

Linear Algebra MAT313 Spring 2023

Professor Sormani

**Lesson 15**

**Proofs with Matrices**

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**If it is after March 23,  
go directly to Lesson 17.  
You will do this lesson later.  
Lesson 17 has priority!**

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**Before you start, find your team's project part 2 document and submit one last step for the project.**

**You will start the group project part 3 after this lesson.**

***If you work with any classmates on this lesson, be sure to write their names on the problems you completed together.***

***You will cut and paste the photos of your notes and completed classwork in a googledoc entitled:***

***MAT313S23-lesson15-lastname-firstname***

***and share editing of that document with me [sormanic@gmail.com](mailto:sormanic@gmail.com). You will also include your homework and any corrections to your homework in this doc.***

If you have a question, type **QUESTION** in your googledoc next to the point in your notes that has a question and email me with the subject MAT313 QUESTION. I will answer your question by inserting a photo into your googledoc or making an extra video.

Today we have two parts:

Part I teaches basic proofs  
and has five required HW problems

Part II (extra credit required for math majors)  
uses sum notation for proofs and has extra credit problems

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**Part I:** Watch [Playlist 313F20-15-PartI](#)

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## Lesson on Proofs with Matrices

In this lesson we will prove:

Thm Distribution of Matrix Mult over Add

$$A \times (B + C) = A \times B + A \times C$$

Thm Associativity of Matrix Multiplication

$$A \times (B \times C) = (A \times B) \times C$$

Defn: The zero matrix,  $O$ , is a matrix which has zeroes everywhere.

Thm  $A \times O = ?$  and  $O \times A = ?$

Defn The identity matrix,  $I$ , is a square matrix with 1's on the diagonal and zeroes elsewhere.

Thm  $A \times I = ?$  and  $I \times A = ?$

What about

$$A \times B \stackrel{?}{=} B \times A?$$

## Defn of Matrix Mult

$$A \in M_{n \times m} \quad \begin{matrix} m \text{ columns} \\ n \text{ rows} \end{matrix}$$

$$B \in M_{m \times l} \quad A \times B \in M_{n \times l}$$

$$[A \times B]_{ik} = \sum_{j=1}^m a_{ij} b_{jk}$$

$$A \times B = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1l} \\ b_{21} & b_{22} & \dots & b_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \dots & b_{ml} \end{pmatrix} = ?$$

$j$  counter goes across the  $i$ th row of  $A$

$j$  counter goes down in the  $k$ th column of  $B$

$[A \times B]_{ik}$  = dot product of  $i$ th row of  $A$  and  $k$ th column of  $B$

## Defn of Matrix Addition

$$[A + B]_{ik} = [A]_{ik} + [B]_{ik}$$

Example:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1+1 & 2+6 \\ 3+0 & 4+1 \end{bmatrix}$$

$$A + B \in M_{n \times m}$$

when  $A, B \in M_{n \times m}$ .

1:37 AM Sat Sep 26

Linear Algebra

Thm Distribution of Matrix Mult over Add.

$$A \times (B + C) = A \times B + A \times C$$

Proof for  $A, B, C \in M_{2 \times 2}$ :

$$\textcircled{1} A \times (B + C) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \times \left( \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \right)$$

$$\textcircled{2} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \times \begin{bmatrix} (b_{11} + c_{11}) & (b_{12} + c_{12}) \\ (b_{21} + c_{21}) & (b_{22} + c_{22}) \end{bmatrix}$$

$$\textcircled{3} = \begin{bmatrix} a_{11}(b_{11} + c_{11}) + a_{12}(b_{21} + c_{21}) \\ a_{21}(b_{11} + c_{11}) + a_{22}(b_{21} + c_{21}) \end{bmatrix}$$

$$\textcircled{4} = \begin{bmatrix} a_{11}b_{11} + a_{11}c_{11} + a_{12}b_{21} + a_{12}c_{21} \\ a_{21}b_{11} + a_{21}c_{11} + a_{22}b_{21} + a_{22}c_{21} \end{bmatrix}$$

Next do right  
hand side  
until they match

Linear Algebra

Statements Justifications

$$\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \textcircled{1} \text{ Given } A, B, C \in M_{2 \times 2} \text{ Defn of } M_{2 \times 2}$$

$\textcircled{2}$  Defn of Matrix Addition

$$\begin{bmatrix} a_{11}(b_{12} + c_{12}) + a_{12}(b_{22} + c_{22}) \\ a_{21}(b_{12} + c_{12}) + a_{22}(b_{22} + c_{22}) \end{bmatrix} \textcircled{3} \text{ Defn of Matrix Mult}$$

$$\begin{bmatrix} a_{11}b_{12} + a_{11}c_{12} + a_{12}b_{22} + a_{12}c_{22} \\ a_{21}b_{12} + a_{21}c_{12} + a_{22}b_{22} + a_{22}c_{22} \end{bmatrix} \textcircled{4} \text{ distribution of mult over add for } \mathbb{R}$$



$$\textcircled{5} \text{ RHS } A \times B + A \times C =$$

$$= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \times \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \times \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

$$\textcircled{6} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

$$+ \begin{bmatrix} a_{11}c_{11} + a_{12}c_{21} & a_{11}c_{12} + a_{12}c_{22} \\ a_{21}c_{11} + a_{22}c_{21} & a_{21}c_{12} + a_{22}c_{22} \end{bmatrix}$$

$$\textcircled{7} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{11}c_{11} + a_{12}c_{21} & a_{11}b_{12} + a_{12}b_{22} + a_{11}c_{12} + a_{12}c_{22} \\ a_{21}b_{11} + a_{22}b_{21} + a_{21}c_{11} + a_{22}c_{21} & a_{21}b_{12} + a_{22}b_{22} + a_{21}c_{12} + a_{22}c_{22} \end{bmatrix}$$

Now check  
this matrix



$\textcircled{5}$  Given  $A, B, C \in M_{2 \times 2}$   
and defn of  $M_{2 \times 2}$ .

$\textcircled{6}$  Defn of Matrix Mult  
and Order of Operations

$\textcircled{7}$  Defn of Matrix  
Addition

$$\begin{bmatrix} a_{11}b_{12} + a_{12}b_{22} + a_{11}c_{12} + a_{12}c_{22} \\ a_{21}b_{12} + a_{22}b_{22} + a_{21}c_{12} + a_{22}c_{22} \end{bmatrix}$$

matches  
step 4

QED

Thm Associativity of Matrix Mult

$$A \times (B \times C) = (A \times B) \times C$$

Proof (for  $A, B, C \in M_{2 \times 2}$ ):

① LHS  $A \times (B \times C) =$

$$= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \times \left( \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \times \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \right)$$

②  $= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \times \begin{bmatrix} (b_{11}c_{11} + b_{12}c_{21}) & (b_{11}c_{12} + b_{12}c_{22}) \\ (b_{21}c_{11} + b_{22}c_{21}) & (b_{21}c_{12} + b_{22}c_{22}) \end{bmatrix}$

③  $= \begin{bmatrix} a_{11}(b_{11}c_{11} + b_{12}c_{21}) + a_{12}(b_{21}c_{11} + b_{22}c_{21}) & a_{11}(b_{11}c_{12} + b_{12}c_{22}) + a_{12}(b_{21}c_{12} + b_{22}c_{22}) \\ a_{21}(b_{11}c_{11} + b_{12}c_{21}) + a_{22}(b_{21}c_{11} + b_{22}c_{21}) & a_{21}(b_{11}c_{12} + b_{12}c_{22}) + a_{22}(b_{21}c_{12} + b_{22}c_{22}) \end{bmatrix}$

④ Simplify

HW1 Fill in  
rest of proof

① Given  $A, B, C \in M_{2 \times 2}$   
Defn of  $M_{2 \times 2}$

② Defn of Matrix  
Mult.

$$\left[ \begin{array}{l} a_{11}(\text{---}) + a_{12}(\text{---}) \\ a_{21}(\text{---}) + a_{22}(\text{---}) \end{array} \right]$$

③ Defn of Matrix Mult.

④ algebra



Defn The zero matrix,  $\mathbf{0}$ , is a matrix which has zeroes everywhere.

Thm  $A \times \mathbf{0} = \mathbf{0}$

Proof: ( $A \in M_{2 \times 3}$   $\mathbf{0} \in M_{3 \times 4}$ )

$$\textcircled{1} A \times \mathbf{0} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \times \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\textcircled{2} = \begin{bmatrix} a_{11} \cdot 0 + a_{12} \cdot 0 + a_{13} \cdot 0 & a_{11} \cdot 0 + \dots & 0 + 0 + 0 & 0 \\ a_{21} \cdot 0 + a_{22} \cdot 0 + a_{23} \cdot 0 & 0 + 0 + 0 & 0 + 0 + 0 & 0 \end{bmatrix}$$

$$\textcircled{3} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \in M_{2 \times 4}$$

HW2  $\mathbf{0} \times A = \mathbf{0}$  when  $A \in M_{2 \times 3}$   $\mathbf{0} \in M_{4 \times 2}$

we will figure out the answer.

$\textcircled{1}$  by given  $A \in M_{2 \times 3}$  & defn  $M_{2 \times 3}$   
by defn  $\mathbf{0}$  matrix in  $M_{3 \times 4}$

$\textcircled{2}$  by defn of matrix mult

$\textcircled{3}$  by arithmetic

2:19 AM Sat Sep 26

Linear Algebra

Defn The Identity Matrix,  $I$ , is a square matrix with 1's on diagonal and 0's elsewhere.

Thm  $A \times I = ?$

Proof:  $I \in M_{3 \times 3}$   $A \in M_{4 \times 3}$

①  $A \times I =$

$$= \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

②

$$= \begin{pmatrix} a_{11} \cdot 1 + a_{12} \cdot 0 + a_{13} \cdot 0 & a_{11} \cdot 0 + a_{12} \cdot 1 + a_{13} \cdot 0 & a_{11} \cdot 0 + a_{12} \cdot 0 + a_{13} \cdot 1 \\ & & \\ & & \\ & & \end{pmatrix}$$

Linear Algebra

$I \in M_{2 \times 2}$   $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$I \in M_{3 \times 3}$   $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

HW3 fill in blank s

①

②

Simplify

**HW1-HW3** are described above.

**HW4** Prove  $I \times B = B$  when  $B$  is  $2 \times 4$  and  $I$  is the correct choice of identity matrix.

**HW5** is to find the errors in the incorrect proof below which has many errors so find all of the errors. Then either fix the proof or find a pair of specific  $2 \times 2$  matrices  $A$  and  $B$  for

which this fails.

2:36 AM Sat Sep 26

Linear Algebra

False Thm  $A \times B = B \times A$   
Proof  $(A, B \in M_{2 \times 2})$

① <sup>LHS</sup>  $A \times B = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \times \begin{pmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{pmatrix}$

②  $= \begin{pmatrix} a_{11}b_{11} + a_{12}b_{12} & a_{11}b_{21} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{12} & a_{21}b_{21} + a_{22}b_{22} \end{pmatrix}$

③ <sup>RHS</sup>  $B \times A = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \times \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

④  $= \begin{pmatrix} b_{11}a_{11} + b_{12}a_{21} & b_{21}a_{11} + b_{22}a_{12} \\ b_{11}a_{21} + b_{12}a_{22} & b_{21}a_{21} + b_{22}a_{22} \end{pmatrix}$

HW5 Mark All errors in the "proof" below.

① by matrix mult

② by matrix mult

③ by matrix mult

④ by matrix mult.

They Match QED

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**Part II:** Watch [Playlist 313F20-15-PartII](#) which is **Extra Credit** highly recommended for math majors and has three extra credit problems.

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Lesson on Proofs with Matrices

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Thm Distribution of Matrix Mult over Add

$$A \times (B + C) = A \times B + A \times C$$

Thm Associativity of Matrix Multiplication

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Thm  $A \times O = ?$  and  $O \times A = ?$

Defn The identity matrix,  $I$ , is a square matrix with 1's on the diagonal and zeroes elsewhere.

Thm  $A \times I = ?$  and  $I \times A = ?$

What about

$$A \times B \stackrel{?}{=} B \times A?$$

Part II Proofs

using  $\sum$  notation

Defn of Matrix Mult

$$[A \times B]_{ik} = \sum_{j=1}^m a_{ij} b_{jk}$$

$\begin{matrix} i\text{th} \\ \text{row} \\ \text{of} \\ A \end{matrix}$ 
 $\begin{pmatrix} a_{i1} & a_{i2} & \dots & a_{im} \end{pmatrix}$ 
 $\cdot$ 
 $\begin{pmatrix} b_{1k} \\ b_{2k} \\ \vdots \\ b_{mk} \end{pmatrix}$ 
 $\begin{matrix} k\text{th} \\ \text{column} \\ \text{of} \\ B \end{matrix}$

Defn of Addition

$$[A + B]_{ik} = [A]_{ik} + [B]_{ik} = a_{ik} + b_{ik}$$



Thm Distribution of Matrix Mult over Add

$$A \times (B + C) = A \times B + A \times C$$

Proof using  $\Sigma$  notation

for  $A \in M_{n \times m}$ ,  $B, C \in M_{m \times l}$ :

$$\textcircled{1} \text{ LHS } [A \times (B + C)]_{ik} = \sum_{j=1}^m [A]_{ij} [B + C]_{jk} \quad \textcircled{1} \text{ by defn of matrix mult}$$

$$\textcircled{2} = \sum_{j=1}^m [A]_{ij} ([B]_{jk} + [C]_{jk}) \quad \textcircled{2} \text{ defn of matrix addition}$$

$$\textcircled{3} = \sum_{j=1}^m ([A]_{ij} [B]_{jk} + [A]_{ij} [C]_{jk}) \quad \textcircled{3} \text{ distribution of mult over addition of reals.}$$

$$\textcircled{4} = \sum_{j=1}^m ([A]_{ij} [B]_{jk}) + \sum_{j=1}^m ([A]_{ij} [C]_{jk}) \quad \textcircled{4} \text{ req. } \Sigma_{j=1}^m$$

Part II Proofs

using  $\Sigma$  notation

Defn of Matrix Mult

$$[A \times B]_{ik} = \sum_{j=1}^m [A]_{ij} [B]_{jk}$$

ith row of A  $a_{i1} \ a_{i2} \ \dots \ a_{im}$   $\cdot$   $\begin{pmatrix} b_{1k} \\ b_{2k} \\ \vdots \\ b_{mk} \end{pmatrix}$  kth column of B

Defn of Addition

$$[A + B]_{ik} = [A]_{ik} + [B]_{ik} = a_{ik} + b_{ik}$$



Thm Distribution of Matrix Mult over Add

$$A \times (B + C) = A \times B + A \times C$$

Proof using  $\Sigma$  notation

for  $A \in M_{n \times m}$ ,  $B, C \in M_{m \times l}$ :

$$\textcircled{1} \text{ LHS } [A \times (B + C)]_{ik} = \sum_{j=1}^m [A]_{ij} [B + C]_{jk} \quad \textcircled{1} \text{ by defn of matrix mult}$$

$$\textcircled{2} = \sum_{j=1}^m [A]_{ij} ([B]_{jk} + [C]_{jk}) \quad \textcircled{2} \text{ defn of matrix addition}$$

$$\textcircled{3} = \sum_{j=1}^m ([A]_{ij} [B]_{jk} + [A]_{ij} [C]_{jk}) \quad \textcircled{3} \text{ distribution of mult over addition of reals.}$$

$$\textcircled{4} = \sum_{j=1}^m ([A]_{ij} [B]_{jk}) + \sum_{j=1}^m ([A]_{ij} [C]_{jk}) \quad \textcircled{4} \text{ req. } \Sigma \text{ law}$$

$\textcircled{5}$  RHS

$$[A \times B + A \times C]_{ik} =$$

$$= [A \times B]_{ik} + [A \times C]_{ik}$$

$\textcircled{5}$  defn of matrix add.

$$\textcircled{6} = \sum_{j=1}^m [A]_{ij} [B]_{jk} + \sum_{j=1}^m [A]_{ij} [C]_{jk} \quad \textcircled{6} \text{ by def matrix Mult.}$$

Steps 4 & 6 Match QED

EC  $A, B \in M_{n \times m}$   $C \in M_{m \times l}$

$$(A + B) \times C = A \times C + B \times C$$

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Thm Associativity of Matrix Multiplication

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Defn: The zero matrix,  $O$ , is a matrix which has zeroes everywhere.

Thm  $A \times O = O$  and  $O \times A = O$

Defn The identity matrix,  $I$ , is a square matrix with 1's on the diagonal and zeroes elsewhere.

Thm  $A \times I = A$  and  $I \times A = A$

What about

$$A \times B \stackrel{?}{=} B \times A \quad \text{FALSE}$$

Thm: If  $A \in M_{n \times m}$  and  $O \in M_{m \times l}$   
then  $A \times O = O \in M_{n \times l}$

Proof:  
 (1)  $[A \times O]_{ik} = \sum_{j=1}^m [A]_{ij} [O]_{jk}$   
 (1) by defn matrix mult

(2)  $= \sum_{j=1}^m [A]_{ij} \cdot 0$  (2) by defn of zero matrix

(3)  $= \sum_{j=1}^m 0$  (3) by  $a \cdot 0 = 0$  for any  $a \in \mathbb{R}$

(4)  $= 0$  (4)  $0 + 0 = 0$

(5)  $= [O]_{ik}$  (5) Defn of zero matrix QED

EC  $O \times A = O$   
 $\begin{matrix} \nearrow & \nwarrow & \nearrow \\ M_{n \times m} & M_{m \times l} & M_{n \times l} \end{matrix}$



## Lesson 15 Proofs with Matrices

In this lesson we will prove:

Thm Distribution of Matrix Mult over Add

$$A \times (B + C) = A \times B + A \times C$$

Thm Associativity of Matrix Multiplication

$$A \times (B \times C) = (A \times B) \times C$$

Defn: The zero matrix,  $O$ , is a matrix which has zeroes everywhere.

Thm  $A \times O = O$  and  $O \times A = O$

Defn The identity matrix,  $I$ , is a square matrix with 1's on the diagonal and zeroes elsewhere.

Thm  $A \times I = A$  and  $I \times A = A$

What about

$$A \times B \stackrel{?}{=} B \times A \quad \text{FALSE}$$



Thm Given  $A \in M_{n \times m}$   $I \in M_{m \times m}$   
 then  $A \times I = A$  (EC3)  $I \times A = A$   $I \in M_{m \times m}$   $A \in M_{n \times m}$

Thoughts  
 $[I]_{ij} = \begin{cases} 1 & \text{if } i=j \text{ (diagonal)} \\ 0 & \text{if } i \neq j \text{ (elsewhere)} \end{cases}$   
 Defn of Identity Matrix

Proof:

$$\textcircled{1} [A \times I]_{ik} = \sum_{j=1}^m [A]_{ij} [I]_{jk} \quad \textcircled{1} \text{ by defn matrix mult}$$

$$\textcircled{2} = [A]_{i1} I_{1k} + [A]_{i2} I_{2k} + \dots + [A]_{ik} I_{kk} + \dots + [A]_{im} I_{mk} \quad \textcircled{2} \text{ defn of } I$$

$$\textcircled{3} = [A]_{i1} \cdot 0 + [A]_{i2} \cdot 0 + \dots + [A]_{ik} \cdot 1 + \dots + [A]_{im} \cdot 0 \quad \textcircled{3} \text{ by defn of identity}$$

$$\textcircled{4} = 0 + 0 + \dots + 0 + [A]_{ik} + 0 + \dots + 0 \quad \textcircled{4} \text{ algebra}$$

$$\textcircled{5} = [A]_{ik} \quad \textcircled{5} \text{ by } 0+a=a \text{ for } a \in \mathbb{R} \quad \text{QED}$$

**Extra Credit:** Prove for arbitrary matrices  $A$  and  $B$  and vector  $v$  using sum notation that  $A(Bv) = (A \times B)v$

**Extra Credit:** Prove for arbitrary matrices  $A$ ,  $B$ , and  $C$  using sum notation that  $A \times (B \times C) = (A \times B) \times C$