Computer Science Assignment: Interpolation and Cubic Splines

Have you ever seen how computer animators capture the motion of an actor and translate that into an animation? Motion capture technology records the movement of carefully chosen points on an actor's body and allows the animator to use this as a skeleton to mold the animation around. There is a quick view of what this looks like here: https://youtu.be/fm-A1lknrxE.

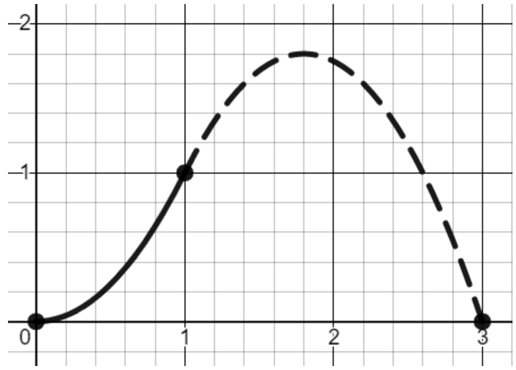
One main question animators face is how to take those points and create smooth surfaces out of them. Mathematically, this process is called **interpolation** and it is a crucial part of a lot of computing, not just animation. This is one factor that allows modern video games to be smooth instead of pixelated. Two main methods computer scientists use to find these smooth curves are Bézier Curves[1] and Cubic Splines.

1. Plot the points (0, 0), (1,1), (3, 1), (4, 5), (6, 1), and (7, 0). Connect these points into a single shape by drawing line segments between them, much like connect-the-dots.
   - What do you notice about the shape, especially at the points given?
   - What would have to be true about the function in order for this shape to be *smooth*?

To start matching the slopes at the connecting points, we need more freedom than line segments give us. We can do this by increasing the degree of the polynomial connecting the points -- using a quadratic ($a + bx + cx^2$) for instance. Having the extra freedom will allow us to match not only the heights but also the slopes.

Suppose that we want a smooth curve connecting the points (0,0), (1,1), and (3,0). If we use quadratics, we can fill in the first space using $y = x^2$.[2]

[1] You can read more about these here: https://en.wikipedia.org/wiki/B%C3%A9zier_curve
[2] https://www.desmos.com/calculator/gpsj840so6



To smoothly fill in the rest, we need a quadratic $y = a + bx + cx^2$ that meets $y = x^2$ **smoothly** at the point (1,1) and that goes through the other point (3,0).

2. Set up a system of three equations that the coefficients a, b, and c for the quadratic $y = a + bx + cx^2$ must satisfy to meet $y = x^2$ smoothly at $(1, 1)$ and that goes through the point $(3, 0)$. Then solve the system and graph the quadratic to verify that this is the correct curve.

The curve $y = x^2$ is not the only curve that can fill in the first part of the curve. There are a lot of choices, and each one leads to a different second curve.

3. Set up a system of three equations that the coefficients a, b, and c for the quadratic $y = a + bx + cx^2$ must satisfy to meet $y = 2x - x^2$ smoothly at $(1, 1)$ and that goes through the point $(3, 0)$. Then solve the system and graph the quadratic to verify that this is the correct curve.

(*Bonus*) It takes a lot of time to find the correct curves by hand. A computer can help do the tedious work. Using the points in question #1 and the [3 equation solver][3], set up and find the quadratics (starting with $y = x^2$) to smoothly connect the points. Plot your quadratics.

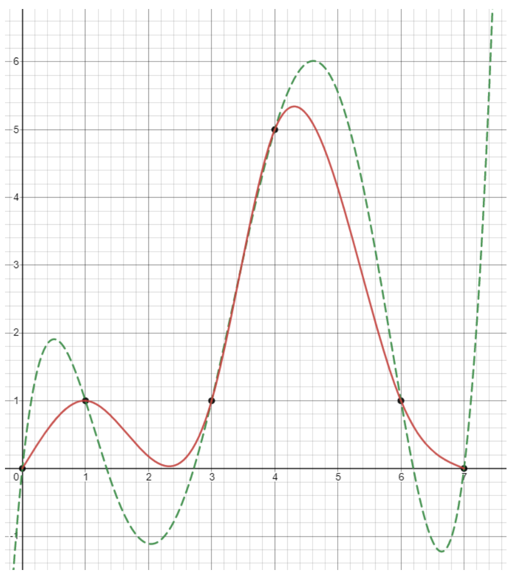
[3] https://www.wolframalpha.com/widgets/view.jsp?id=54af80f0c43c8717d710f39be0642aaa

The Next Step:

A **natural cubic spline** fits a cubic between each pair of points so that each curve flows naturally into the next. Each cubic has four constants ($a + bx + cx^2 + dx^3$) so there is a lot of flexibility in how the curve is shaped. These curves are chosen so that not only do the heights and the slopes match, but also the second derivatives match at each connecting point. This gives a very nice smooth curve that goes through the points and does not stray too far from what you would draw naturally. The following graph shows the natural cubic spline (in red) for the six points in question #1 as well as the fifth-degree polynomial

$$y = \frac{1}{18}x^5 - \frac{23}{24}x^4 + \frac{67}{12}x^3 - \frac{895}{72}x^2 + \frac{35}{4}x$$

that also fits the points. The spline does a better job matching the points[4].



[4] https://www.desmos.com/calculator/pp0jqbswkr