Linear Algebra MAT313 Spring 2023
Professor Sormani
Lesson 11
Eigenvectors and Eigenvalues

Part I: Eigenvectors and Eigenvalues
Part II: Power Method for finding Eigenvalues

Before you start, find your team's project part 2 document and submit a step for the project.

If you work with any classmates on this lesson, be sure to write their names on the problems you completed together.

You will cut and paste the photos of your notes and completed classwork in a googledoc entitled:

MAT313S23-lesson11-lastname-firstname

and share editing of that document with me sormanic@gmail.com. You will also include your homework and any corrections to your homework in this doc.

If you have a question, type QUESTION in your googledoc next to the point in your notes that has a question and email me with the subject MAT313 QUESTION. I will answer your question by inserting a photo into your googledoc or making an extra video.

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This lesson has two parts:
Part I: Eigenvectors and Eigenvalues
Part II: Power Method
There are ten homework problems.
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Part I: Eigenvectors and Eigenvalues
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Begin by reading what an eigenvalue and an eigenvector is right here:

Given a square $n \times n$ matrix A

$\vec{v}$ is an eigenvector of A with eigenvalue λ

if $$A\vec{v} = \lambda \vec{v}$$

our $n \times n$ matrix

vector in $\mathbb{R}^n$

real number

$\vec{v}$ cannot be the $\vec{0}$ vector. λ can be zero.

An eigenvector can never be the zero vector, but an eigenvalue can be zero.

Classwork 1

$$A = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$$

Check that $\vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector and find its eigenvalue.

Solution:

$$\begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3(1)+1(0) \\ 0(1)+2(0) \end{pmatrix} = \begin{pmatrix} 3+0 \\ 0+0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

$$\lambda \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda(1) \\ \lambda(0) \end{pmatrix} = \begin{pmatrix} \lambda \\ 0 \end{pmatrix} \leftarrow \text{so } \lambda = 3$$

$$\begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Classwork ②

$A = \begin{pmatrix}3 & 1\\ 0 & 2\end{pmatrix}$ Check that $\vec{v} = \begin{pmatrix}-1\\ 1\end{pmatrix}$ is an eigenvector and find its eigenvalue.

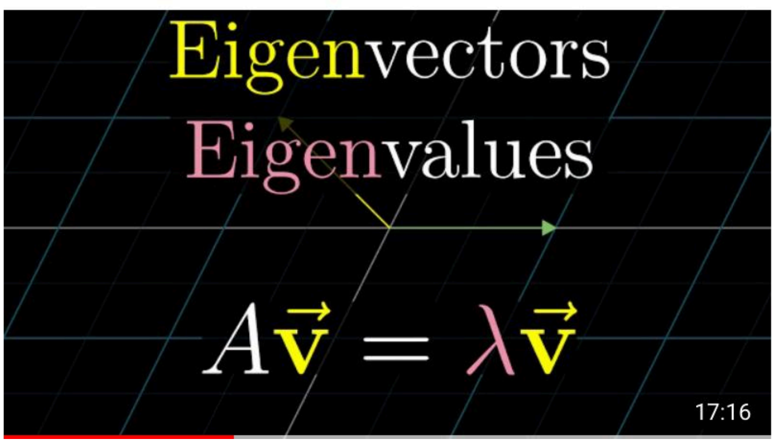
Solution:

$$\begin{pmatrix}3 & 1\\ 0 & 2\end{pmatrix}\begin{pmatrix}-1\\ 1\end{pmatrix} = \begin{pmatrix}3(-1)+1(1)\\ 0(-1)+2(1)\end{pmatrix} = \begin{pmatrix}-3+1\\ 0+2\end{pmatrix} = \begin{pmatrix}-2\\ 2\end{pmatrix}$$

$$\lambda\begin{pmatrix}-1\\ 1\end{pmatrix} = \begin{pmatrix}\lambda(-1)\\ \lambda(1)\end{pmatrix} = \begin{pmatrix}-\lambda\\ \lambda\end{pmatrix} \leftarrow \text{so } \lambda = 2$$

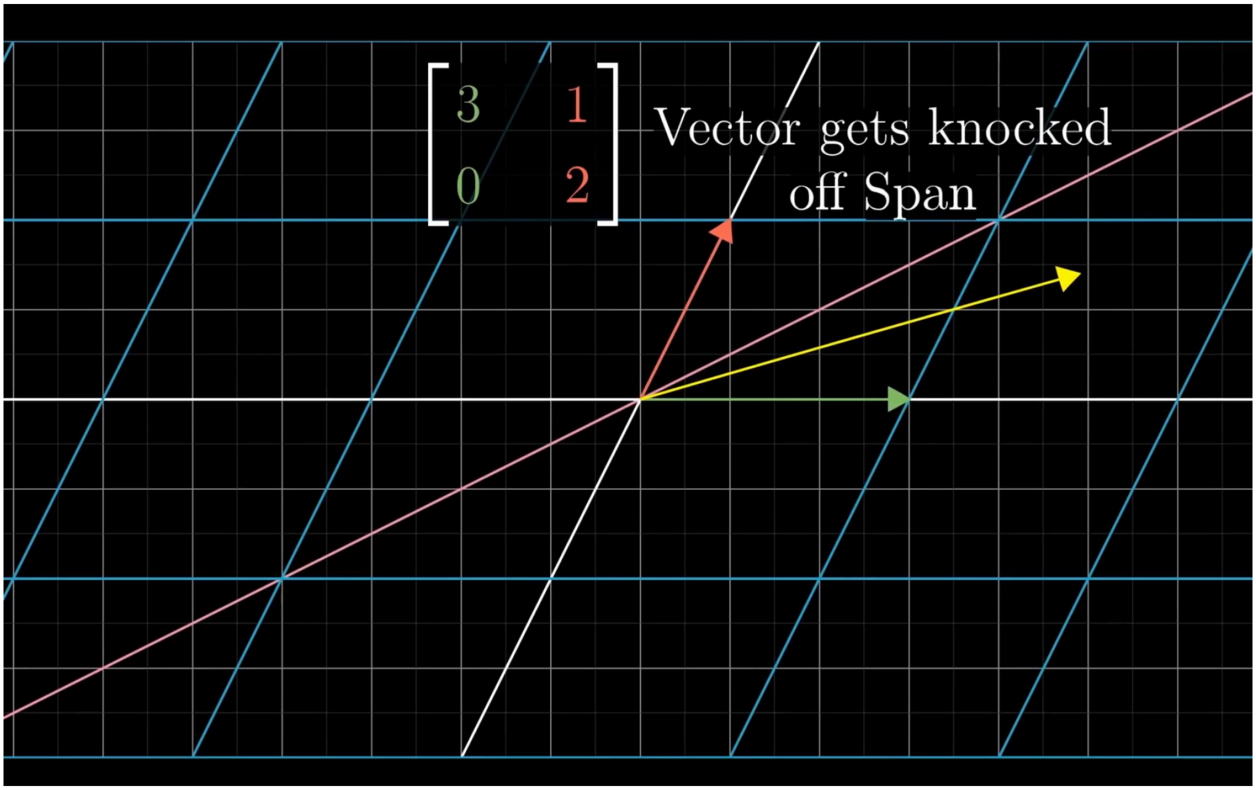
$$\begin{pmatrix}3 & 1\\ 0 & 2\end{pmatrix}\begin{pmatrix}-1\\ 1\end{pmatrix} = 2\begin{pmatrix}-1\\ 1\end{pmatrix}$$

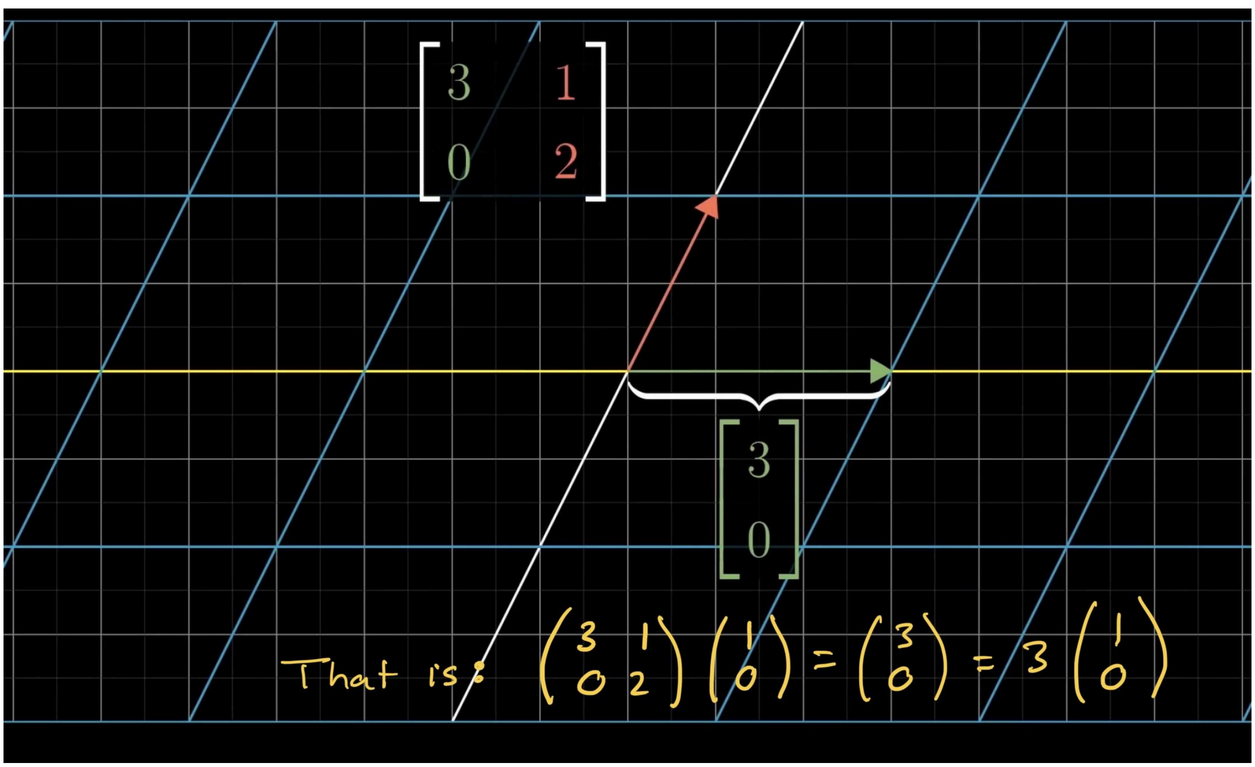
Below we see this example in a three blue one brown video:

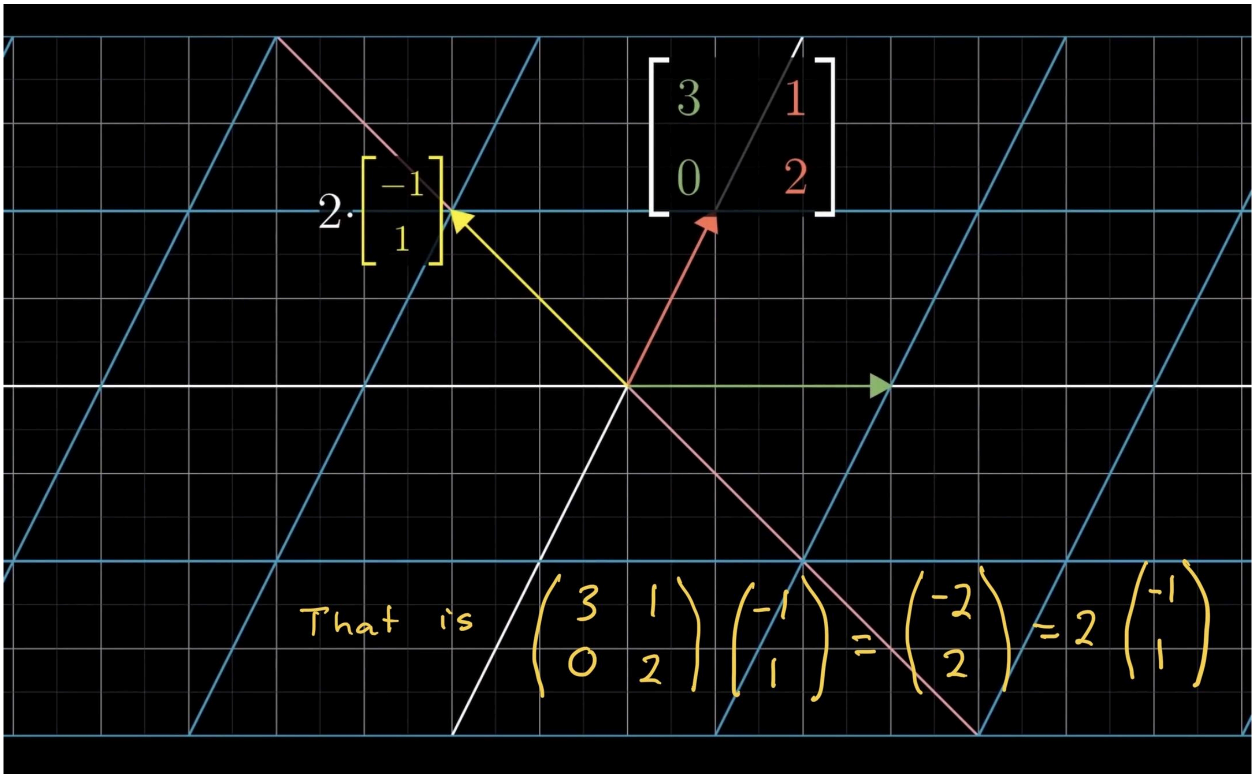


Eigenvectors and eigenvalues |
3.3M views · 6 years ago

3Blue1Brown


$\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$
Vector gets knocked off Span


$\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$
$\begin{bmatrix} 3 \\ 0 \end{bmatrix}$
That is: $\begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$



Please watch only the first five minutes of the Three Blue One Brown video on eigenvalues and eigenvectors where the above images are animated. Do not watch the whole video as it covers more advanced topics we will learn next month.

HW1 Show the matrix $\begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 4 & 2 \end{pmatrix}$

has three eigenvectors

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \quad v_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

and find the eigenvalues

HW2 Check your answer has $\lambda_1 = 5$, $\lambda_2 = -2$, $\lambda_3 = 6$

Ask me for help if this did not work.

Do not ask a tutor for help with this. The tutor will show you extra complicated things you do not need yet. You will learn them later. Just practice multiplying the matrix times each eigenvector.

Solution to HW1 to check your answers

Try First!

HW1 Show the matrix $\begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 4 & 2 \end{pmatrix}$

has three eigenvectors

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \quad v_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

and find the eigenvalues

Check your answer has $\lambda_1 = 5$, $\lambda_2 = -2$, $\lambda_3 = 6$

Ask me for help if this did not work.

Do not ask a tutor for help with this. The tutor will show you extra complicated things you do not need yet. You will learn them later. Just practice multiplying the matrix times each eigenvector.

$$\begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 4 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 5\cdot 1 + 0\cdot 0 + 0\cdot 0 \\ 0\cdot 1 + 2\cdot 0 + 4\cdot 0 \\ 0\cdot 1 + 4\cdot 0 + 2\cdot 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} = 5 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 4 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 5\cdot 0 + 0\cdot 1 + (-1)\cdot 0 \\ 5\cdot 0 + 2\cdot 1 + -1\cdot 4 \\ 5\cdot 0 + 4\cdot 1 + -1\cdot 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2-4 \\ 4-2 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix} = -2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 4 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5\cdot 0 + 0\cdot 1 + 0\cdot 1 \\ 0\cdot 0 + 2\cdot 1 + 4\cdot 1 \\ 0\cdot 0 + 4\cdot 1 + 2\cdot 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2+4 \\ 4+2 \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \\ 6 \end{pmatrix} = 6 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

If your matrix multiplication is incorrect, go back to the previous lesson and review it.

Now try the following classwork:

Classwork 3

Consider a diagonal matrix $D = \begin{pmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{pmatrix}$

What is $D\hat{i}$? Is $\hat{i}$ an eigenvector?

What is $D\hat{j}$? Is $\hat{j}$ an eigenvector?

What is $D\hat{k}$? Is $\hat{k}$ an eigenvector?

Solution: Remember $\hat{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\hat{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $\hat{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

What is $D\hat{i}$? Is $\hat{i}$ an eigenvector?

$$\begin{pmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} d_{11} \cdot 1 + 0 \cdot 0 + 0 \cdot 0 \\ 0 \cdot 1 + d_{22} \cdot 0 + 0 \cdot 0 \\ 0 \cdot 1 + 0 \cdot 0 + d_{33} \cdot 0 \end{pmatrix} = \begin{pmatrix} d_{11} \\ 0 \\ 0 \end{pmatrix} = d_{11} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

So yes $\hat{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is an eigenvector with eigenvalue $= d_{11}$

What is $D\hat{j}$? Is $\hat{j}$ an eigenvector?

You do the work . . .

What is $D\hat{k}$? Is $\hat{k}$ an eigenvector?

You do the work . . .

Theorem: A 3x3 diagonal matrix D has $D\hat{i} = d_{11}\hat{i}$ $D\hat{j} = d_{22}\hat{j}$ and $D\hat{k} = d_{33}\hat{k}$

Definition: $A^2 \vec{v} = A \underbrace{A \vec{v}}_{\text{first find this vector}}$

then multiply by A again

Example

$$\begin{pmatrix}1&2\\0&1\end{pmatrix}^2\begin{pmatrix}5\\6\end{pmatrix} = \begin{pmatrix}1&2\\0&1\end{pmatrix}\underbrace{\begin{pmatrix}1&2\\0&1\end{pmatrix}\begin{pmatrix}5\\6\end{pmatrix}}$$

$$= \begin{pmatrix}1&2\\0&1\end{pmatrix}\begin{pmatrix}1\cdot5+2\cdot6\\0\cdot5+1\cdot6\end{pmatrix} = \begin{pmatrix}1&2\\0&1\end{pmatrix}\begin{pmatrix}17\\6\end{pmatrix}$$

$$= \begin{pmatrix}1\cdot17+2\cdot6\\0\cdot17+1\cdot6\end{pmatrix} = \begin{pmatrix}17+12\\6\end{pmatrix} = \begin{pmatrix}29\\6\end{pmatrix}$$

Definition: $A^k \vec{v} = \underbrace{A \times A \times \ldots \times A}_{k \text{ times}} \vec{v}$

Example:

$$\begin{pmatrix}1&2\\0&1\end{pmatrix}^5\begin{pmatrix}5\\6\end{pmatrix} = \begin{pmatrix}1&2\\0&1\end{pmatrix}\begin{pmatrix}1&2\\0&1\end{pmatrix}\begin{pmatrix}1&2\\0&1\end{pmatrix}\begin{pmatrix}1&2\\0&1\end{pmatrix}\underbrace{\begin{pmatrix}1&2\\0&1\end{pmatrix}\begin{pmatrix}5\\6\end{pmatrix}}$$

$$= \begin{pmatrix}1&2\\0&1\end{pmatrix}\begin{pmatrix}1&2\\0&1\end{pmatrix}\begin{pmatrix}1&2\\0&1\end{pmatrix}\underbrace{\begin{pmatrix}1&2\\0&1\end{pmatrix}\begin{pmatrix}17\\6\end{pmatrix}}$$

$$= \begin{pmatrix}1&2\\0&1\end{pmatrix}\begin{pmatrix}1&2\\0&1\end{pmatrix}\underbrace{\begin{pmatrix}1&2\\0&1\end{pmatrix}\begin{pmatrix}29\\6\end{pmatrix}}$$

$$= \begin{pmatrix}1&2\\0&1\end{pmatrix}\begin{pmatrix}1&2\\0&1\end{pmatrix}\begin{pmatrix}1\cdot29+2\cdot6\\0\cdot29+1\cdot6\end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 41 \\ 6 \end{pmatrix}}$$

$$= \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \cdot 41 + 2 \cdot 6 \\ 0 \cdot 41 + 1 \cdot 6 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 53 \\ 6 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \cdot 53 + 2 \cdot 6 \\ 0 \cdot 53 + 1 \cdot 6 \end{pmatrix} = \begin{pmatrix} 65 \\ 6 \end{pmatrix}$$

Note that $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ is a skew matrix

and $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^5$ means skew five times

so the x term $5 \to 17 \to 29 \to 41 \to 53 \to 65$

but the y term did not change $6 \to \cdots \to 6$

Classwork ⑤ $\quad D = \begin{pmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{pmatrix} \quad \vec{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

Find $D^2 \vec{v}$ and $D^3 \vec{v}$ and $D^4 \vec{v}$

What is $D^k \vec{v}$?

Solution:

$$D^2\vec{v} = \begin{pmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{pmatrix} \underbrace{\begin{pmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}}$$

$$= \begin{pmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{pmatrix} \begin{pmatrix} d_{11}a \\ d_{22}b \\ d_{33}c \end{pmatrix} = \begin{pmatrix} d_{11}d_{11}a \\ d_{22}d_{22}b \\ d_{33}d_{33}c \end{pmatrix} = \begin{pmatrix} d_{11}^2 a \\ d_{22}^2 b \\ d_{33}^2 c \end{pmatrix}$$

$$D^3\vec{v} = D\left(D^2\vec{v}\right)$$

$$= \begin{pmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{pmatrix} \begin{pmatrix} d_{11}^2 a \\ d_{22}^2 b \\ d_{33}^2 c \end{pmatrix} = \begin{pmatrix} d_{11}^3 a \\ d_{22}^3 b \\ d_{33}^3 c \end{pmatrix}$$

Similarly $\quad D^4\vec{v} = \begin{pmatrix} d_{11}^4 a \\ d_{22}^4 b \\ d_{33}^4 c \end{pmatrix}$

And $\quad D^k\vec{v} = \begin{pmatrix} d_{11}^k a \\ d_{22}^k b \\ d_{33}^k c \end{pmatrix}$

This is a very special property of diagonal matrices

We can do something similar if we know the eigenvectors of a matrix:
First recall this theorem from Lesson 9:

<u>Thm</u>: $A(k\vec{v}) = k(A\vec{v})$

for all matrices $A \in \mathcal{M}_{n\times m}$

all real numbers $k \in \mathbb{R}$

and all vectors $\vec{v} \in \mathbb{R}^m$

We can use it to prove a theorem about eigenvalues:

Theorem: If $A\vec{v} = \lambda\vec{v}$

then $A^2\vec{v} = \lambda^2\vec{v}$

Proof:

1. $A^2\vec{v} = A * A\vec{v}$ — ① by defn of $A^2\vec{v}$
2. $= A \cdot \lambda\vec{v}$ — ② by given
3. $= \lambda A\vec{v}$ — ③ by $Ak\vec{v} = kA\vec{v}$ from Lesson 9
4. $= \lambda\lambda\vec{v}$ — ④ by given
5. $= \lambda^2\vec{v}$ — ⑤ by defn of λ^2

QED

$\boxed{\text{HW3}}$ Prove that

$$A\vec{v} = \lambda\vec{v} \implies A^3\vec{v} = \lambda^3\vec{v}$$

Theorem: $A\vec{v}=\lambda\vec{v} \implies A^k\vec{v}=\lambda^k\vec{v}$

So if we want to find $\begin{pmatrix}3&1\\0&2\end{pmatrix}^5\begin{pmatrix}-1\\1\end{pmatrix}$

We can use classwork ①: $\begin{pmatrix}3&1\\0&2\end{pmatrix}\begin{pmatrix}-1\\1\end{pmatrix}=2\begin{pmatrix}-1\\1\end{pmatrix}$

So $\begin{pmatrix}3&1\\0&2\end{pmatrix}^5\begin{pmatrix}-1\\1\end{pmatrix}=2^5\begin{pmatrix}-1\\1\end{pmatrix}=\begin{pmatrix}-2^5\\2^5\end{pmatrix}=\begin{pmatrix}-32\\32\end{pmatrix}$

$\boxed{\text{HW4}}$ Find $\begin{pmatrix}3&1\\0&2\end{pmatrix}^6\begin{pmatrix}1\\0\end{pmatrix}$ using this method.

Another Theorem from Lesson 9:

313F22-Lesson9

<u>Thm</u>: $A(\vec{v}+\vec{w})=A\vec{v}+A\vec{w}$
$\forall$ matrices $A\in M_{n\times m}$ and $\vec{v},\vec{w}\in\mathbb{R}^m$

We can use it to prove a theorem about eigenvectors:

$\boxed{\text{Theorem}}$ If $A\vec{v}_1 = \lambda_1 \vec{v}_1$ and $A\vec{v}_2 = \lambda_2 \vec{v}_2$

Then $A(c_1\vec{v}_1 + c_2\vec{v}_2) = \lambda_1 c_1 \vec{v}_1 + \lambda_2 c_2 \vec{v}_2$

Proof:

① $A(c_1\vec{v}_1 + c_2\vec{v}_2) = Ac_1\vec{v}_1 + Ac_2\vec{v}_2$ ① by $A(\vec{v}+\vec{\omega}) = A\vec{v} + A\vec{\omega}$

② $= c_1 A\vec{v}_1 + c_2 A\vec{v}_2$ ② by $Ak\vec{v} = kA\vec{v}$

③ $= c_1 \lambda_1 \vec{v}_1 + c_2 \lambda_2 \vec{v}_2$ ③ by given

④ $= \lambda_1 c_1 \vec{v}_1 + \lambda_2 c_2 \vec{v}_2$ ④ by $ab = ba$ for reals

QED

$\boxed{\text{HW5}}$ Prove that:

If $A\vec{v}_1 = \lambda_1 \vec{v}_1$ and $A\vec{v}_2 = \lambda_2 \vec{v}_2$

Then $A^2(c_1\vec{v}_1 + c_2\vec{v}_2) = \lambda_1^2 c_1 \vec{v}_1 + \lambda_2^2 c_2 \vec{v}_2$

$\boxed{\text{Theorem}}$ If $A\vec{v}_1 = \lambda_1 \vec{v}_1$ and $A\vec{v}_2 = \lambda_2 \vec{v}_2$

Then $A^k(c_1\vec{v}_1 + c_2\vec{v}_2) = \lambda_1^k c_1 \vec{v}_1 + \lambda_2^k c_2 \vec{v}_2$

Classwork: Use this theorem to find $\begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}^{10} \begin{pmatrix} 5 \\ 6 \end{pmatrix}$

Warning $\begin{pmatrix} 5 \\ 6 \end{pmatrix}$ is not an eigenvector!

First we recall the eigenvalues we found in Classwork ①

$$\begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

To use the thm we need c_1 and c_2: $\begin{pmatrix} 5 \\ 6 \end{pmatrix} = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

This is a system: $\boxed{\begin{matrix} -1c_1 + 0c_2 = 5 \\ 1c_1 + 1c_2 = 6 \end{matrix}}$

$$\left[\begin{array}{cc|c} -1 & 0 & 5 \\ 1 & 1 & 6 \end{array}\right] \xrightarrow{\rho_1 \to -\rho_1} \left[\begin{array}{cc|c} 1 & 0 & -5 \\ 1 & 1 & 6 \end{array}\right]$$

$$\xrightarrow{\rho_2 \to \rho_2 - \rho_1} \left[\begin{array}{cc|c} 1 & 0 & -5 \\ 0 & 1 & 11 \end{array}\right] \qquad \boxed{\begin{matrix} c_1 = -5 \\ c_2 = 11 \end{matrix}}$$

Check: $-5\begin{pmatrix} -1 \\ 1 \end{pmatrix} + 11 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ -5 \end{pmatrix} + \begin{pmatrix} 0 \\ 11 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$

So now

$$\begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}^{10} \begin{pmatrix} 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}^{10} \left(-5 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + 11 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$$

$2^{10} = ?$

$2^1 = 2$
$2^2 = 4$
$2^3 = 8$
$2^4 = 16$
$2^5 = 32$
$2^6 = 64$
$2^7 = 128$
$2^8 = 256$
$2^9 = 512$
$2^{10} = 1024$

$$= 2^{10}(-5)\begin{pmatrix}-1\\1\end{pmatrix} + 2^{10}\cdot 11\begin{pmatrix}0\\1\end{pmatrix}$$

$$= 1024(-5)\begin{pmatrix}-1\\1\end{pmatrix} + 1024(11)\begin{pmatrix}0\\1\end{pmatrix}$$

$$= \begin{pmatrix}5120\\-5120\end{pmatrix} + \begin{pmatrix}0\\11264\end{pmatrix}$$

$$= \begin{pmatrix}5120\\6144\end{pmatrix}$$

This may seem like a lot of work but is easier than doing it the long way.

It is almost as easy as a diagonal matrix.

Check your answer has $\lambda_1 = 5$, $\lambda_2 = -2$, $\lambda_3 = 6$

Ask me for help if this did not work.

$\boxed{\text{HW6}}$ Use your work from $\boxed{\text{HW1}}$ where you showed

$$\begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 4 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 5 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 4 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 6 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

to find

$$\begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 4 & 2 \end{pmatrix}^3 \begin{pmatrix} 8 \\ 6 \\ 6 \end{pmatrix}$$

Hint: find c_1 and c_2 s.t.

$$c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 6 \\ 6 \end{pmatrix}$$

$\boxed{\text{HW6}}$ Use your work from $\boxed{\text{HW1}}$ where you showed

$$\begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 4 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 5 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 4 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 6 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

to find

$$\begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 4 & 2 \end{pmatrix}^3 \begin{pmatrix} 8 \\ 6 \\ 6 \end{pmatrix}$$

Hint: find c_1 and c_2 s.t.

$$c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 6 \\ 6 \end{pmatrix}$$

Solution

$$\boxed{\begin{aligned} c_1 \cdot 1 + c_2 \cdot 0 &= 8 \\ 0\, c_1 + c_2 \cdot 1 &= 6 \\ 0 \cdot c_1 + 1 \cdot c_2 &= 6 \end{aligned}} \Rightarrow \begin{aligned} c_1 &= 8 \\ c_2 &= 6 \end{aligned}$$

$$\begin{aligned} \begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 4 & 2 \end{pmatrix}^3 \begin{pmatrix} 8 \\ 6 \\ 6 \end{pmatrix} &= \begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 4 & 2 \end{pmatrix}^3 \left(8 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 6 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right) \\ &= 8 \cdot 5^3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 6 \cdot 6^3 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \\ &= 8 \cdot 125 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 6 \cdot 216 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \\ &= 1000 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 1296 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 1000 \\ 1296 \\ 1296 \end{pmatrix} \end{aligned}$$

After you do HW6 Check your answer here

$$\begin{array}{r} \overset{3}{216} \\ \times\ 6 \\ \hline 1296 \end{array}$$

$\boxed{\text{Theorem:}}$ Suppose $A \in \mathbb{R}^{n\times n}$

has $A\vec{v}_i = \lambda_i \vec{v}_i$ for $i=1$ to n

then $$A^k\left(\sum_{i=1}^{n} c_i \vec{v}_i\right) = \sum_{i=1}^{n} c_i \lambda_i^k \vec{v}_i$$

and if you can solve the system

$$c_1\vec{v}_1 + \dots + c_n\vec{v}_n = \vec{w}$$

then $$A^k \vec{w} = c_1 \lambda_1^k v_1 + \dots + c_n \lambda_n^k \vec{v}_n$$

$\boxed{\text{Theorem}}$ Suppose $A \in \mathbb{R}^{n\times n}$

has $A\vec{v}_i = \lambda_i \vec{v}_i$ for $i = 1$ to n

then $A^k\left(\sum_{i=1}^{n} c_i \vec{v}_i\right) = \sum_{i=1}^{n} c_i \lambda_i^k \vec{v}_i$

and if you can solve the system

$$c_1\vec{v}_1 + \dots + c_n\vec{v}_n = \vec{w}$$

This is the same as saying the matrix whose columns are $\vec{v}_1, \dots \vec{v}_n$ is a nonsingular matrix

then $A^k\vec{w} = c_1\lambda_1^k v_1 + \dots + c_n\lambda_n^k\vec{v}_n$

$\boxed{\text{Note}}$ The 3×3 matrix in $\boxed{\text{HW1}}$

is especially nice because you

can always solve the system

$$c_1\begin{pmatrix}1\\0\\0\end{pmatrix} + c_2\begin{pmatrix}0\\1\\-1\end{pmatrix} + c_3\begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}w_1\\w_2\\w_3\end{pmatrix}$$

$$\begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 4 & 2 \end{pmatrix}^k \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$$

We know this because

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & \omega_1 \\ 0 & 1 & 1 & \omega_2 \\ 0 & -1 & 1 & \omega_3 \end{array}\right) \xrightarrow[\rho_3 \to \rho_3 + \rho_2]{} \left(\begin{array}{ccc|c} 1 & 0 & 0 & \omega_1 \\ 0 & 1 & 1 & \omega_2 \\ 0 & 0 & 2 & \omega_2 + \omega_3 \end{array}\right)$$

$$\xrightarrow{\rho_3 \to \frac{1}{2}\rho_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & \omega_1 \\ 0 & 1 & 1 & \omega_2 \\ 0 & 0 & 1 & (\omega_2 + \omega_3)/2 \end{array}\right)$$

$$\xrightarrow{\rho_2 \to \rho_2 - \rho_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & \omega_1 \\ 0 & 1 & 0 & \omega_2 - (\omega_2 + \omega_3)/2 \\ 0 & 0 & 1 & (\omega_2 + \omega_3)/2 \end{array}\right)$$

always has the solution

$$c_1 = \omega_1$$

$$c_2 = \omega_2 - (\omega_2 + \omega_3)/2$$

$$c_3 = (\omega_2 + \omega_3)/2$$

So
$$\begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 4 & 2 \end{pmatrix}^k \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 4 & 2 \end{pmatrix}^k \left(c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right)$$

$$= c_1 (5)^k \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 (-2)^k \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + c_3 (6)^k \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

can always be found for any $\vec{\omega}$.

Warning Not all matrices have real eigenvalues and eigenvectors!

Classwork: $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ rotation by $90°$

Let us show A has no eigenvectors!

Try to solve $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$

We get $\begin{pmatrix} 0x - 1y \\ 1x + 0y \end{pmatrix} = \begin{pmatrix} \lambda x \\ \lambda y \end{pmatrix}$

So $\begin{matrix} -y = \lambda x \\ x = \lambda y \end{matrix} \Longrightarrow \begin{matrix} y = -\lambda x \\ x = \lambda y \end{matrix}$

Then $x = \lambda y = \lambda(-\lambda x)$

So $x = -\lambda^2 x$

So $-\lambda^2 = 1 \Longrightarrow \lambda^2 = -1$

There is no real number λ that works!

Complex numbers can be used

$$\lambda^2 = -1 \implies \lambda = \pm i$$

For $\lambda = i$ we get $x = iy$

so $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} i \\ 1 \end{pmatrix}$ might be an eigenvector

Check it:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} i \\ 1 \end{pmatrix} = \begin{pmatrix} 0\,i + (-1)\,1 \\ 1\,i + 0\,1 \end{pmatrix} = \begin{pmatrix} -1 \\ i \end{pmatrix}$$

$$i\begin{pmatrix} i \\ 1 \end{pmatrix} = \begin{pmatrix} i \cdot i \\ i \cdot 1 \end{pmatrix} = \begin{pmatrix} -1 \\ i \end{pmatrix}$$ match!

So $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} i \\ 1 \end{pmatrix} = i\begin{pmatrix} i \\ 1 \end{pmatrix}$

$\boxed{\text{HW7}}$

Show the eigenvector for $\lambda = -i$ is $\begin{pmatrix} i \\ -1 \end{pmatrix}$

If you think about the three blue one brown video you watched, you will remember that eigenvectors describe directions that are not changed under the linear transformation. If the linear transformation is a rotation by 90 degrees, then all directions are changed. This is why the eigenvalues and eigenvectors are not real.

If you wish, watch the first five minutes of the [Three Blue One Brown video]() a second time. There are many many applications of eigenvectors and eigenvalues and it is good to have an understanding how they describe linear transformations. We will learn the material covered in the rest of the video later.

Extra Credit: Prove that if v is an eigenvector for a matrix A, and you rescale v by a real number r, then rv is also an eigenvector of A with the same eigenvalue.

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Part II: Methods to find Eigenvalues and Eigenvectors

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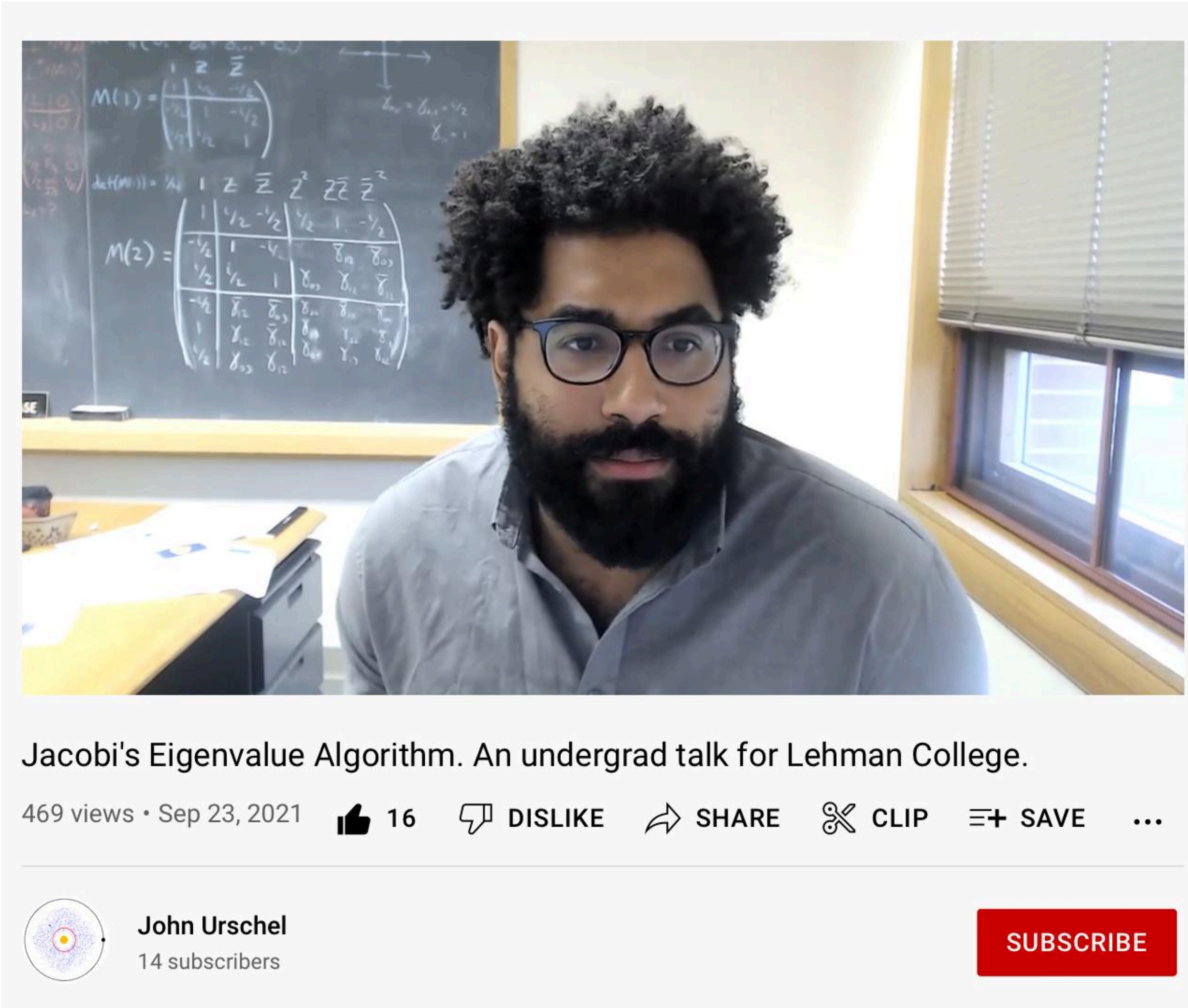
There are various techniques for finding the eigenvalues and eigenvectors of matrices. For large matrices, algorithms have been developed and coded into computers. See for example the wikipedia article about eigenvalue algorithms which you may consult if one comes up in a future course or in the workplace.

We will learn Power Iteration today.

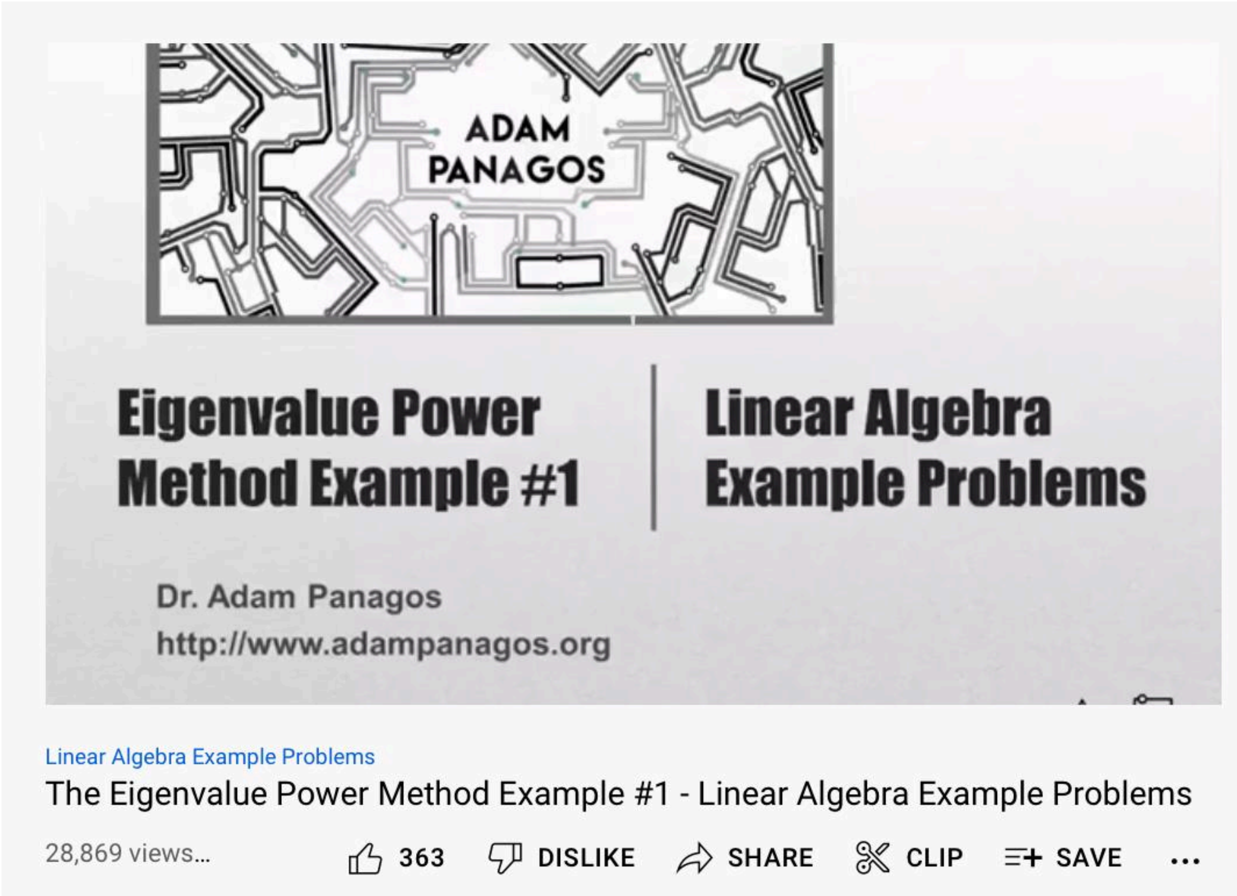
Another method is to use the Characteristic Polynomial which we will learn next month.

A third is the Jacobi Eigenvalue Algorithm. We will have a guest speaker, Dr. Urschel, present this method:

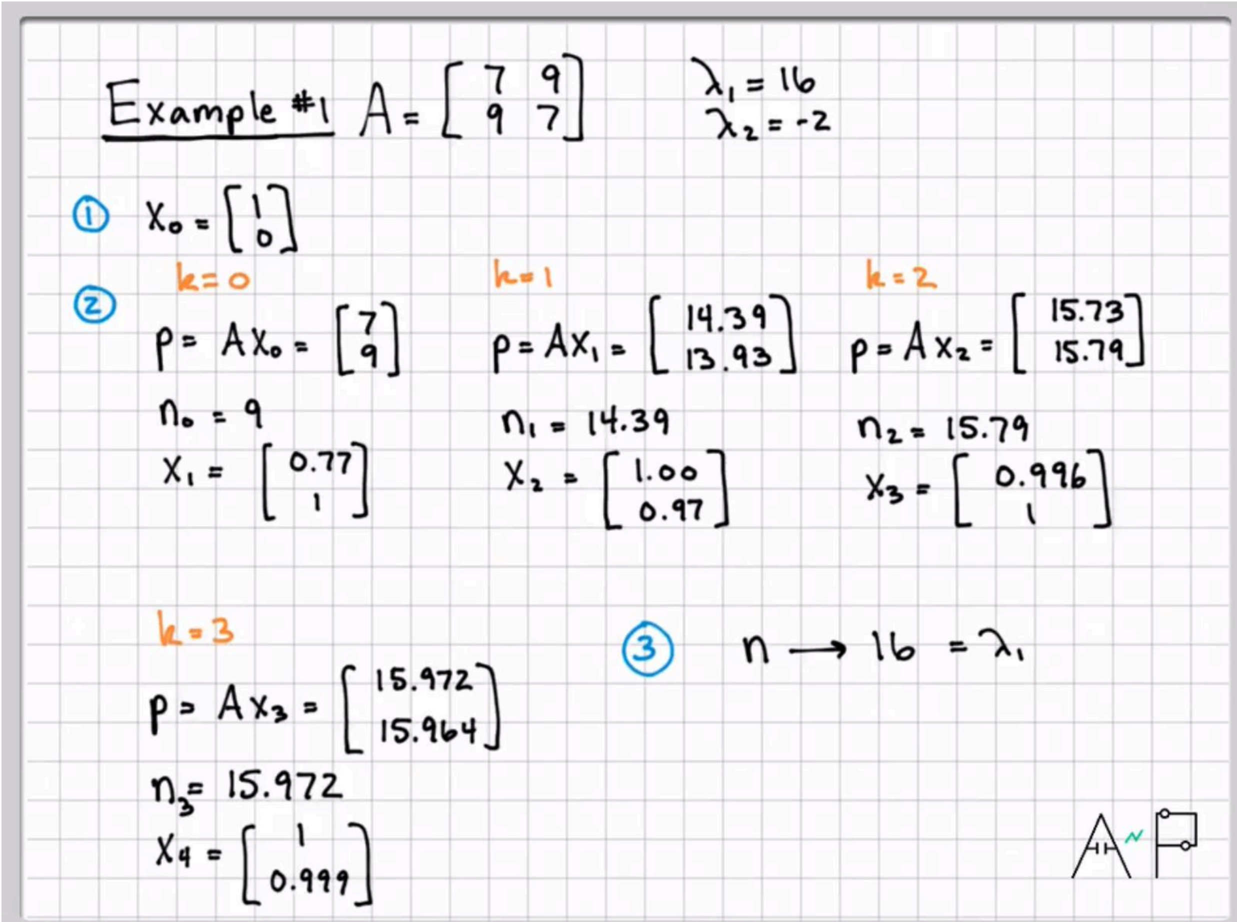


His talk is a little too advanced for the class to follow yet, so we will wait until later in the semester to include it in the course.

So let's learn power iteration now from Dr. Panagos:



Watch Dr. Panagos on youtube: https://youtu.be/yBiQh1vsCLU



In fact we can adapt the power method iteratively to find all the eigenvalues of a good matrix, but we will learn this later.

When does the eigenvalue power method work?

It works if the matrix $A \in \mathbb{R}^{n \times n}$ has $A\vec{v}_i = \lambda_i \vec{v}_i$ for $i = 1$ to n

such that the system

$$c_1 \vec{v}_1 + \dots + c_n \vec{v}_n = \vec{\omega}$$

can always be solved

The power method is using

$$A^k \vec{\omega} = c_1 \lambda_1^k \vec{v}_1 + \dots + c_n \lambda_n^k \vec{v}_n$$

If $|\lambda_1| > |\lambda_2| \geq \dots \geq |\lambda_n|$

Then $c_1 \lambda_1^k \vec{v}_1$ gets much larger than the other terms.

The division in the algorithm keeps it away from $\to \infty$.

We won't prove this but you can think about it.

$\boxed{\text{HW8}}$ Use the power method to find the largest eigenvalue of $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$ starting with $\vec{x}_0 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Example #1 $A = \begin{bmatrix} 7 & 9 \\ 9 & 7 \end{bmatrix}$ $\quad \lambda_1 = 16$, $\lambda_2 = -2$

① $X_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

② k=0

$p = AX_0 = \begin{bmatrix} 7 \\ 9 \end{bmatrix}$

$n_0 = 9$

$X_1 = \begin{bmatrix} 0.77 \\ 1 \end{bmatrix}$

k=1

$p = AX_1 = \begin{bmatrix} 14.39 \\ 13.93 \end{bmatrix}$

$n_1 = 14.39$

$X_2 = \begin{bmatrix} 1.00 \\ 0.97 \end{bmatrix}$

k=2

$p = AX_2 = \begin{bmatrix} 15.73 \\ 15.79 \end{bmatrix}$

$n_2 = 15.79$

$X_3 = \begin{bmatrix} 0.996 \\ 1 \end{bmatrix}$

k=3

$p = AX_3 = \begin{bmatrix} 15.972 \\ 15.964 \end{bmatrix}$

$n_3 = 15.972$

$X_4 = \begin{bmatrix} 1 \\ 0.999 \end{bmatrix}$

③ $n \to 16 = \lambda_1$

Hint for HW8

follow this method

HW8 Use the power method to find the largest eigenvalue of $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$ starting with $x_0 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Solution for HW8 is below to check your work

① $\vec{x}_0 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

② k=0

$A\vec{x}_0 = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3\cdot 1 + 0\cdot 2 \\ 0\cdot 1 + 2\cdot 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

n_0 is the largest number here

so $n_0 = 4$

$\vec{x}_1 = \frac{1}{n_0} A\vec{x}_0 = \frac{1}{4}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 3/4 \\ 1 \end{pmatrix}$

k=1

$A\vec{x}_1 = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 3/4 \\ 1 \end{pmatrix} = \begin{pmatrix} 3(3/4) + 0\cdot 1 \\ 0(3/4) + 2\cdot 1 \end{pmatrix} = \begin{pmatrix} 9/4 \\ 2 \end{pmatrix} = \begin{pmatrix} 2.25 \\ 2 \end{pmatrix}$

n_1 is the largest number here

so $n_1 = 2.25$

$\vec{x}_2 = \frac{1}{2.25} A\vec{x}_1 = \frac{1}{2.25}\begin{pmatrix} 2.25 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 8/9 \end{pmatrix}$

k=2 $A\vec{x}_2 = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 8/9 \end{pmatrix} = \begin{pmatrix} 3 \\ 16/9 \end{pmatrix}$

$n_2 = 3$ $\quad \vec{x}_3 = \frac{1}{3} A\vec{x}_2 = \frac{1}{3}\begin{pmatrix} 3 \\ 16/9 \end{pmatrix} = \begin{pmatrix} 1 \\ 16/27 \end{pmatrix}$

k=3 $A\vec{x}_3 = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 16/27 \end{pmatrix} = \begin{pmatrix} 3 \\ 32/27 \end{pmatrix}$

$n_3 = 3$ $\quad \vec{x}_4 = \frac{1}{n_3} A\vec{x}_3 = \begin{pmatrix} 1 \\ 32/81 \end{pmatrix}$

same as n_2 so 3 is the eigenvalue

k=4 $A\vec{x}_3 = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 32/81 \end{pmatrix} = \begin{pmatrix} 3 \\ 64/81 \end{pmatrix}$

$n_4 = 3$ $\quad \vec{x}_4 = \frac{1}{3} A\vec{x}_3 = \frac{1}{3}\begin{pmatrix} 3 \\ 64/81 \end{pmatrix} \approx \begin{pmatrix} 1 \\ 21/81 \end{pmatrix}$ ← I approximated here

k=5 $A\vec{x}_4 = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 21/81 \end{pmatrix} = \begin{pmatrix} 3 \\ 10/81 \end{pmatrix}$

$n_4 = 3$ $\quad \vec{x}_5 = \frac{1}{3} A\vec{x}_4 = \begin{pmatrix} 1 \\ 7/81 \end{pmatrix}$

Notice $\vec{x}_k = \begin{pmatrix} 1 \\ \text{something small} \end{pmatrix} \xrightarrow{\text{as } k \to \infty} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Check $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ ✓

Why does HW8 work?

these are the eigenvectors

these are the eigenvalues

$$\begin{pmatrix}3&0\\0&2\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}3&0\\0&2\end{pmatrix}\left(1\begin{pmatrix}1\\0\end{pmatrix}+2\begin{pmatrix}0\\1\end{pmatrix}\right) = 1\cdot 3\begin{pmatrix}1\\0\end{pmatrix} + 2\cdot 2\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}3\\4\end{pmatrix}$$

$n_1 = 4$

$$\text{So } \vec{x}_1 = \frac{1}{4}\left(1\cdot 3\begin{pmatrix}1\\0\end{pmatrix} + 2\cdot 2\begin{pmatrix}0\\1\end{pmatrix}\right)$$

$$\begin{pmatrix}3&0\\0&2\end{pmatrix}\vec{x}_1 = \begin{pmatrix}3&0\\0&2\end{pmatrix}\frac{1}{4}\left(1\cdot 3\begin{pmatrix}1\\0\end{pmatrix} + 2\cdot 2\begin{pmatrix}0\\1\end{pmatrix}\right)$$

$$= \frac{1}{4}\left(1\cdot 3^2\begin{pmatrix}1\\0\end{pmatrix} + 2\cdot 2^2\begin{pmatrix}0\\1\end{pmatrix}\right) = \begin{pmatrix}9/4\\2\end{pmatrix}$$

$n_2 = \frac{9}{4}$

$$\text{So } \vec{x}_2 = \left(\frac{\cancel{4}}{9}\right)\left(\frac{1}{\cancel{4}}\right)\left(1\cdot 3^2\begin{pmatrix}1\\0\end{pmatrix} + 2\cdot 2^2\begin{pmatrix}0\\1\end{pmatrix}\right)$$

$$\begin{pmatrix}3&0\\0&2\end{pmatrix}\vec{x}_2 = \frac{1}{9}\left(1\cdot 3^3\begin{pmatrix}1\\0\end{pmatrix} + 2\cdot 2^3\begin{pmatrix}0\\1\end{pmatrix}\right) = \begin{pmatrix}3\\2^4/9\end{pmatrix}$$

$n_3 = 3$

$$\text{So } \vec{x}_3 = \frac{1}{3}\frac{1}{9}\left(1\cdot 3^3\begin{pmatrix}1\\0\end{pmatrix} + 2\cdot 2^3\begin{pmatrix}0\\1\end{pmatrix}\right)$$

$\cdots$

$$\text{So } \vec{x}_k = \frac{1}{3^k}\left(1\cdot 3^k\begin{pmatrix}1\\0\end{pmatrix} + 2\cdot 2^k\begin{pmatrix}0\\1\end{pmatrix}\right)$$

$$= \begin{pmatrix}1\\0\end{pmatrix} + \frac{2\cdot 2^k}{3^k}\begin{pmatrix}0\\1\end{pmatrix}$$

as $k \to \infty$ $\quad\downarrow\quad\downarrow$

$$\begin{pmatrix}1\\0\end{pmatrix} + 0\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$$

first eigenvector dominates

HW 9 Use the power method to find the largest eigenvalue of $\begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$ starting with $\vec{x}_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

HW 10 What goes wrong if you try to use the power method to find the largest eigenvalue of $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ starting with any vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

HW 10 What goes wrong if you try to use the power method to find the largest eigenvalue of $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ starting with any vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Solution to HW 10

Try First!

$\vec{x}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$k=1$ $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$n_1 = 1$

$\vec{x}_1 = \frac{1}{1}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$k=2$ $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

$n_2 = |-1| = 1$

$\vec{x}_2 = \frac{1}{1}\begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

$k=3$ $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

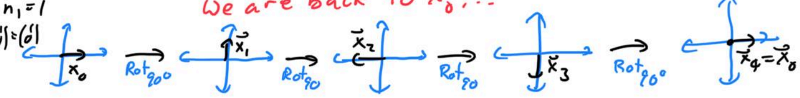
$n_3 = |-1| = 1$ ← It seems maybe the eigenvalue is 1?

$\vec{x}_3 = \frac{1}{1}\begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

$k=4$ $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ ← WAIT! We are back to $\vec{x}_0$!!!

$n_1 = 1$

$\vec{x}_4 = \frac{1}{1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$



No eigenvector!

Once you have completed this lesson, you can contribute again to the group project, and then start preparing for Quiz 4.