Analysis
Lesson 17
Uniform Convergence and Pointwise Convergence and the Arzela-Ascoli Theorem

This Lesson has five parts. The classwork is required. You should also try the 10 homework problems at least starting all the proofs.

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Part I Uniform Convergence
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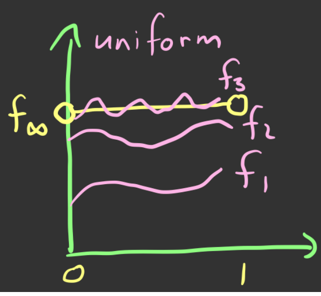
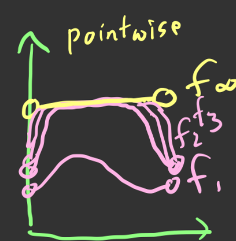
Watch the UnifConv-1to7 Playlist and do the classwork.

Uniform vs Pointwise Convergence of Functions

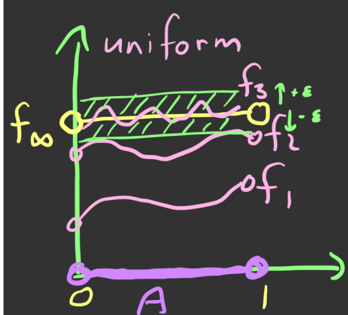


$$f_j \longrightarrow f_\infty$$

Domain $(0,1)$



Defn: A sequence of functions $f_j : A \to \mathbb{R}$ **converges uniformly** to $f_\infty : A \to \mathbb{R}$ on the set A if

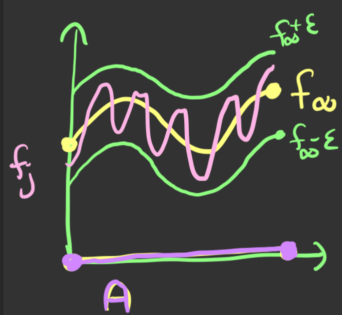


$$\forall \varepsilon > 0 \quad \exists N_\varepsilon \in \mathbb{R} \text{ s.t } \forall j \geq N_\varepsilon$$

we have $\left| f_j(x) - f_\infty(x) \right| < \varepsilon \quad \forall x$

$$\sup_{x \in A} \left| f_j(x) - f_\infty(x) \right| < \varepsilon$$

Defn: A sequence of functions $f_j : A \to \mathbb{R}$ converges uniformly to $f_\infty : A \to \mathbb{R}$ on the set A if



$\forall \varepsilon > 0 \ \exists N_\varepsilon \in \mathbb{R}$ s.t $\forall j \geq N_\varepsilon$

we have $|f_j(x) - f_\infty(x)| < \varepsilon \quad \forall x \in A$

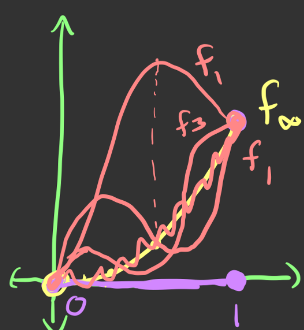
$$\sup_{x \in A} |f_j(x) - f_\infty(x)| < \varepsilon$$

← the same →

$$f_\infty(x) - \varepsilon < f_j(x) < f_\infty(x) + \varepsilon \quad \forall x \in A$$

Classwork: Prove $f_j(x) = x^2 + \frac{1}{j}\sin(j\pi x)$ converges uniformly to $f_\infty(x) = x^2$ on $[0,1]$.

Show: $\forall \varepsilon > 0\ \exists N_\varepsilon$ s.t. $\forall j \geq N_\varepsilon\ |f_j(x) - f_\infty(x)| < \varepsilon\ \forall x \in [0,1]$



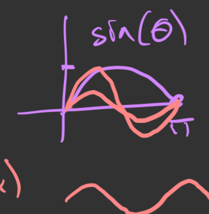
First few terms:

$f_1(x) = x^2 + \frac{1}{1}\sin(1\pi x) = x^2 + \sin(\pi x)$

$f_2(x) = x^2 + \frac{1}{2}\sin(2\pi x)$

$f_3(x) = x^2 + \frac{1}{3}\sin(3\pi x)$

$f_{10}(x) = x^2 + \frac{1}{10}\sin(10\pi x)$



Proof next!

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MAT320S21

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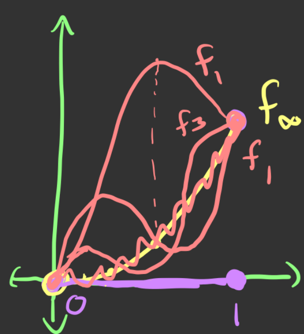
Classwork: Prove $f_j(x) = x^2 + \frac{1}{j}\sin(j\pi x)$ converges uniformly to $f_\infty(x) = x^2$ on $[0,1]$.

Show: $\forall \varepsilon > 0\ \exists N_\varepsilon$ s.t. $\forall j \geq N_\varepsilon\ |f_j(x) - f_\infty(x)| < \varepsilon\ \forall x \in [0,1]$



Proof Structure:

* Cannot depend on x

① Given any $\varepsilon > 0$ Choose $N_\varepsilon = \square$

② Whenever $j \geq N_\varepsilon$ we have ....

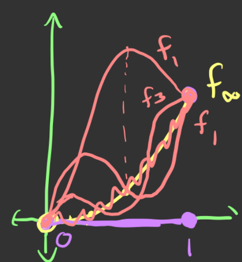
Solve up

Justify Down

(final) $\left| x^2 + \frac{1}{j}\sin(j\pi x) - x^2 \right| < \varepsilon\ \ \forall x \in [0,1]$

Classwork: Prove $f_j(x) = x^2 + \frac{1}{j}\sin(j\pi x)$ converges uniformly to $f_\infty(x) = x^2$ on $[0,1]$.

Show: $\forall \varepsilon > 0 \ \exists N_\varepsilon$ s.t. $\forall j \geq N_\varepsilon \ |f_j(x) - f_\infty(x)| < \varepsilon \ \forall x \in [0,1]$



Proof Structure:

① Given any $\varepsilon > 0$ Choose $N_\varepsilon = \boxed{}$ (* Cannot depend on x)

② Whenever $j \geq N_\varepsilon$ we have . . . .

(solve upwards)

$\frac{1}{\varepsilon} < j$

$1 < j \cdot \varepsilon$

$|\sin(j\pi x)| \leq 1 < j \cdot \varepsilon \quad \forall x \in [0,1]$ (Using $|\sin\theta| \leq 1$ for all $\theta \in \mathbb{R}$)

$|\sin(j\pi x)| < j \cdot \varepsilon \quad \forall x \in [0,1]$

$\frac{1}{j}|\sin(j\pi x)| < \varepsilon \quad \forall x \in [0,1]$

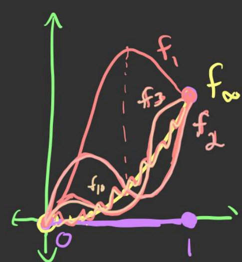
$\left|\frac{1}{j}\sin(j\pi x)\right| < \varepsilon \quad \forall x \in [0,1]$

(final) $\left|x^2 + \frac{1}{j}\sin(j\pi x) - x^2\right| < \varepsilon \quad \forall x \in [0,1]$

↑ Be sure to copy this as you solve upwards.

Classwork: Prove $f_j(x) = x^2 + \frac{1}{j}\sin(j\pi x)$ converges uniformly to $f_\infty(x) = x^2$ on $[0,1]$.

Show: $\forall \varepsilon > 0\ \exists N_\varepsilon$ s.t. $\forall j \geq N_\varepsilon$ $|f_j(x) - f_\infty(x)| < \varepsilon$ $\forall x \in [0,1]$



Proof Structure:

* ~~Cannot depend~~ Does not depend on x

① Given any $\varepsilon > 0$ Choose $N_\varepsilon = \boxed{\frac{1}{\varepsilon} + 1}$ GOOD

② Whenever $j \geq N_\varepsilon$ we have . . . .

$\frac{1}{\varepsilon} < j$

③ $1 < j \cdot \varepsilon$

Using $|\sin\theta| \leq 1$ for all $\theta \in \mathbb{R}$

④ $|\sin(j\pi x)| \leq 1 < j \cdot \varepsilon$ $\forall x \in [0,1]$

⑤ $|\sin(j\pi x)| < j \cdot \varepsilon$ $\forall x \in [0,1]$

⑥ $\frac{1}{j}|\sin(j\pi x)| < \varepsilon$ $\forall x \in [0,1]$

⑦ $\left|\frac{1}{j}\sin(j\pi x)\right| < \varepsilon$ $\forall x \in [0,1]$

(final) $\left|x^2 + \frac{1}{j}\sin(j\pi x) - x^2\right| < \varepsilon$ $\forall x \in [0,1]$

① $\varepsilon > 0$ so $\frac{1}{\varepsilon}$ is defined

② Choice of N_ε in step 1

③ $a < b$ and $\varepsilon > 0$ $\Rightarrow ac < bc$

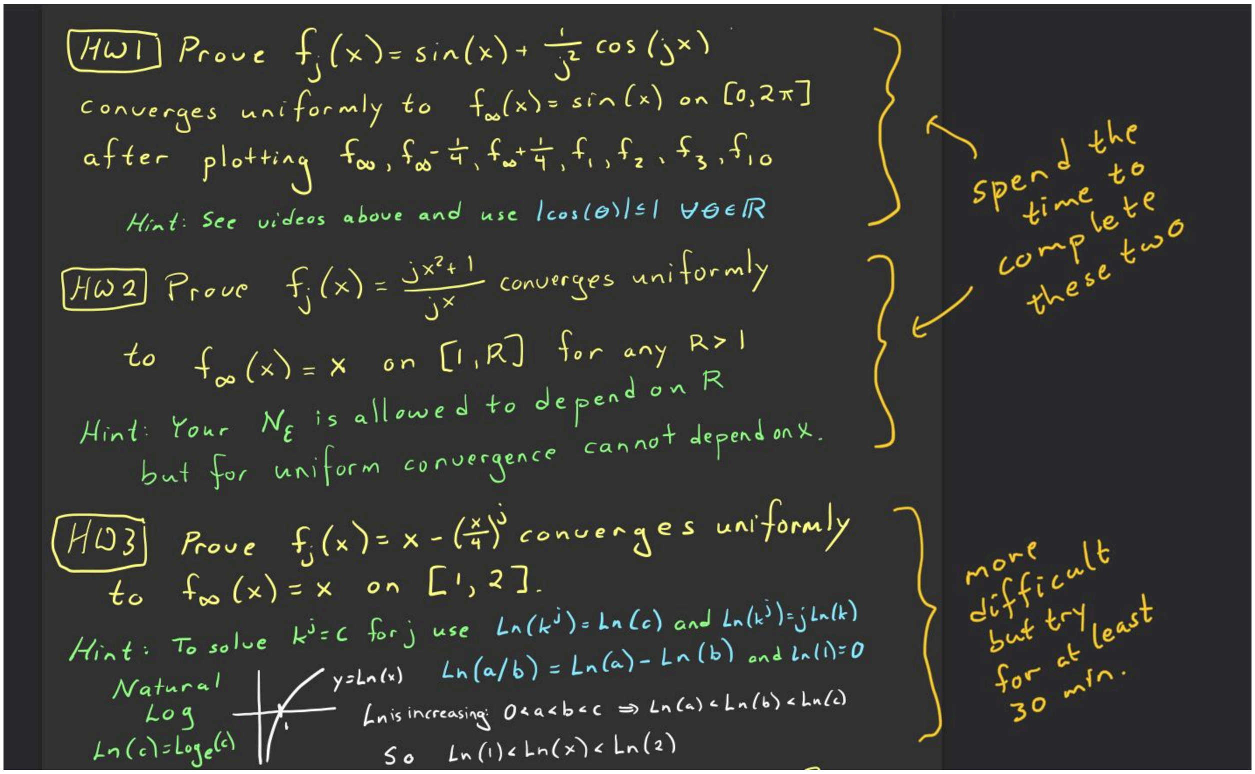
④ $|\sin\theta| \leq 1$ $\forall \theta \in \mathbb{R}$

⑤ by step 4

⑥ $a < b$ and $c = \frac{1}{j} > 0$ $\Rightarrow ac < bc$

⑦ $a|b| = |ab|$ when $a \geq 0$

(final) algebra $x^2 - x^2 = 0$



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Part 2: When a proof of Uniform Convergence fails

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Watch Unif-Conv-8to14 and do the classwork.

Classwork: Prove $f_j(x) = x^j$ converges uniformly to $f_\infty(x) = 0$ on $[0, \frac{1}{2}]$ but not on $[0, 1]$.

pause + try

Classwork: Prove $f_j(x) = x^j$ converges uniformly to $f_\infty(x) = 0$ on $[0, \frac{1}{2}]$ but not on $[0,1]$.

$\forall \varepsilon > 0 \ \exists N_\varepsilon$ s.t $\forall j \geq N_\varepsilon \ |x^j - 0| < \varepsilon \ \forall x \in [0, \frac{1}{2}]$

Proof Structure:

① Given $\varepsilon > 0$ Choose $N_\varepsilon = \boxed{}$

↑ cannot depend on x for $x \in [0, \frac{1}{2}]$

May depend on 0 and $\frac{1}{2}$.

② Whenever $j \geq N_\varepsilon$ we have...

↑ Solve up for j

(final) $|x^j - 0| < \varepsilon \ \forall x \in [0, \frac{1}{2}]$

Classwork: Prove $f_j(x) = x^j$ converges uniformly to $f_\infty(x) = 0$ on $[0, \frac{1}{2}]$ but not on $[0,1]$.

$\forall \varepsilon > 0\ \exists N_\varepsilon$ s.t. $\forall j \geq N_\varepsilon\ |x^j - 0| < \varepsilon\ \forall x \in [0, \frac{1}{2}]$

Proof (1) Given $\varepsilon > 0$ Choose $N_\varepsilon = \frac{Ln(\varepsilon)}{Ln(x)} + 1$ ~~X~~

(2) Whenever $j \geq N_\varepsilon$ we have...

$j > Ln(\varepsilon)/Ln(x)$

$j\, Ln(x) < Ln(\varepsilon)$

$Ln(x^j) < Ln(\varepsilon)$

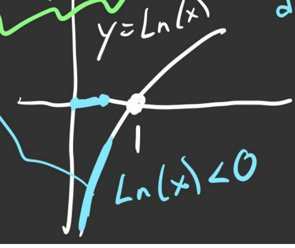
$x^j < \varepsilon$ (because $x > 0$)

(final) $|x^j - 0| < \varepsilon\ \forall x \in [0, \frac{1}{2}]$

Cannot depend on x

cannot depend on x for $x \in [0, \frac{1}{2}]$

May depend on 0 and $\frac{1}{2}$



$\forall \varepsilon > 0 \ \exists N_\varepsilon$ s.t $\forall j \geq N_\varepsilon \ |x^j - 0| < \varepsilon \ \forall x \in [0, \frac{1}{2}]$

Proof (1) Given $\varepsilon > 0$ Choose $N_\varepsilon = \left\lceil \frac{Ln(\varepsilon)}{Ln(\frac{1}{2})} \right\rceil + 1 = \lceil -Ln(2) Ln(\varepsilon) \rceil + 1$

(2) Whenever $j \geq N_\varepsilon$ we have $j > \frac{Ln(\varepsilon)}{Ln(1/2)}$

email me if you have another N_ε.

now solve for j:

$j \, Ln(\frac{1}{2}) < Ln(\varepsilon)$

$Ln(\frac{1}{2}) < \frac{1}{j} Ln(\varepsilon)$

switch < to > because $Ln(\frac{1}{2}) < 0$

$Ln(\frac{1}{2}) = Ln(2^{-1}) = -Ln(2)$

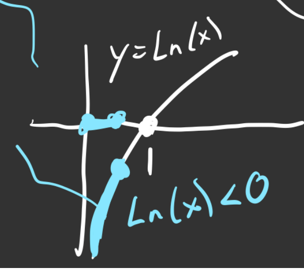
$Ln(x) \leq Ln(\frac{1}{2}) < \frac{1}{j} Ln(\varepsilon) \ \forall x \in [0, 1/2]$

$Ln(x) < \frac{1}{j} Ln(\varepsilon)$

$j \, Ln(x) < Ln(\varepsilon) \ \forall x \in [0, \frac{1}{2}]$

$Ln(x^j) < Ln(\varepsilon) \ \forall x \in [0, \frac{1}{2}]$

$x^j < \varepsilon \ \forall x \in [0, \frac{1}{2}]$



(final) $|x^j - 0| < \varepsilon \ \forall x \in [0, \frac{1}{2}]$

$\forall \varepsilon > 0 \;\; \exists N_\varepsilon$ s.t $\forall j \geq N_\varepsilon \;\; |x^j - 0| < \varepsilon \;\; \forall x \in [0, \frac{1}{2}]$

Proof ① Given $\varepsilon > 0$ Choose $N_\varepsilon = \boxed{\frac{\ln(\varepsilon)}{\ln(\frac{1}{2})} + 1}$

② Whenever $j \geq N_\varepsilon$ we have $j > \frac{\ln(\varepsilon)}{\ln(1/2)}$

$$j \ln(\tfrac{1}{2}) < \ln(\varepsilon)$$

$$\ln(\tfrac{1}{2}) < \tfrac{1}{j} \ln(\varepsilon)$$

$$\ln(x) \leq \ln(\tfrac{1}{2}) < \tfrac{1}{j} \ln(\varepsilon) \;\; \forall x \in [0, 1/2]$$

$$\ln(x) < \tfrac{1}{j} \ln(\varepsilon)$$

$$j \ln(x) < \ln(\varepsilon) \;\; \forall x \in [0, \tfrac{1}{2}]$$

$$\ln(x^j) < \ln(\varepsilon) \;\; \forall x \in [0, \tfrac{1}{2})$$

$$x^j < \varepsilon \;\; \forall x \in [0, \tfrac{1}{2}]$$

(final) $|x^j - 0| < \varepsilon \;\; \forall x \in [0, \frac{1}{2}]$

$\forall \varepsilon > 0 \ \exists N_\varepsilon$ s.t $\forall j \geq N_\varepsilon \ |x^j - 0| < \varepsilon \ \forall x \in [0, \tfrac{1}{2}]$

Proof ① Given $\varepsilon > 0$ Choose $N_\varepsilon = \left\lceil \frac{\ln(\varepsilon)}{\ln(\frac{1}{2})} \right\rceil + 1$

② Whenever $j \geq N_\varepsilon$ we have $j > \frac{\ln(\varepsilon)}{\ln(1/2)}$

③ $j \ln(\tfrac{1}{2}) < \ln(\varepsilon)$

④ $\ln(\tfrac{1}{2}) < \tfrac{1}{j}\ln(\varepsilon)$

⑤ $\ln(x) \leq \ln(\tfrac{1}{2}) < \tfrac{1}{j}\ln(\varepsilon) \ \forall x \in [0, 1/2]$

⑥ $\ln(x) < \tfrac{1}{j}\ln(\varepsilon)$

⑦ $j \ln(x) < \ln(\varepsilon) \ \forall x \in [0, \tfrac{1}{2}]$

⑧ $\ln(x^j) < \ln(\varepsilon) \ \forall x \in [0, \tfrac{1}{2})$

⑨ $x^j < \varepsilon \ \forall x \in [0, \tfrac{1}{2}]$

(final) $|x^j - 0| < \varepsilon \ \forall x \in [0, \tfrac{1}{2}]$

Justify Down

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$\forall \varepsilon > 0 \ \exists N_\varepsilon$ s.t $\forall j \geq N_\varepsilon \ |x^j - 0| < \varepsilon \ \forall x \in [0, \frac{1}{2}]$

Proof ① Given $\varepsilon > 0$ Choose $N_\varepsilon = \left\lceil \frac{Ln(\varepsilon)}{Ln(\frac{1}{2})} \right\rceil + 1$

① $\varepsilon > 0$ so $Ln(\varepsilon)$ is defined and $Ln(\frac{1}{2}) \neq 0$ so division is OK

② Whenever $j \geq N_\varepsilon$ we have $j > \frac{Ln(\varepsilon)}{Ln(1/2)}$

② by Choice of N_ε in step 1

③ $j\,Ln(\frac{1}{2}) < Ln(\varepsilon)$

③ $a > b$ and $c = Ln(\frac{1}{2}) < 0 \Rightarrow ac < bc$

④ $Ln(\frac{1}{2}) < \frac{1}{j} Ln(\varepsilon)$

④ $a < b$ and $c = 1/j > 0 \Rightarrow ac < bc$

⑤ $Ln(x) \leq Ln(\frac{1}{2}) < \frac{1}{j} Ln(\varepsilon) \ \forall x \in [0, 1/2]$

ERROR $Ln(0)$ is not DEFINED! ⑤ in Step 5

⑥ $Ln(x) < \frac{1}{j} Ln(\varepsilon)$

⑦ $j\,Ln(x) < Ln(\varepsilon) \ \forall x \in [0, \frac{1}{2}]$

⑧ $Ln(x^j) < Ln(\varepsilon) \ \forall x \in [0, \frac{1}{2}]$

⑨ $x^j < \varepsilon \ \forall x \in [0, \frac{1}{2}]$

final $|x^j - 0| < \varepsilon \ \forall x \in [0, \frac{1}{2}]$

Classwork: Prove $f_j(x) = x^j$ converges uniformly to $f_\infty(x) = 0$ on $[0, \frac{1}{2}]$ but not on $[0,1]$.

$\forall \varepsilon > 0\ \exists N_\varepsilon$ s.t. $\forall j \geq N_\varepsilon\ |x^j - 0| < \varepsilon\ \forall x \in [0, \frac{1}{2}]$

Proof ① Given $\varepsilon > 0$. Choose $N_\varepsilon = \left\lceil \frac{\ln(\varepsilon)}{\ln(\frac{1}{2})} + 1 \right\rceil$

② Whenever $j \geq N_\varepsilon$ we have $j > \frac{\ln(\varepsilon)}{\ln(1/2)}$

③ $j \ln(\frac{1}{2}) < \ln(\varepsilon)$

④ $\ln(\frac{1}{2}) < \frac{1}{j} \ln(\varepsilon)$

⑤ $\ln(x) \leq \ln(\frac{1}{2}) < \frac{1}{j}\ln(\varepsilon)\ \forall x \in (0, 1/2]$ ← CORRECT NOW

⑥ $\ln(x) < \frac{1}{j}\ln(\varepsilon)\ \forall x \in (0, \frac{1}{2}]$

⑦ $j \ln(x) < \ln(\varepsilon)\ \forall x \in (0, \frac{1}{2}]$

⑧ $\ln(x^j) < \ln(\varepsilon)\ \forall x \in (0, \frac{1}{2}]$

⑨ $x^j < \varepsilon\ \forall x \in [0, \frac{1}{2}]$

(final) $|x^j - 0| < \varepsilon\ \forall x \in [0, \frac{1}{2}]$

Reasons:

① $\varepsilon > 0$ so $\ln(\varepsilon)$ is defined and $\ln(\frac{1}{2}) \neq 0$ so division is OK

② by choice of N_ε in step 1

③ $a > b$ and $c = \ln(\frac{1}{2}) < 0 \Rightarrow ac < bc$

④ $a < b$ and $c = 1/j > 0 \Rightarrow ac < bc$

⑤ $\ln$ is increasing on $(0, \frac{1}{2}]$

⑥ by step 5

⑦ $a < b$ and $c = j > 0 \Rightarrow ac < bc$

⑧ $\ln(a^c) = c\ln(a)$

⑨ $a < b \Rightarrow e^a < e^b$ because e^x is increasing and $e^{\ln(a)} = a$ when $x > 0$

and when $x = 0$ we have $0^j < \varepsilon$ because $\varepsilon > 0$

(final) $x^j \geq 0$ so $|x^j| = x^j$ and $x^j - 0 = x^j$

Proof is now correct for $[0, \frac{1}{2}]$. QED

SCREEN RECORDING now

Screen Recording video saved to Photos

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Classwork: Prove $f_j(x) = x^j$ converges uniformly to $f_\infty(x) = 0$ on $[0, \frac{1}{2}]$ but not on $[0, 1]$

$\forall \varepsilon > 0 \; \exists N_\varepsilon$ s.t $\forall j \geq N_\varepsilon \; |x^j - 0| < \varepsilon \; \forall x \in [0, \frac{1}{2}]$

← What goes wrong?

Show Line final line

Proof ① Given $\varepsilon > 0$ Choose $N_\varepsilon = \frac{\ln(\varepsilon)}{\ln(\frac{1}{2})} + 1$

① $\varepsilon > 0$ so $\ln(\varepsilon)$ is defined and $\ln(\frac{1}{2}) \neq 0$ so division not OK

② Whenever $j \geq N_\varepsilon$ we have $j > \frac{\ln(\varepsilon)}{\ln(\frac{1}{2})}$

② by choice of N_ε in step 1

③ $j \ln(\frac{1}{2}) < \ln(\varepsilon)$

③ $a > b$ and $c = \ln(\frac{1}{2}) < 0 \Rightarrow ac < bc$ False

④ $\ln(\frac{1}{2}) < \frac{1}{j}\ln(\varepsilon)$ False Trouble

④ $a < b$ and $c = 1/j > 0 \Rightarrow ac < bc$

⑤ $\ln(x) \leq \ln(\frac{1}{2}) < \frac{1}{j}\ln(\varepsilon) \; \forall x \in (0, \frac{1}{2}]$

⑤ Ln is increasing on $(0, \frac{1}{2}]$ ($y = \ln(x)$)

⑥ $\ln(x) < \frac{1}{j}\ln(\varepsilon) \; \forall x \in (0, \frac{1}{2}]$

⑥ by step 5

⑦ $j \ln(x) < \ln(\varepsilon) \; \forall x \in (0, \frac{1}{2}]$

⑦ $a < b$ and $c = j > 0 \Rightarrow ac < bc$

⑧ $\ln(x^j) < \ln(\varepsilon) \; \forall x \in (0, \frac{1}{2}]$

⑧ $\ln(a^c) = c\ln(a)$

⑨ $x^j < \varepsilon \; \forall x \in [0, \frac{1}{2}]$

⑨ $a < b \Rightarrow e^a < e^b$ because e^x is increasing and $e^{\ln(a)} = a$ when $x > 0$

and when $x = 0$ we have $0^j < \varepsilon$ because $\varepsilon > 0$

(final) $|x^j - 0| < \varepsilon \; \forall x \in [0, \frac{1}{2}]$

(final) $x^j \geq 0$ so $|x^j| = x^j$ and $x^j - 0 = x^j$

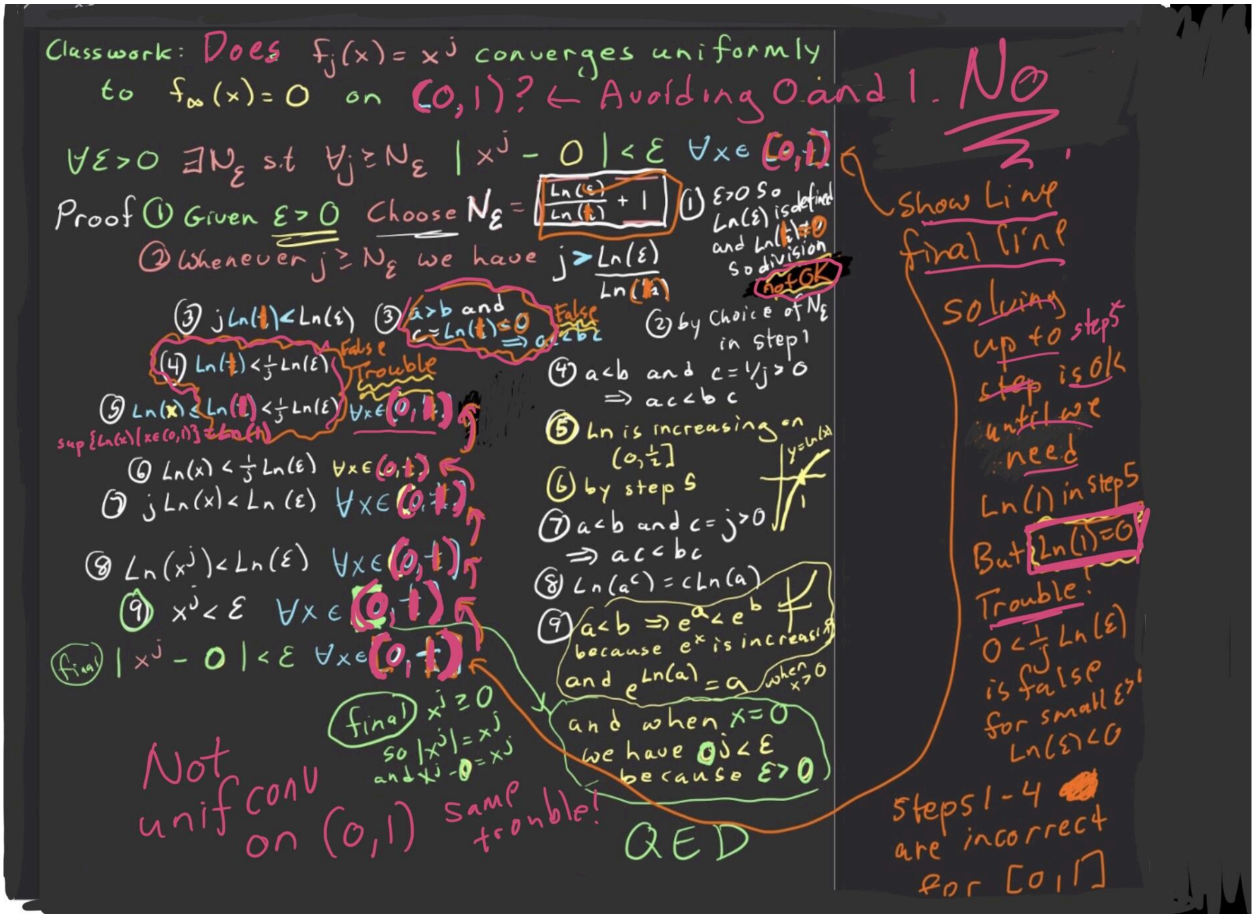
Proof fails for $[0, 1]$ → ~~$0 \in [\,]$~~

solving up to step is OK until we need $\ln(1)$ in step 5

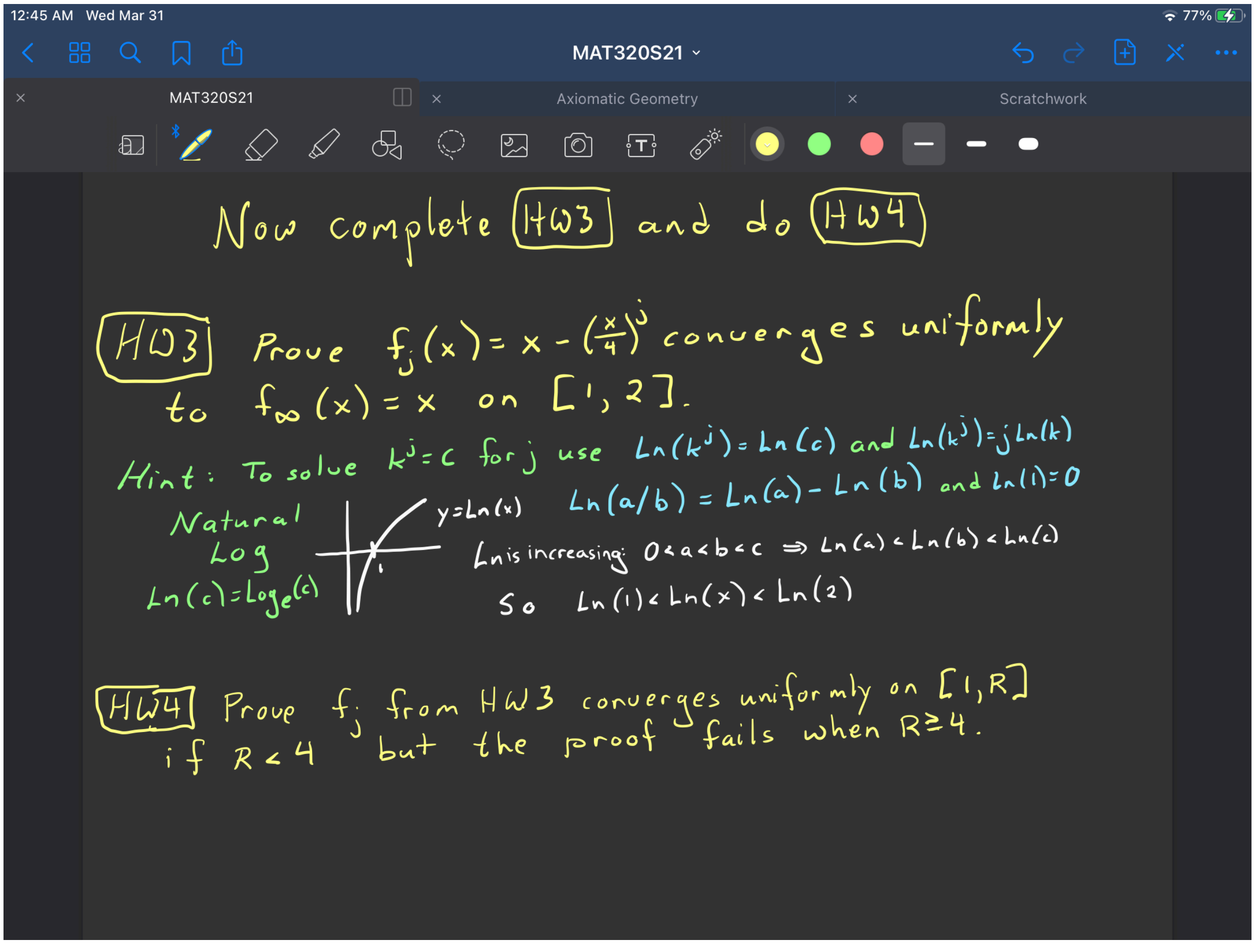
But $\ln(1) = 0$ Trouble!

$0 < \frac{1}{j}\ln(\varepsilon)$ is false for small $\varepsilon > 0$ $\ln(\varepsilon) < 0$

Steps 1-4 are incorrect for $[0, 1]$



If you are very short on time, you may skip these homework problems, but the classwork above is required. It is important to do the



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Part 2 Pointwise Convergence

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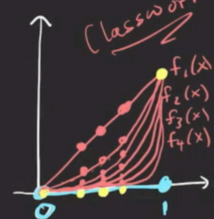
Watch [Playlist Pointwise-1to6] and do the classwork.

Defn: a sequence of functions

$f_j : A \to \mathbb{R}$ converges pointwise on A

to $f_\infty : A \to \mathbb{R}$ if $\forall a \in A$ $\underbrace{f_j(a)}_{\text{points}} \to \underbrace{f_\infty(a)}_{\text{points}}$.

Classwork



Example

$f_j(x) = x^j$ converges ptwise on $[0,1]$?

$f_1(x) = x^1 = x$

$f_2(x) = x^2$

$f_3(x) = x^3$

$f_4(x) = x^4$

$f_j(0) = 0^j = 0 \to 0$
so $f_\infty(0) = 0$

$f_j(1) = 1^j = 1 \to 1$
so $f_\infty(1) = 1$

$f_j(\frac{1}{2}) = (\frac{1}{2})^j \to 0 = f_\infty(\frac{1}{2})$

$f_j(\frac{1}{3}) = (\frac{1}{3})^j \to 0 = f_\infty(\frac{1}{3})$

$f_j(\frac{2}{3}) = \frac{2^j}{3^j} \to 0 = f_\infty(\frac{2}{3})$

$f_j(x)$ converges pointwise

to $$f_\infty(x) = \begin{cases} 0 & \text{for } x \in [0,1) \\ 1 & \text{at } x = 1 \end{cases}$$

on $[0,1]$

Classwork: Does $f_j(x) = x^j$ converges uniformly to $f_\infty(x) = 0$ on $(0,1)$? ← Avoiding 0 and 1. No

$\forall \varepsilon > 0 \; \exists N_\varepsilon$ s.t $\forall j \geq N_\varepsilon \; |x^j - 0| < \varepsilon \; \forall x \in (0,1)$

Proof ① Given $\varepsilon > 0$ Choose $N_\varepsilon = \frac{Ln(\varepsilon)}{Ln(\frac{1}{2})} + 1$

② Whenever $j \geq N_\varepsilon$ we have $j > \frac{Ln(\varepsilon)}{Ln(\frac{1}{2})}$

③ $j\,Ln(\frac{1}{2}) < Ln(\varepsilon)$

④ $Ln(\frac{1}{2}) < \frac{1}{j} Ln(\varepsilon)$ False Trouble

⑤ $Ln(x) < Ln(\frac{1}{2}) < \frac{1}{j} Ln(\varepsilon) \; \forall x \in (0,\frac{1}{2})$

$\sup\{Ln(x) \mid x \in (0,1)\} = Ln(1)$

⑥ $Ln(x) < \frac{1}{j} Ln(\varepsilon) \; \forall x \in (0,\frac{1}{2})$

⑦ $j\,Ln(x) < Ln(\varepsilon) \; \forall x \in (0,\frac{1}{2})$

⑧ $Ln(x^j) < Ln(\varepsilon) \; \forall x \in (0,\frac{1}{2})$

⑨ $x^j < \varepsilon \; \forall x \in (0,\frac{1}{2})$

(final) $|x^j - 0| < \varepsilon \; \forall x \in (0,\frac{1}{2})$

(final) $x^j \geq 0$ so $|x^j| = x^j$ and $x^j - 0 = x^j$

Justifications:

① $\varepsilon > 0$ so $Ln(\varepsilon)$ is defined and $Ln(\frac{1}{2}) \neq 0$ so division not OK

② by Choice of N_ε in step 1

③ $a > b$ and $c = Ln(\frac{1}{2}) < 0 \Rightarrow ac < bc$ False

④ $a < b$ and $c = 1/j > 0 \Rightarrow ac < bc$

⑤ Ln is increasing on $(0,\frac{1}{2}]$ ($y = Ln(x)$)

⑥ by step 5

⑦ $a < b$ and $c = j > 0 \Rightarrow ac < bc$

⑧ $Ln(a^c) = c\,Ln(a)$

⑨ $a < b \Rightarrow e^a < e^b$ because e^x is increasing and $e^{Ln(a)} = a$ when $x > 0$

and when $x = 0$ we have $0^j < \varepsilon$ because $\varepsilon > 0$

QED

Not unif conv on $(0,1)$ same trouble!

Show Line final line

solving up to step 5 step is OK until we need $Ln(1)$ in step 5

But $Ln(1) = 0$ Trouble!

$0 < \frac{1}{j} Ln(\varepsilon)$ is false for small ε's $Ln(\varepsilon) < 0$

Steps 1-4 are incorrect for $[0,1]$

SCREEN RECORDING now
Screen Recording video saved to Photos

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Axiomatic Geometry | Scratchwork

Classwork: Prove $f_j(x) = x^j$ converges uniformly to $f_\infty(x) = 0$ on $[0, \frac{1}{2}]$ but not on $[0, 1]$

$\forall \varepsilon > 0 \; \exists N_\varepsilon$ s.t. $\forall j \geq N_\varepsilon \; |x^j - 0| < \varepsilon \; \forall x \in [0, \frac{1}{2}]$

Proof ① Given $\varepsilon > 0$ Choose $N_\varepsilon = \frac{\ln(\varepsilon)}{\ln(\frac{1}{2})} + 1$

① $\varepsilon > 0$ so $\ln(\varepsilon)$ is defined and $\ln(\frac{1}{2}) \neq 0$ so division ~~not OK~~

② Whenever $j \geq N_\varepsilon$ we have $j > \frac{\ln(\varepsilon)}{\ln(\frac{1}{2})}$

② by choice of N_ε in step 1

③ $j \ln(\frac{1}{2}) < \ln(\varepsilon)$

③ $a > b$ and $c = \ln(\frac{1}{2}) < 0 \Rightarrow ac < bc$ False

④ $\ln(\frac{1}{2}) < \frac{1}{j} \ln(\varepsilon)$ False Trouble

④ $a < b$ and $c = 1/j > 0 \Rightarrow ac < bc$

⑤ $\ln(x) \leq \ln(\frac{1}{2}) < \frac{1}{j} \ln(\varepsilon) \; \forall x \in (0, \frac{1}{2}]$

⑤ $\ln$ is increasing on $(0, \frac{1}{2}]$ ($y = \ln(x)$)

⑥ $\ln(x) < \frac{1}{j} \ln(\varepsilon) \; \forall x \in (0, \frac{1}{2}]$

⑥ by step 5

⑦ $j \ln(x) < \ln(\varepsilon) \; \forall x \in (0, \frac{1}{2}]$

⑦ $a < b$ and $c = j > 0 \Rightarrow ac < bc$

⑧ $\ln(x^j) < \ln(\varepsilon) \; \forall x \in (0, \frac{1}{2}]$

⑧ $\ln(a^c) = c \ln(a)$

⑨ $x^j < \varepsilon \; \forall x \in [0, \frac{1}{2}]$

⑨ $a < b \Rightarrow e^a < e^b$ because e^x is increasing and $e^{\ln(a)} = a$ when $x > 0$

and when $x = 0$ we have $0^j < \varepsilon$ because $\varepsilon > 0$

(final) $|x^j - 0| < \varepsilon \; \forall x \in [0, \frac{1}{2}]$

(final) $x^j \geq 0$ so $|x^j| = x^j$ and $x^j - 0 = x^j$

Proof fails for $[0, 1]$ → ~~$0 < \varepsilon$~~

← What goes wrong?

Show Line final line

solving up to step is ok until we need $\ln(1)$ in step 5

But $\ln(1) = 0$

Trouble!

$0 < \frac{1}{j} \ln(\varepsilon)$ is false for small $\varepsilon > 0$, $\ln(\varepsilon) < 0$

Steps 1-4 are incorrect for $[0, 1]$

Classwork Prove $f_j(x) = x^j$ converges pointwise to

$$f_\infty(x) = \begin{cases} 0 & \text{for } x \in [0,1) \leftarrow x = \frac{1}{2} \\ 1 & \text{for } x = 1 \end{cases}$$

Show $\forall a \in [0,1]$ we have $f_j(a) \to f_\infty(a)$

Very easy proof at the endpoints of $[0,1]$:

Case I, $a=0$: $\forall \varepsilon > 0 \ \exists N_\varepsilon = 1$ s.t $\forall j \geq N_\varepsilon$ $|f_j(0) - f_\infty(0)| = |0^j - 0| = 0 < \varepsilon$.
super short proof.

Case II, $a=1$: $\forall \varepsilon > 0 \ \exists N_\varepsilon = 1$ s.t $\forall j \geq N_\varepsilon$ $|f_j(1) - f_\infty(1)| = |1^j - 1| = 0 < \varepsilon$.
super short proof

What about $a = \frac{1}{2}$?

Show $\forall \varepsilon > 0 \ \exists N_\varepsilon$ s.t $\forall j \geq N_\varepsilon$ $|f_j(\tfrac{1}{2}) - f_\infty(\tfrac{1}{2})| < \varepsilon$

$$|(\tfrac{1}{2})^j - 0| < \varepsilon$$

Proof:

(1) Given any $\varepsilon > 0$ Choose $N_\varepsilon = \boxed{\frac{Ln(\varepsilon)}{Ln(\frac{1}{2})} + 1}$

(2) Whenever $j \geq N_\varepsilon$ we have

$$j > Ln(\varepsilon)/Ln(\tfrac{1}{2})$$

(3) $j\,Ln(\frac{1}{2}) < Ln(\varepsilon)$

(4) $Ln(\frac{1}{2})^j < Ln(\varepsilon)$

(5) $(\frac{1}{2})^j < \varepsilon$

(final) $|(\frac{1}{2})^j - 0| < \varepsilon$

Classwork fill in the justifications!

Completes $a = 1/2$

Classwork: Prove $f_j(x) = x^j$ converges pointwise to

$$f_\infty(x) = \begin{cases} 0 & \text{for } x \in [0,1) \leftarrow \text{Case I + Case III} \\ 1 & \text{for } x = 1 \leftarrow \text{Case II} \end{cases}$$

Prove $\forall a \in [0,1]$ we have $f_j(a) \to f_\infty(a)$

Proof:

Case I: $a=0$: $\forall \varepsilon > 0\ \exists N_\varepsilon = 1$ s.t. $\forall j \geq N_\varepsilon$ $|f_j(0) - f_\infty(0)| = |0^j - 0| = 0 < \varepsilon$. (super short proof.)

Case II: $a=1$: $\forall \varepsilon > 0\ \exists N_\varepsilon = 1$ s.t. $\forall j \geq N_\varepsilon$ $|f_j(1) - f_\infty(1)| = |1^j - 1| = 0 < \varepsilon$. (super short proof)

Case III: $a \in (0,1)$:

Show $\forall \varepsilon > 0\ \exists N_\varepsilon$ s.t. $\forall j \geq N_\varepsilon$ $|f_j(a) - f_\infty(a)| < \varepsilon$, i.e. $|(a)^j - 0| < \varepsilon$

Proof of Case III $\forall a \in (0,1)$

(1) Given any $\varepsilon > 0$ Choose $N_\varepsilon = \frac{\ln(\varepsilon)}{\ln(a)} + 1$

(2) Whenever $j \geq N_\varepsilon$ we have

$$j > \ln(\varepsilon) / \ln(a)$$

(3) $j \ln(a) < \ln(\varepsilon)$

(4) $\ln(a)^j < \ln(\varepsilon)$

(5) $(a)^j < \varepsilon$

(final) $|(a)^j - 0| < \varepsilon$

Classwork fill in the justifications!

Pointwise convergence N_ε depends on a, ε

Notice we prove in Case III that x^j converges pointwise on $(0,1)$ to the const function $f_\infty(x) = 0$

Recall we could not prove unif. conv.

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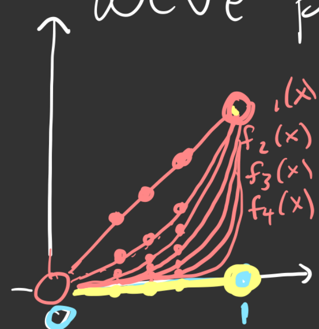
MAT320S21

MAT320S21 | Axiomatic Geometry | Scratchwork



We've proven $f_j(x) = x^j$ converges pointwise to $f_\infty(x) = 0$ on $A = (0,1)$

$\forall a \in A$ $\underbrace{f_j(a)}_{\text{points}} \to \underbrace{f_\infty(a)}_{\text{points}}$.

$\forall a \in A$ (fixed) $\forall \varepsilon > 0$ $\exists N_{\varepsilon,a}$ s.t. $\forall j \geq N_{\varepsilon,a}$ $|f_j(a) - f_\infty(a)| < \varepsilon$

↑ may depend on the point a

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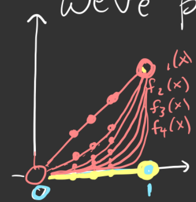
MAT320S21

MAT320S21 | Axiomatic Geometry | Scratchwork



We've proven $f_j(x) = x^j$ converges pointwise to $f_\infty(x) = 0$ on $A = (0,1)$

$\forall a \in A \quad f_j(a) \to f_\infty(a)$ (points) (points)

$\forall a \in A$ (fixed) $\forall \varepsilon > 0 \ \exists N_{\varepsilon, a}$ s.t $\forall j \geq N_{\varepsilon, a} \ |f_j(a) - f_\infty(a)| < \varepsilon$

($N_{\varepsilon,a}$ may depend on the point a)

Failed to prove uniform convergence

$\forall \varepsilon > 0 \ \exists N_\varepsilon$ s.t $\forall j \geq N_\varepsilon \ |f_j(a) - f_\infty(a)| < \varepsilon \quad \forall a \in (0,1)$

(N_ε which does not depend on a)

Can we prove it does not converge uniformly?

We've proven $f_j(x) = x^j$

converges pointwise

to $f_\infty(x) = 0$

on $A = (0,1)$

$\forall a \in A \quad f_j(a) \to f_\infty(a)$.

points   points

$\forall a \in A \quad \forall \varepsilon > 0 \quad \exists N_{\varepsilon, a}$ s.t $\forall j \geq N_{\varepsilon, a} \; |f_j(a) - f_\infty(a)| < \varepsilon$

fixed

may depend on the point a

Failed to prove uniform convergence

$\forall \varepsilon > 0$ N_ε s.t $\forall j \geq N_\varepsilon \; |f_j(x) - f_\infty(x)| < \varepsilon$

$\forall x \in (0,1)$

which does not depend on a

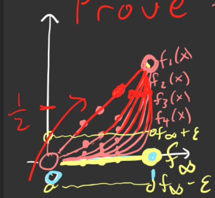
$f_\infty(x) - \varepsilon < f_j(x) < f_\infty(x) + \varepsilon$

each f_j has a bad pt! $\forall x \in A = (0,1)$

We can prove there is no unif conv.

Each f_j has an $a_j \in (0,1)$ s.t $f_j(a_j) > \frac{1}{2}$

Prove $f_j(x) = x^j$ does not converge uniformly to $f_\infty(x) = 0$ on $A = (0,1)$



Thoughts:

$\forall \varepsilon > 0$ $\exists N_\varepsilon$ (which does not depend on x) s.t. $\forall j \geq N_\varepsilon$ $|f_j(x) - f_\infty(x)| < \varepsilon$ $\forall x \in (0,1)$

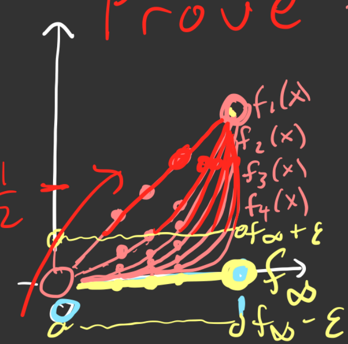
$f_\infty(x) - \varepsilon < f_j(x) < f_\infty(x) + \varepsilon$ $\forall x \in A = (0,1)$

each f_j has a bad pt!

We can prove there is no unif conv.

Each f_j has an $a_j \in (0,1)$ s.t. $f_j(a_j) > \frac{1}{2}$

Prove $f_j(x) = x^j$ does not converge uniformly to $f_\infty(x) = 0$ on $A = (0,1)$



Proof by Contradiction:

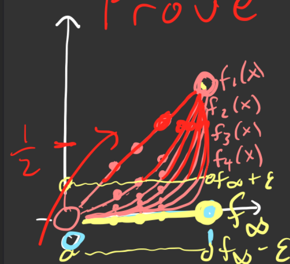
① Assume on the contrary that ① Indirect Hyp.

$\forall \varepsilon > 0 \ N_\varepsilon$ s.t $\forall j \geq N_\varepsilon \ |f_j(x) - f_\infty(x)| < \varepsilon \ \forall x \in (0,1)$

② $f_\infty(x) - \varepsilon < f_j(x) < f_\infty(x) + \varepsilon \ \forall x \in (0,1)$

Towards a ⊗

Prove $f_j(x) = x^j$ does not converge uniformly to $f_\infty(x) = 0$ on $A = (0,1)$



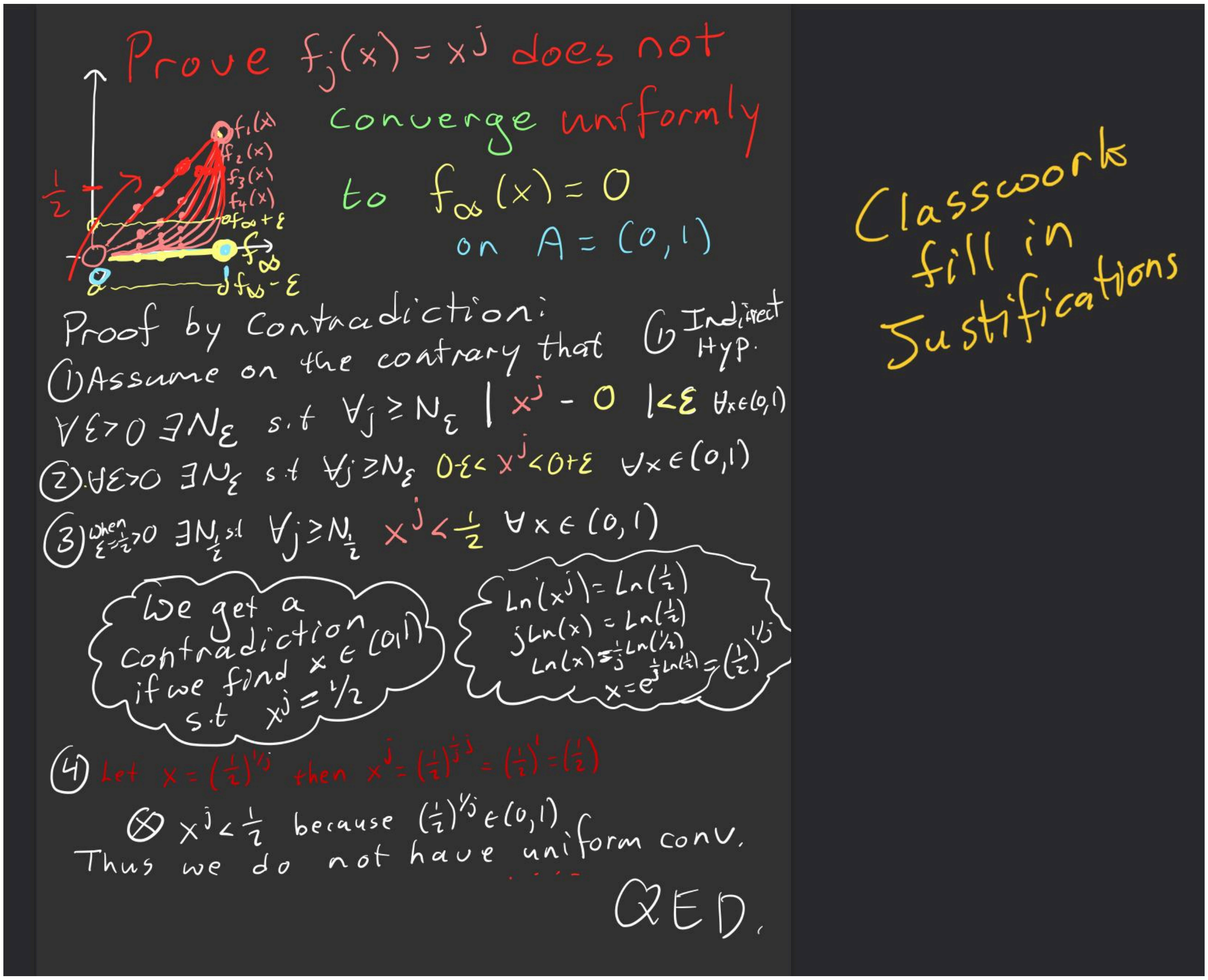
Proof by Contradiction: (1) Indirect Hyp.

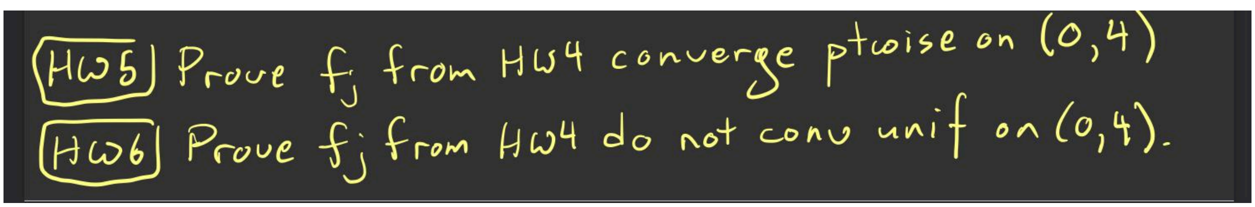
(1) Assume on the contrary that $\forall \varepsilon>0\ \exists N_\varepsilon$ s.t $\forall j \geq N_\varepsilon\ |x^j - 0| < \varepsilon\ \forall x \in (0,1)$

(2) $\forall \varepsilon>0\ \exists N_\varepsilon$ s.t $\forall j \geq N_\varepsilon\ 0-\varepsilon < x^j < 0+\varepsilon\ \forall x \in (0,1)$

(3) when $\varepsilon = \frac{1}{2} > 0\ \exists N_{\frac{1}{2}}$ s.t $\forall j \geq N_{1/2}\ x^j < \frac{1}{2}$ (3) True $\forall \varepsilon > 0$ so take $\varepsilon = \frac{1}{2}$

Classwork } contradiction





You must do classwork and at least start these two homework problems.

****

Part 4 Stronger and Weaker Convergence

****

Watch [Playlist Pointwise-7to8] and complete classwork.

Uniform Convergence is stronger than Pointwise Convergence.

Pointwise Convergence is weaker.

Unif Conv $\Longrightarrow$ Pointwise Conv.

but examples like $f_j(x) = x^j$ have ptwise conv to $f_\infty(x) = 0$ on $[0,1)$

Pointwise but not unif. conv.

Uniform Convergence is <u>stronger</u> than Pointwise Convergence.

Pointwise Convergence is <u>weaker</u>.

Thm

Unif Conv $\Rightarrow$ Pointwise Conv.

<u>Given</u>: $f_j \xrightarrow{unif} f_\infty$ on A <u>Show</u> $f_j \xrightarrow{pt} f_\infty$ on A

<u>Given</u>: $\forall \varepsilon > 0 \ \exists N_\varepsilon$ s.t $\forall j \geq N_\varepsilon \ |f_j(x) - f_\infty(x)| < \varepsilon \ \forall x \in A$

<u>Show</u>: $\forall a \in A \ \forall \varepsilon > 0 \ \exists N_{\varepsilon,a}$ s.t $\forall j \geq N_{\varepsilon,a} \ |f_j(a) - f_\infty(a)| < \varepsilon$

<u>Proof</u>: ① Given any $a \in A$ and given any $\varepsilon > 0$

Structure

Choose $N_{\varepsilon,a} = \boxed{N_\varepsilon}$ ① given

② Whenever $j \geq N_{\varepsilon,a}$ we have $j \geq N_\varepsilon$

so $|f_j(x) - f_\infty(x)| < \varepsilon \ \forall x \in A$. ② given

(final) $|f_j(a) - f_\infty(a)| < \varepsilon$ (final) $a \in A$ QED.

Corollary If $f_j \xrightarrow{unif} f_\infty$ on A then $f_j(a) \rightarrow f_\infty(a) \ \forall a \in A$.

HW5 Prove f_j from HW4 converge ptwise on $(0,4)$

HW6 Prove f_j from HW4 do not conv unif on $(0,4)$.

Gaussian Functions

What about $f_j(x) = \frac{j}{\sqrt{2\pi}} e^{-(jx)^2/2}$?

Does this conv unif? Ptwise?

Gaussian

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

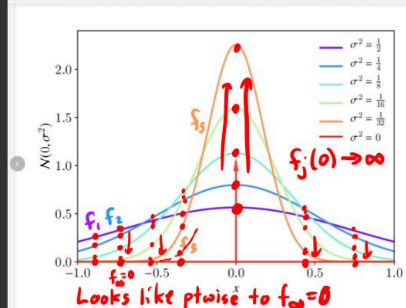
where σ is the standard deviation and μ the mean

$\sigma_j = 1/j \quad \mu = 0$

What domain?

Standard Deviation $= \frac{1}{j}$

Mean $= 0$



Gaussian bell function (normal distribution) N (0, σ 2) with varying variance σ 2 . For σ 2 → 0 the function approaches the Dirac-δ distribution. The arrow indicates a function value of ∞. The function family has non-compact support.

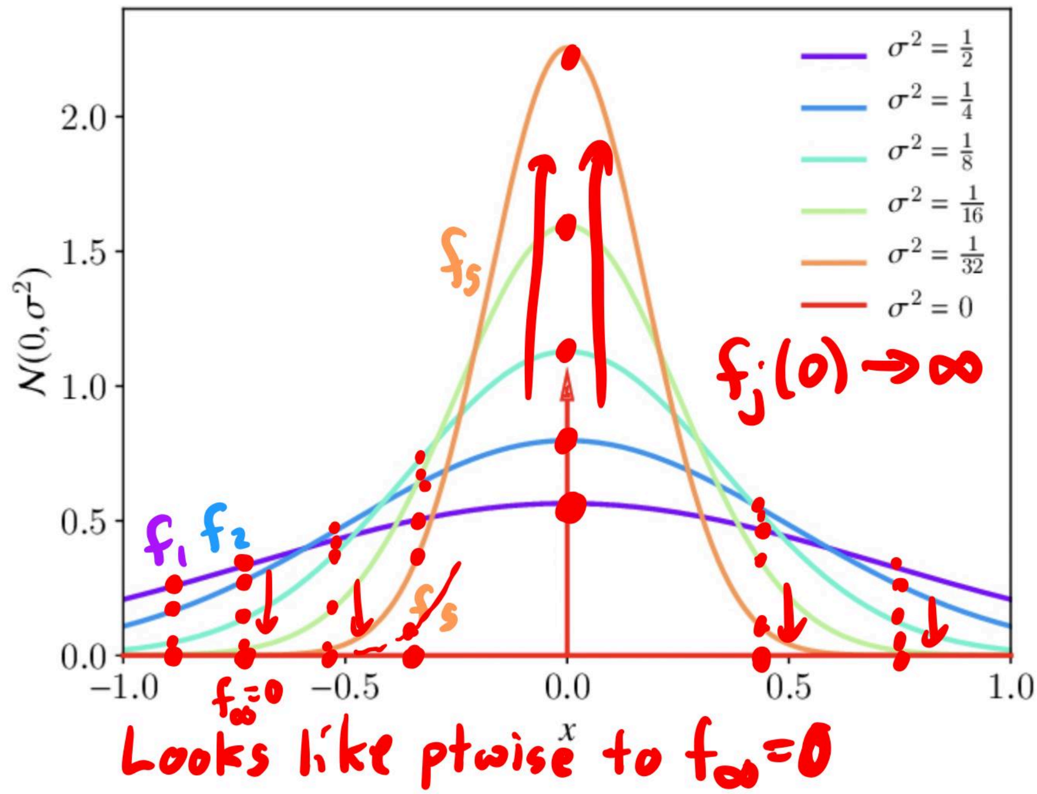
HW7 Prove that for $a \neq 0$

$f_j(a) \to f_\infty(a) = 0$

HW8 Prove that for $a = 0$

$f_j(a) \to \infty$ diverges!

Does not converge ptwise on on A where $0 \in A$.



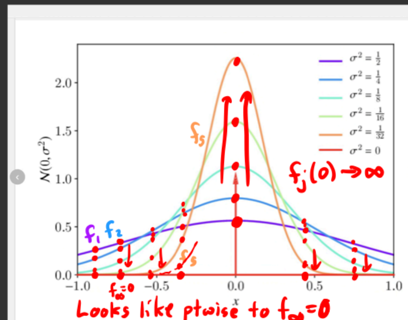
Gaussian bell function (normal distribution) N (0, σ 2) with varying variance σ 2 . For σ 2 → 0 the function approaches the Dirac-δ distribution. The arrow indicates a function value of ∞. The function family has non-compact support.

What about $f_j(x) = \frac{j}{\sqrt{2\pi}} e^{-(jx)^2/2}$?

Does this conv unif? Ptwise?

Gaussian
$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$
where σ is the standard deviation and μ the mean

What domain?



Gaussian bell function (normal distribution) N(0, σ 2) with varying variance σ 2 . For σ 2 → 0 the function approaches the Dirac-δ distribution. The arrow indicates a function value of ∞. The function family has non-compact support.

$\sigma_j = 1/j$ $\mu = 0$

Standard Deviation $= \frac{1}{j}$

Mean $= 0$

HW7 Prove that for $a \neq 0$

$f_j(a) \to f_\infty(a) = 0$

HW8 Prove that for $a = 0$

$f_j(a) \to \infty$ diverges!

Does not converge ptwise on A where $0 \in A$.

Many physics + probability classes say $f_j \to f_\infty$ where f_∞ is the Dirac Delta distribution

even weaker notion of convergence taught in doctoral real analysis.

Hint for HW7: **See a similar problem here with videos of its solution.**
Hint for HW8: **Use the definition of diverging sequences from before Exam 1.**

****
Part 4: The Arzela-Ascoli Theorem
****

Watch Playlist AA-1to3.

The Arzela - Ascoli Theorem:

If $f_j : [0,1] \to \mathbb{R}$ are

uniformly bounded $\exists B > 0$ s.t $|f_j(x)| \leq B$ $\forall j \in \mathbb{N}$ $\forall x \in [0,1]$

bound

$-B \leq f_j(x) \leq B$

and

equicontinuous $\forall \varepsilon > 0\ \exists \delta_\varepsilon > 0$ s.t $\underset{x, c \in [0,1]}{|x-c| < \delta_\varepsilon} \Rightarrow |f_j(x) - f_j(c)| < \varepsilon$

← does not depend on c or j.

Then $\exists f_\infty : [0,1] \to \mathbb{R}$ such that

a subsequence $f_{j_k} \xrightarrow{\text{unif}} f_\infty$ on $[0,1]$

and

f_∞ is continuous.

Warning: $f_j(x) = x^j \xrightarrow{\text{ptwise}} f_\infty(x) = \begin{cases} 0 & \text{on } [0,1) \\ 1 & \text{at } x = 1 \end{cases}$

↑ not continuous

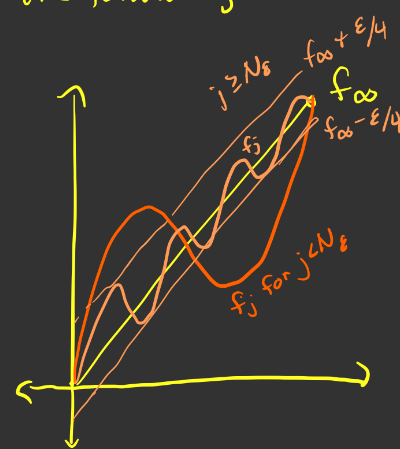
↑ not equicontinuous

↑ not uniform

We don't have more time in today's lesson for the proof but for homework try the following:

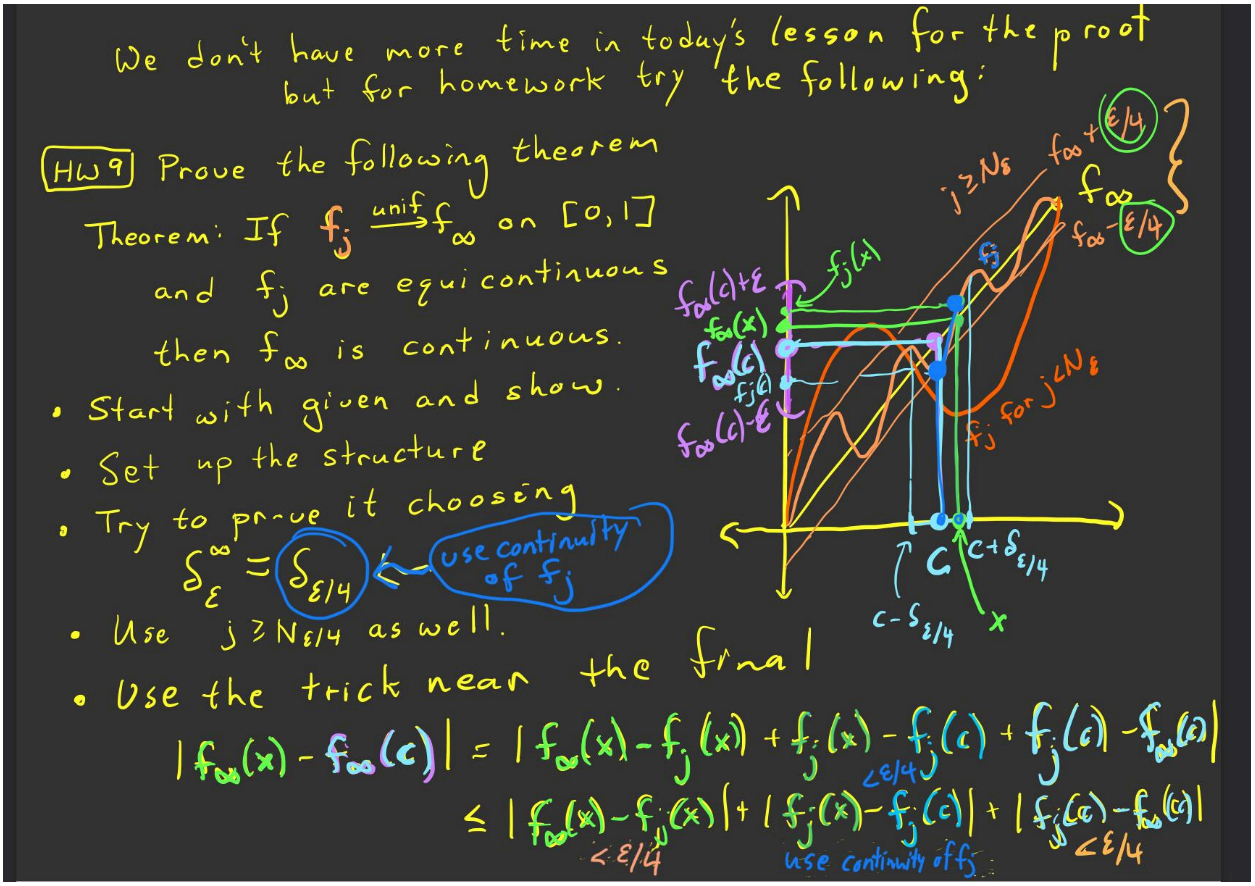
HW 9 Prove the following theorem

Theorem: If $f_j \xrightarrow{unif} f_\infty$ on $[0,1]$ and f_j are equicontinuous then f_∞ is continuous.



- Start with given and show.
- Set up the structure
- Try to prove it choosing $\delta_\varepsilon^\infty = \delta_{\varepsilon/4}$
- Use $j \geq N_{\varepsilon/4}$ as well.
- Use the trick near the final

$$|f_\infty(x) - f_\infty(c)| = |f_\infty(x) - f_j(x) + f_j(x) - f_j(c) + f_j(c) - f_\infty(c)|$$
$$\leq |f_\infty(x) - f_j(x)| + |f_j(x) - f_j(c)| + |f_j(c) - f_\infty(c)|$$



****
Part 5: C([0,1])
****

No videos for this part.

C([0,1]) is the set of all continuous functions f:[0,1] to the reals.

Recall the definition of a metric space from our [Lesson on Metric Spaces].

Thm: C([0,1]) is a metric space with the metric
d(f,h)= sup{ |f(x)-h(x)| : x in [0,1] }
which exists because continuous functions are bounded and bounded sets have suprema.

HW10: prove this theorem use the triangle inequality for absolute values and the theorems about suprema of sums of functions. This is difficult.

Note that sequences of continuous functions converge in C([0,1]) if and only if they converge uniformly on [0,1]. This is very easy if you review the beginning of Part 1.

C([0,1]) is a vector space with addition of functions and scalar multiplication of real numbers times functions as taught in some linear algebra classes. See my Lesson on Vector Spaces to learn this if you wish.