General Maths Unit 4

Chapter 14

Networks and Decision Maths Module
Flow, Matching and Scheduling Problems

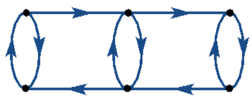
- Network Flow (Ex 14A)
- Matching and Allocation Problems (Ex 14B)
- Precedence Tables and allocation Problems (Ex 14C)
- Scheduling Problems (Ex 14D)
- Crashing (Ex 14E)

Flow Problems – Ex 14a

Directed Graphs

- Weighted graphs are when there is numerical information on the connections between the vertices. When there is **numerical information we call it a network.**

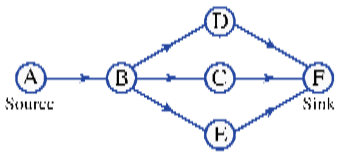
- A **directed graph or digraph** records directions information on networks using arrows on the edges. The arrows show the direction of the flow.



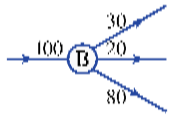
Minimum Flow
Network Flow
- Examples of networks are: road and rail systems, airline flight paths,
- Networks that allow the transfer of information are: internet, telephone lines
- All these show network **flow** – some can flow in both directions some in one direction only. Arrows show the flow
- Quantity of flow is also important.

Flow capacities
- Starting vertex is called the **source** – all flows commence from here
- The flow goes through the network to the end vertex(s) which is called the **sink**.
- Flow capacity of an edge – amount of flow that an edge can allow through if it is not connected to any other edges
- **Inflow** of a vertex – total of the flows of all edges leading into the node
- **Outflow** of a vertex – **minimum** value obtained when one compares the inflow and the sum of the capacities of all edges leaving the node.

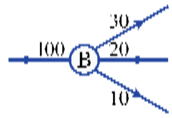
Examples



All flow commences at A. It is therefore the source. All flow converges on F indicating it is the sink.



B still has an inflow of 100 but now the capacity of the edges leaving B is 130 (80 + 20 + 30). The outflow from B is now 100.



B has an inflow of 100. The flow capacity of the edges leaving B is 30 + 20 + 10 = 60. The outflow is the smaller of these two total values, which is 60.

Summary

source → flow through network → sink

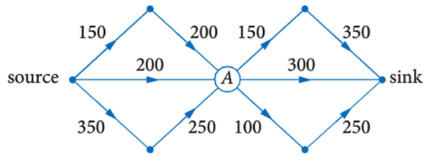
Example

The directed graph shows the flow capacities in litres per minute. Determine

a the inflow capacity of vertex A

b the outflow capacity of vertex A

c the maximum flow out of vertex A.



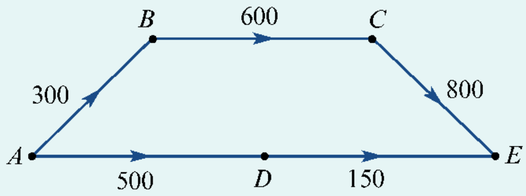
Maximum flow

Maximum flow

If pipes of different capacities are connected one after the other, the *maximum flow* through the pipes is equal to the *minimum* capacity of the individual pipes.

Example

In the digraph shown on the right, the vertices A, B, C, D and E represent towns. The edges of the graph represent roads and the weights of those edges are the maximum number of cars that can travel on the road each hour. The roads allow only one-way travel.



a Find the maximum traffic flow from A to E through town C.

b Find the maximum traffic flow from A to E overall.

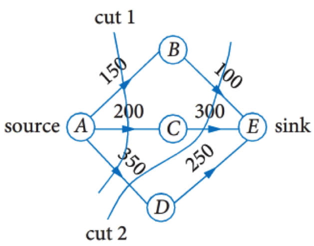
c A new road is being built to allow traffic from town D to town C. This road can carry 500 cars per hour.

  i Add this road to the digraph.

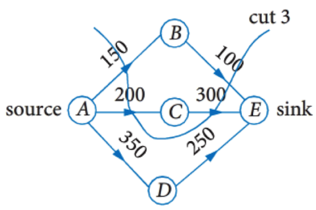
  ii Find the maximum traffic flow from A to E overall after this road is built.

The capacity of a cut

A cut through a network must stop all flow from the start (source) to the end (sink)



valid cuts

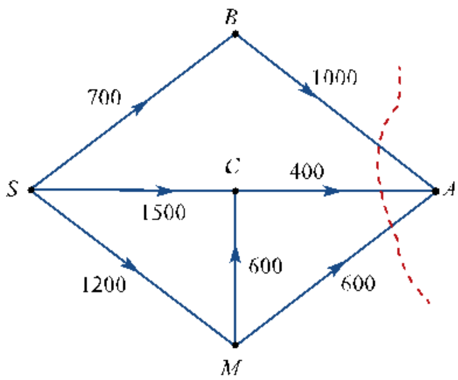


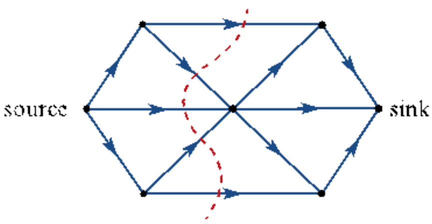
invalid cuts

Cuts

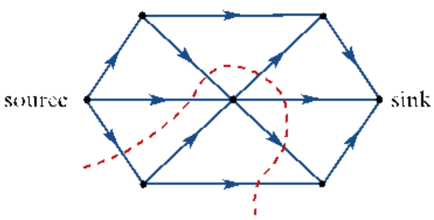
It is difficult to determine the maximum flow by inspection for directed networks that involve many vertices and edges. We can simplify the search for maximum flow by searching for **cuts** within the digraph.

A cut divides the network into two parts, completely separating the source from the sink. It is helpful to think of cuts as imaginary breaks within the network that completely block the flow through that network. For the network or water pipes shown in this diagram, the dotted line is a cut. This cut completely blocks the flow of water from the source (S) to the sink (A).





The dotted line on the graph above is a cut because it separates the source and the sink completely. No material can flow from the source to the sink.

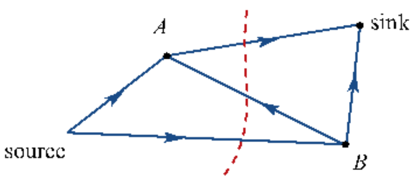


The dotted line on the graph above is *not* a cut because material can still flow from the source to the sink. Not all of the pathways from source to sink have been blocked by the cut.

Capacity of a cut

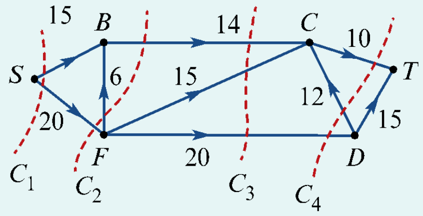
The **cut capacity** is the sum of all the capacities of the edges that the cut passes through, taking into account the direction of flow. The capacity of an edge is only counted if it flows from the source side to the sink side of the cut.

In the simple network shown, the cut passes through three edges. The edge B to A is not counted in the capacity of the cut because the flow for that edge is from the sink side to the source side of the cut.



Example

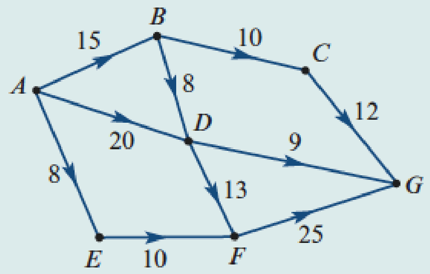
Calculate the capacity of the four cuts shown in the network on the right. The source is vertex S and the sink is vertex T.



Calculating Maximum Flow by Tracking Through a Network

Example 4 Calculating maximum flow

The koala sanctuary in Cowes allows visitors to walk through their park. The park is represented by a network below, where each edge represents one-way tracks for visitors through the park. The direction of travel on each track is shown by an arrow. The numbers on the edges indicate the maximum number of people who are permitted to walk along each track each hour.



a Starting at A, how many people are permitted to walk to G each hour?

b Given that one group of nine people would like to walk from A to G together as a group, list all the different routes they could take so that the entire group of nine will stay together for the duration of their walk.

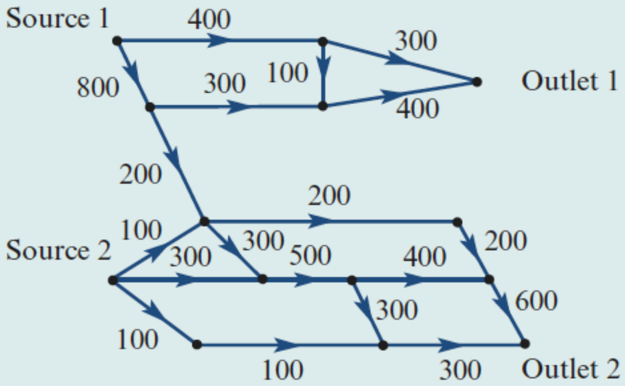
c What is the largest group of people that could walk through the koala sanctuary if they must stay together in a group for the entire duration of the walk?

Calculating Maximum Flow from more than one source

Example 5 Calculating maximum flow from more than one source

Water enters a network of pipes at either Source 1 or Source 2 and flows out at either Outlet 1 or Outlet 2.

The numbers next to the arrows represent the maximum rate, in kilolitres per minute, at which water can flow through each pipe.



Determine the maximum rate, in kilolitres per minute, at which water can flow from these pipes into the ocean at Outlet 1 and Outlet 2.

Note that although this method gives us the maximum flow for each outlet, we cannot always add these values up to find the total maximum flow through the system, because we might not be able to achieve maximum flow for every outlet at the same time.

Summary

Cut, cut capacity and minimum cut capacity

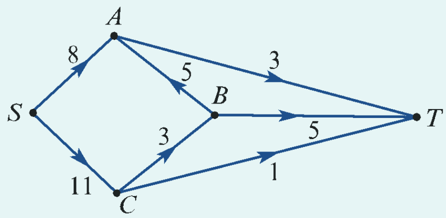
A *cut* is an imaginary line across a directed graph that completely separates the *source* (start of the flow) from the *sink* (destination of the flow).

The *cut capacity* is the sum of the capacities of the edges that are cut. Only edges that flow from the source side of the cut to the sink side of the cut are included in a cut capacity calculation.

The ***minimum cut*** *capacity* possible for a graph equals the *maximum flow* through the graph.

Example

Determine the maximum flow from S to T for the digraph shown on the right.

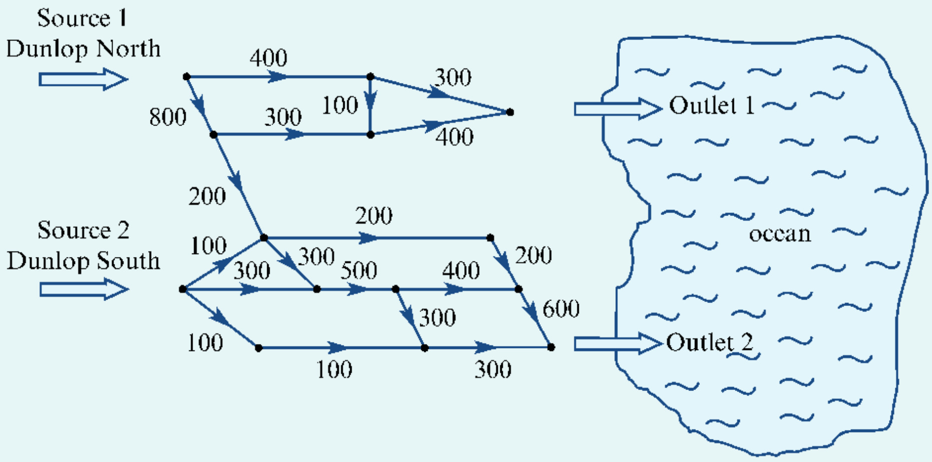


Example

Storm water enters a network of pipes at either Dunlop North (Source 1) or Dunlop South (Source 2) and flows into the ocean at either Outlet 1 or Outlet 2.

On the network diagram below, the pipes are represented by straight lines with arrows that indicate the direction of the flow of water. Water cannot flow through a pipe in the opposite direction.

The numbers next to the arrows represent the maximum rate, in kilolitres per minute, at which storm water can flow through each pipe.

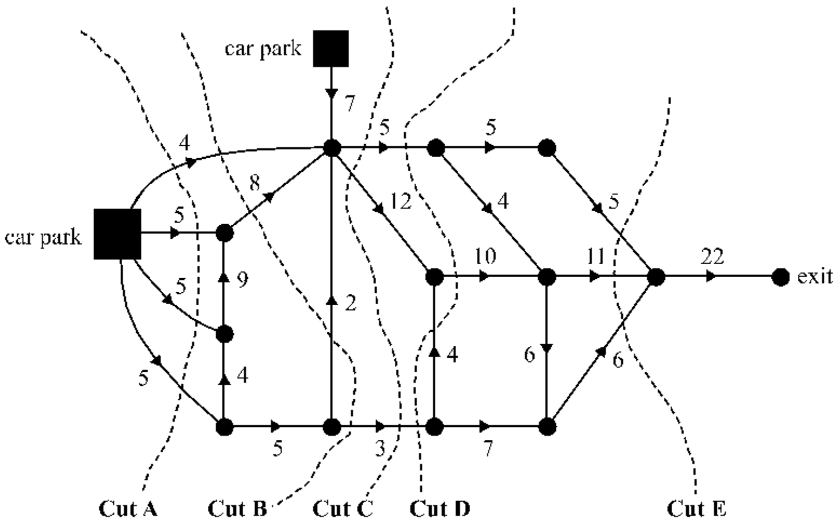


Determine the maximum rate, in kilolitres per minute, at which water can flow from these pipes into the ocean at Outlet 1 and Outlet 2.

Exam Questions
2014

Question 9

A network of tracks connects two car parks in a festival venue to the exit, as shown in the directed graph below.



The arrows show the direction that cars can travel along each of the tracks and the numbers show each track's capacity in cars per minute.

Five cuts are drawn on the diagram.

The maximum number of cars per minute that will reach the exit is given by the capacity of

A. Cut A

B. Cut B

C. Cut C

D. Cut D

E. Cut E

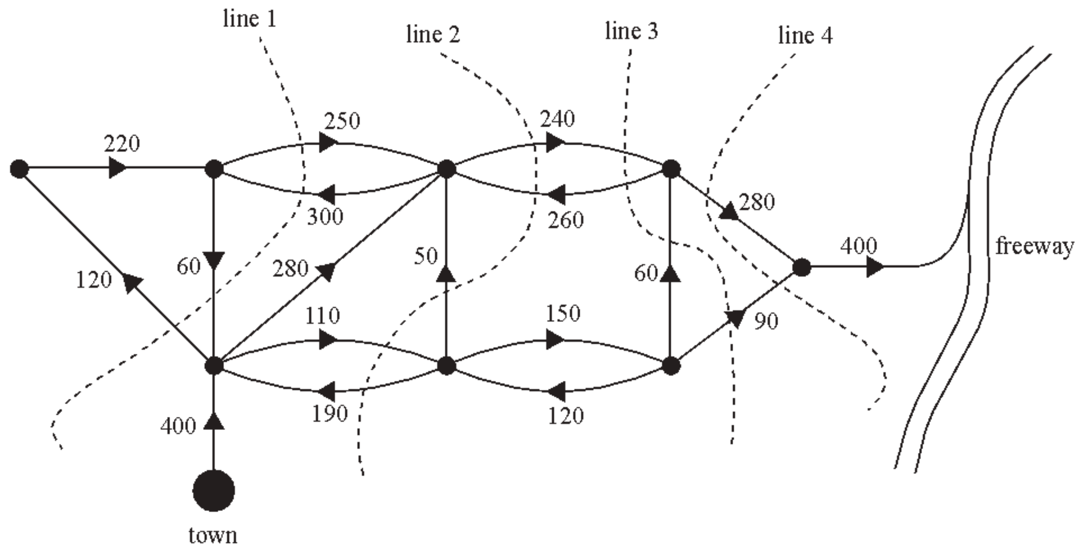
2012

Question 7

Vehicles from a town can drive onto a freeway along a network of one-way and two-way roads, as shown in the network diagram below.

The numbers indicate the maximum number of vehicles per hour that can travel along each road in this network. The arrows represent the permitted direction of travel.

One of the four dotted lines shown on the diagram is the minimum cut for this network.



The maximum number of vehicles per hour that can travel through this network from the town onto the freeway is

A. 310

B. 330

C. 350

D. 370

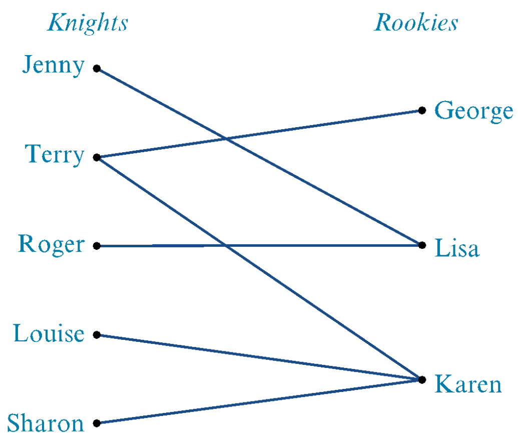
E. 390

Ex 14a

Matching and allocation problems – Ex 14b

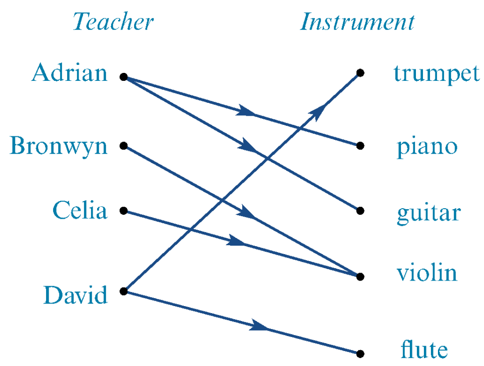
Bipartite Graphs

- A graph which connects one vertex from one group to a vertex in the other group.



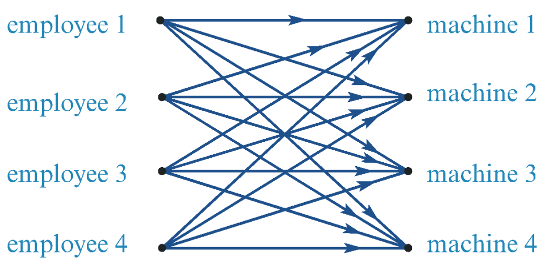
Directed bipartite graphs

- One vertex from one group is matched or allocated to one or more vertices of the second group – used for allocations



Complete weighted bipartite graphs

- There is an equal number of vertices in each group



The Assignment Problem and bipartite graphs
Optimal allocation – Key terms and definitions

- Used when there are a number of jobs which can be done and the same number of people do the jobs, but in different time (or at a different cost)
- **Optimal allocation is allocating the jobs so that the time or cost is minimised.**
- Sometimes it may be obvious, however sometimes requires more work.
- An allocation matrix has the **source nodes as rows** and the **demand nodes as columns** and is generally a square matrix.

Skills needed to determine optimal allocation

Row reduction of a matrix – this involves finding the smallest number in each row and subtracting this number from all numbers in that row.
Very important skill!!!
Example
Perform row reduction on the following matrix which represents time in hours…

$$\begin{bmatrix} 16 & 14 & 20 & 13 \\ 15 & 16 & 17 & 16 \\ 19 & 13 & 13 & 18 \\ 22 & 26 & 20 & 24 \end{bmatrix}$$

Column reduction - this involves finding the smallest number in each column and subtracting this number from all numbers in that row.

Example
Perform column reduction on the matrix you ended up with above after you performed row reduction

Steps to perform optimal allocation

1a Perform row reduction by finding the samllest number in each row, and subtracting this number from all numbers in that row.

$$\begin{bmatrix} 4 & 3 & 7 & 3 \\ 9 & 4 & 6 & 5 \\ 5 & 6 & 7 & 8 \\ 4 & 8 & 3 & 5 \end{bmatrix}$$

1b Attempt and allocation by covering all the zeros in the row-reduced matrix. Proceed to step 3 if the minimum number of lines equals the number of rows in the matrix.

2a Produce a bipartite graph

2b List all the possible allocations and determine the smallest total.

The Hungarian algorithm

****Unfortunately, there are two cases where the basic algorithm will not producte th optimal allocation:

1. When there are not enough zeros to draw the right number of straight lines in step 1b
2. When there are too many zeros so that one person may be allocated more than one job in step 2a.

The Hungarian Algorithm!!

- Used for optimum allocations

Step 1

Perform row and column reduction

$$\begin{bmatrix} 17 & 24 & 42 & 21 \\ 25 & 18 & 19 & 20 \\ 29 & 14 & 31 & 22 \\ 11 & 20 & 17 & 14 \end{bmatrix}$$

Step 2

Try and cover up all the zeros with the least amount of lines.
If the number of lines equals the number of jobs – set up a bipartite graph and proceed with allocations.
If not step 3.

Step 3

Find the smallest uncovered number.

Step 4

Subtract the smallest uncovered number from all the uncovered numbers.

Step 5
Try to cover all zeros with the least amount of lines. If not possible repeat the above two steps!
If number of jobs equals number of lines…

Step 6
Produce a bipartite graph

Step 7
List all possible allocations and determine the smallest total, by looking back at your original matrix.

2012 – Exam 2

Question 3

Four tasks, *W*, *X*, *Y* and *Z*, must be completed.

Four workers, Julia, Ken, Lana and Max, will each do one task.

Table 1 shows the time, in minutes, that each person would take to complete each of the four tasks.

Table 1

	Worker			
Task	**Julia**	**Ken**	**Lana**	**Max**
W	26	21	22	25
X	31	26	21	38
Y	29	26	20	27
Z	38	26	26	35

The tasks will be allocated so that the total time of completing the four tasks is a minimum.

The Hungarian method will be used to find the optimal allocation of tasks.

Step 1 of the Hungarian method is to subtract the minimum entry in each row from each element in the row.

Table 2

	Worker			
Task	**Julia**	**Ken**	**Lana**	**Max**
W	5	0	1	4
X	10	5	0	
Y	9	6	0	7
Z	12	0	0	9

a. Complete step 1 for task *X* by writing down the number missing from the shaded cell in Table 2.

1 mark

The second step of the Hungarian method ensures that all columns have at least one zero.

The numbers that result from this step are shown in Table 3 below.

Table 3

	Worker			
Task	**Julia**	**Ken**	**Lana**	**Max**
W	0	0	1	0
X	5	5	0	13
Y	4	6	0	3
Z	7	0	0	5

b. Following the Hungarian method, the smallest number of lines that can be drawn to cover the zeros is shown dashed in Table 3.

These dashed lines indicate that an optimal allocation cannot be made yet.

Give a reason why.

1 mark

c. Complete the steps of the Hungarian method to produce a table from which the optimal allocation of tasks can be made.

Two blank tables have been provided for working if needed.

	Worker			
Task	**Julia**	**Ken**	**Lana**	**Max**
W				
X				
Y				
Z				

	Worker			
Task	**Julia**	**Ken**	**Lana**	**Max**
W				
X				
Y				
Z				

1 mark

d. Write the name of the task that each person should do for the optimal allocation of tasks.

Worker	**Task**
Julia	
Ken	
Lana	
Max	

2 marks

Ex 14b

Precedence Tables and Activity Networks – Ex 14c

- In many projects there are many individual activities which need to be completed before a project is finished. Often one activity cannot be performed until other activities are completed. For example, the plasterer cannot begin work on a house until the frame is up and the house is sealed.
- For any project, if activity A must be completed before activity B can begin, then activity A is said to be an immediate predecessor for activity B. There can be multiple predecessors. This can be shown in a precedence table.
- The information in a precedence table can be used to draw an activity network, which shows the order of activities, including predecessors from start to finish of the project.
- The activities in the project are represented by the edges, so the edges must be labelled not the vertices.

Activity	*Immediate predecessors*
A	–
B	–
C	A
D	B
E	B
F	C, D
G	E, F

Draw an acitivity network for the above table

Guidelines to follow when drawing a network diagram

- Use a vertex to represent the start of the network.
- Look for any activities that do not have any predecessors. These will be your starting activities.
- Multiple predecessors to an activity will all end at the same vertex.
- An activity should not be represented by more than one edge in the network.
- Two vertices can be connected by one edge only.
- A vertex indicating the completion of the project needs to be included in the network.

Example

Draw an activity network from the precedence table shown on the right.

In this solution, the activity network will be drawn from the finish back to the start.

Activity	*Immediate predecessors*
A	–
B	A
C	A
D	A
E	B
F	C
G	D
H	E, F, G

Dummy Activities

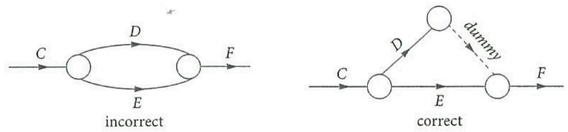
- A dummy activity needs to be added to a network to ensure that no two vertices are connected by multiple edges or to maintain precedence structure.
- A dummy activity has zero time and is shown as a directed edge with a broken line.

Types

1. Multiple Edges

A network cannot be drawn with multiple edges connecting the same vertices. In the example below, the directed graph cannot be drawn with two vertices both connected by activities D and E. A dummy activity needs to be included. It can occur before or after activities D or E to overcome this problem.

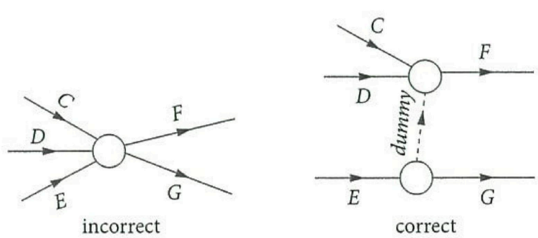
Activity	Immediate predecessor
D	C
E	C
F	D, E



2. Logic difficulties in the precedence structure

The second use of dummy activities is to ensure the logic of the network is maintained. The first network has activity G preceded by activities C, D and E, which does not follow the logice of the activity table. A second vertex and a

Activity	Immediate predecessor
F	C, D, E
G	E



dummy activity need to be included so activities F and G do not both start from the same vertex and activities C, D and E do not finish at the same vertex.

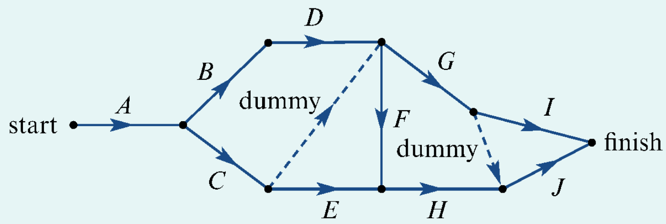
Example

Draw an activity network from the precedence table shown on the right.

Activity	*Immediate predecessors*
A	–
B	–
C	A
D	B
E	C, D
F	C
G	E, F

Example

Write down a precedence table for the activity network shown on the right.



Ex 14c

Scheduling Problems – Ex 14d

Critical Path Analysis

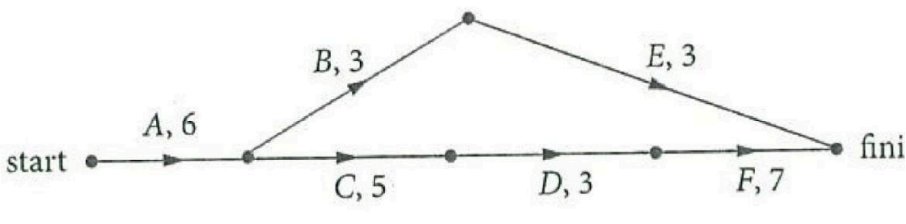
Critical path analysis is a step-by-step project management technique that is used to examine every activity in a project and how each affects the project completion time.

Forward Scanning to determine EST

By forward scanning we can calculate the **earliest start times** for each activity
The **EST** is the earliest that any activity can be started after all prior activities have been completed.

To determine the earliest start times

1. Begin with the first vertex, which has an earliest start time of 0. We will use the



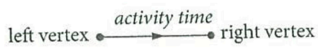
convention of writing the earliest starting times in the box next to each vertex.

2. ESTs are calculated from left to right
3. Add the activity time to EST of the previous vertex. If more than one activity leads to the vertex, the **HIGHEST** figure obtained becomes the new EST.
4. Continue until the finish is reached.

Example

Determine the earliest start times (EST) for each activity in the network. Activity times are in hours.

Backward Scanning to determine LST

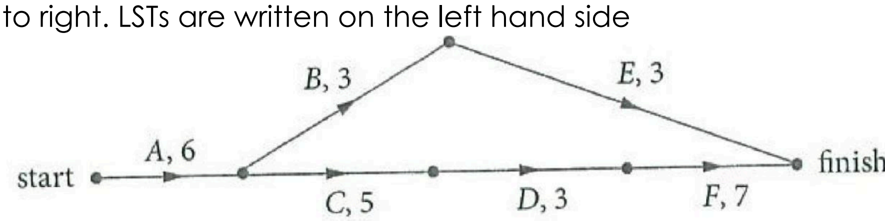


Backward scanning is the process used to find the **latest start time (LST)** for an activity. This is the latest time you can start the activity without affecting the project completion time.

The LST at the left vertex = the LST at the right vertex – the activity time

To determine the LST

1. Commence LST calculations at the 'finish' vertex. At the **finish vertex LST=EST.**
2. Work backwards from left to right. LSTs are written on the left hand side of the box in a different color.
3. To find the LST at the left vertex, work backwards, subtracting the activity time for the LST at the right vertex.



4. Where there is more than one path to the previous vertex, take the SMALLEST result as the LST.
5. The LST at the first vertex must be zero.

**Must fill out EST before finding LST!!

Example

Determine the latest start times (LST) fore each activity in the network. Activity times shown are in hours.

Identifying Critical Paths

There are often multiple paths between the start and finish vertices of a network. The **critical path** is the longest time path between the start and finish and determines the project completion time.

Special points…

- On the critical path, activities have equal EST and LST values
- Activities on the critical path are called critical activities
- Activities not on the critical path are called non-critical activities.

Steps used to find the Critical Path

1. Draw 2 boxes at each node (vertex)

EST *Use forward scanning	LST *Use backward scanning

2. Start at the beginning by filling in all the boxes on the left in one color – this is forward scanning
   *When more than two paths to a node record both values and put the **largest** one in the box

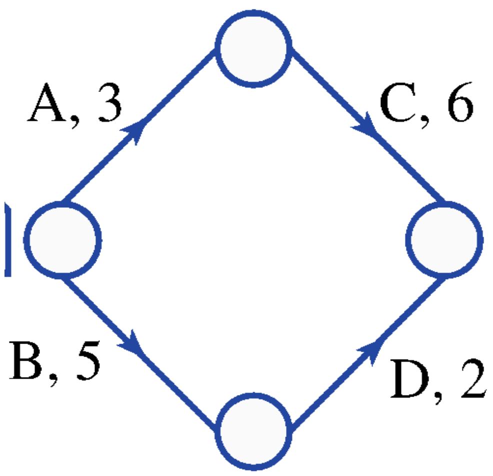
3. Go to the finish and work backwards filling in the left hand boxes in a different color – this is forward scanning
   *When more than two paths to a node record both values and put the **smallest** one in the box

4. The critical path occurs when the numbers in both boxes are the same.

5. Float time involves the difference in time between those paths that cannot be delayed and those that can.

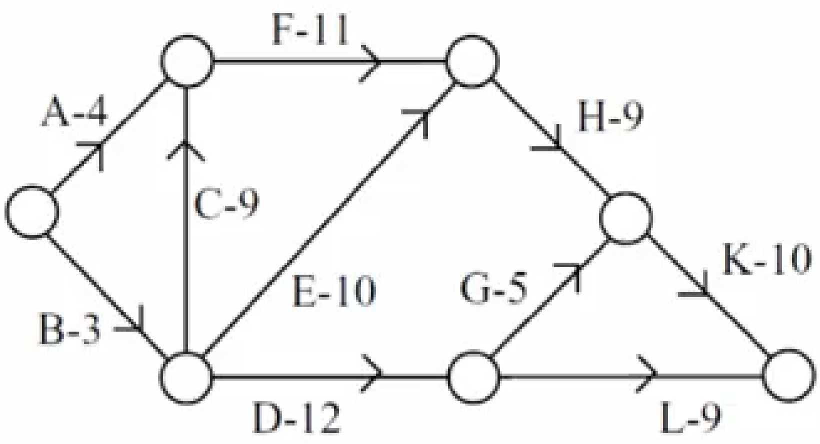
Example (Easy)

The network diagram below has been constructed for a project manager. Use forward and backward scanning to clearly display the **critical path** and to list any **float times**.



Example 2

The network diagram shown below is for the construction of a fashion exhibition stage. Activity times are shown in hours.



Question 7

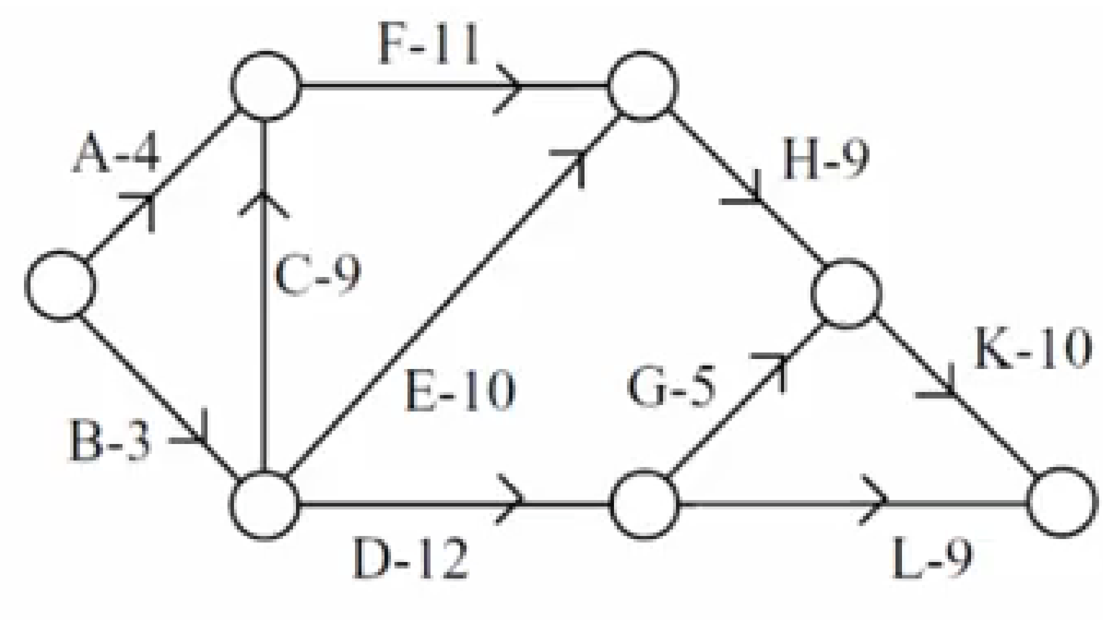
The immediate predecessor(s) of activity E is/are

Question 8

The earliest time, after the start of the project, that activity H can begin is

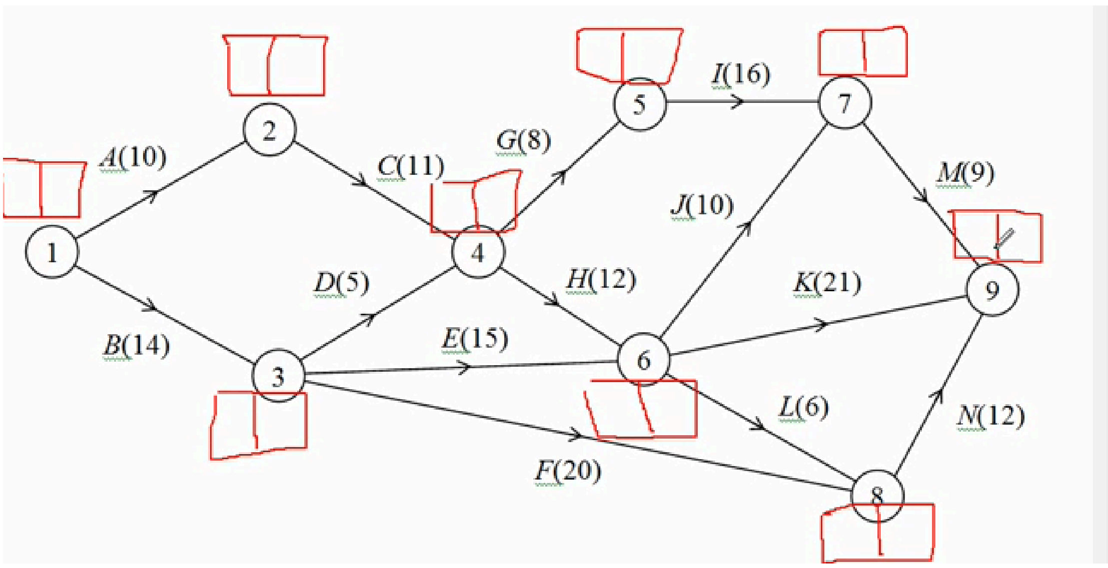
Question 9

The critical path for the entire project is



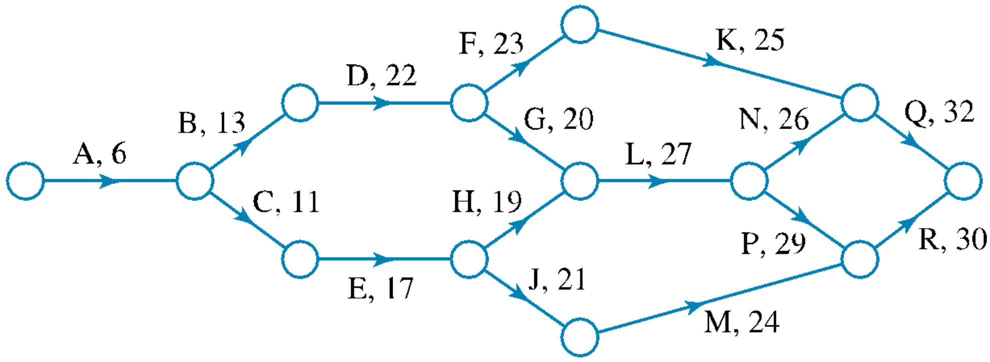
Another Example

Find the critical path



Example 2

The project plan for a new computer software program is shown in the figure below. Time is measured in days. Determine the earliest completion time.



Ex 14d

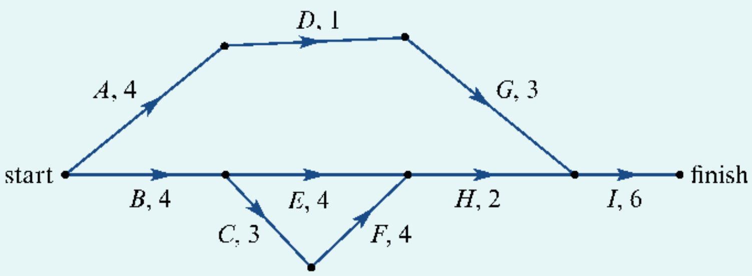
Crashing – Ex 14e

- **Crashing** involves shortening the critical path to speed up the completion time of a project. This results in a new critical path being formed.

Example

Example 9 Crashing

A community centre is to be built on a new housing estate. Nine activities have been identified for this building project. The activity network shown on the right shows the activities and their completion times in weeks.



a Determine the minimum time, in weeks, to complete this project.

b Determine the slack time, in weeks, for activity D.

c Write down the critical path of this project.

The builders of the community centre are able to speed up the project.

Some of the activities can be reduced in time at an additional cost.

The activities that can be reduced in time are A, C, E, F and G.

d Which of these activities, if reduced in time individually, would not result in an earlier completion of the project?

The owner of the estate is prepared to pay the additional cost to achieve early completion.

The cost of reducing the time of each activity is \$5000 per week.

The maximum reduction in time for each one of the five activities, A, C, E, F and G is 2 weeks.

e Determine the minimum time, in weeks, for the project to be completed now that certain activities can be reduced in time.

f Determine the minimum additional cost of completing the project in this reduced time.

based on VCAA (2007)

Ex 14e

Congratulations!!!!! We have finished the course!!!!!!