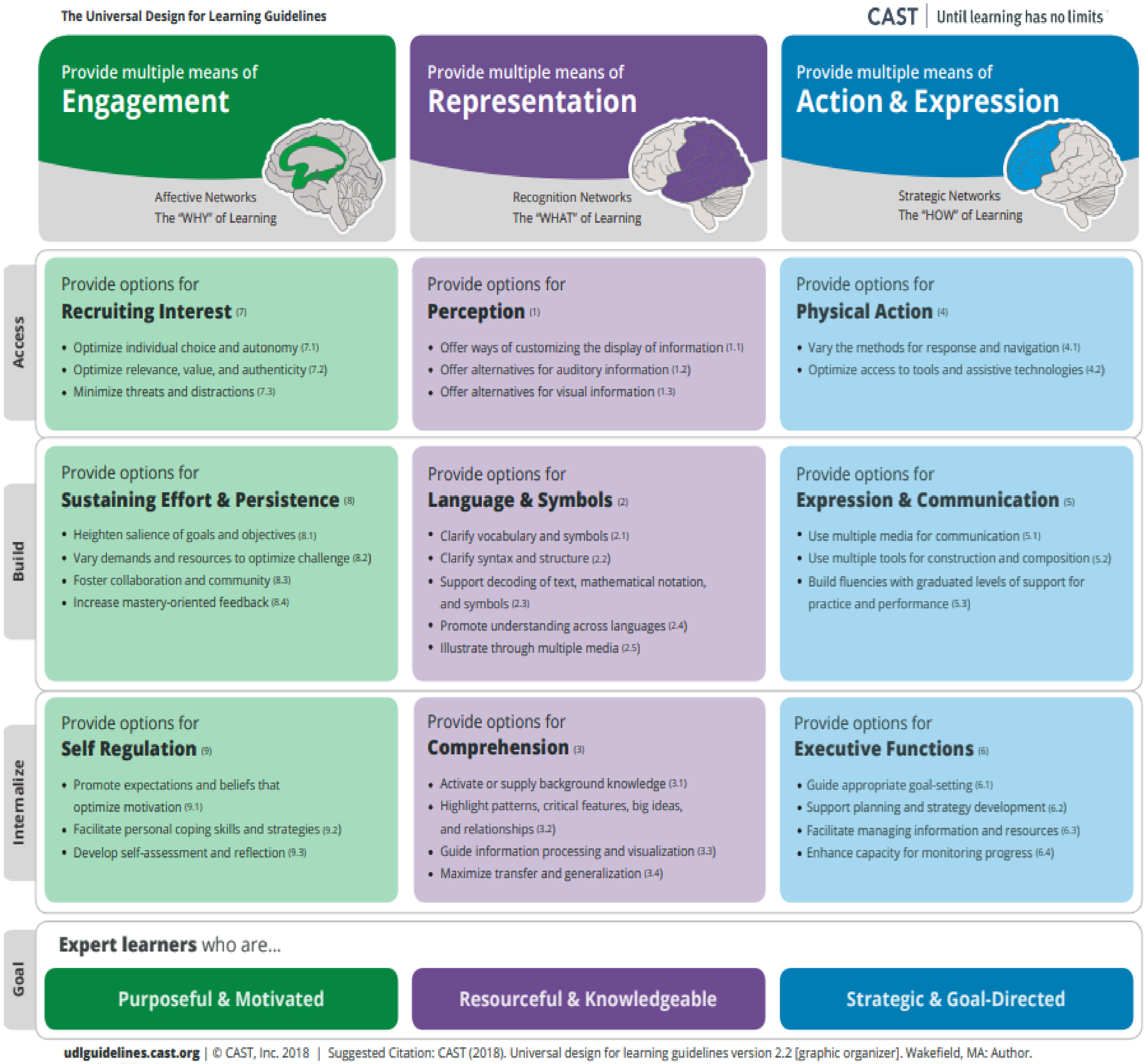
This chapter has recommended five ways of organizing classrooms to encourage equitable outcomes and active student engagement. To encourage truly equitable and engaging mathematics classrooms we need to broaden perceptions of mathematics beyond methods and answers so that students come to view mathematics as a connected, multi-dimensional subject that is about sense making and reasoning, to which they can contribute and belong. To achieve this, we need to change classroom approaches from work on short questions to instruction that engages students in rich, deep tasks that honor students' ideas and thinking and draws on their cultural backgrounds as resources. Several documents and frameworks have been referenced that offer ways to support linguistically and culturally diverse English learners and students with learning differences. In all cases these communicate principles of good teaching that can be extended to all students. The five ways of organizing classrooms are those supported by research and practice as ideas that will encourage a diverse group of students to see themselves as mathematically capable, with growth mindsets, and develop a curiosity and love of learning that will encourage them throughout their schooling. Note: the assessment practices that support this vision of mathematics teaching and learning, and the development of growth mindsets are shared in Chapter 11.

Websites sharing free open tasks:

- https://www.artofmathematics.org/
- https://mathpickle.com/
- https://nrich.maths.org/
- https://teacher.desmos.com/
- http://www.visualpatterns.org/
- http://www.youcubed.org

Universal Design for Learning Guidelines

(Source: http://udlguidelines.cast.org/)



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