Modelling

$Value\ of\ credit\ risky\ derivative\ =\ Value\ of\ credit\ riskless\ derivative\ +\ counterparty\ credit\ risk\ adjustmen$

This can be written as

$$\text{Eq(1)}\quad \hat{A}(t, S_t, J_t^B, J_t^C)\ =\ A(t, S_t)\ +\ U(t, S_t, J_t^B, J_t^C)$$

where
$J_t^B, J_t^C \in \{0, 1\}$ are jump processes

Also, we assume

The risk free bond P accrues at the riskless rate r $\qquad dP_t / P_t\ =\ r_t dt$

The bank's risky zero-recovery bond P^B accrues at rate r^B $\qquad dP_t^B / P_t^B\ =\ r_t^B dt\ -\ dJ_t^B$

Counterparty's risky 0-recovery bond P^B accrues at rate r^C $\qquad dP_t^B / P_t^B\ =\ r_t^B dt\ -\ dJ_t^B$

The asset S drifts with μ and diffuses with volatility σ $\qquad dS\ =\ \mu S dt\ +\ \sigma S\, dW_t$

We also assume that $dW_t \quad dJ_t^B\ \&\ dJ_t^C$ are independent

Equation (1) has the following boundary conditions:

1. $\hat{A}(t, S_t, 0, 1)\ =\ R_C\, max(A(t,\ S_t), 0)\ +\ min(A(t,\ S_t), 0)$

   This boundary condition states that if the counterparty defaults ($J_t^B\ =\ 0,\ J_t^C\ =\ 1$) with recovery rate R_C then the bank receives either nothing or R_C times the value of the asset at time t

2. $\hat{A}(t, S_t, 1,\ 0)\ =\ R_C\, min(A(t,\ S_t), 0)\ +\ max(A(t,\ S_t), 0)$

   Here, it states that if the bank default then the bank pays the full value of the asset

3. $\hat{A}(T, S_T, 1,\ 0)\ =\ A(T, S)$

   The payoff at expiry

A hedged self financing portfolio can be represented as

$$\Pi_t = \Delta_t S_t + \alpha_t P_t^B + \beta_t P_t^C + \chi_t = -\hat{A}_t$$

In the above hedged portfolio, the proceeds of the risky derivative are reinvested into

- a hedge amount for the asset $\Delta_t S_t$
- the bank's own bond $\alpha_t P_t^B$
- the counterparty's bond $\beta_t P_t^C$
- cash χ_t

The change in the portfolio is given by

Eq(2) $$d\Pi_t = \Delta_t dS_t + \alpha_t dP_t^B + \beta_t dP_t^C + d\chi_t = -d\hat{A}_t$$

Assuming that the cash can be rebalanced between the three assets or funded at the rate r^F

$$d\chi_t = d\chi_t^S + d\chi_t^F + d\chi_t^C$$

where

$$d\chi_t^S = \Delta_t(c_t - q_t) S_t dt$$

c is the income or dividend from the asset and q if the funding cost

$$d\chi_t^C = -\beta_t r_t P_t^C dt$$

assumed to be financed at repo rate and accrue at riskless rate

$$d\chi_t^F = [r_t max(-\hat{A}_t - \alpha_t P_t^B, 0) + r_t^F min(-\hat{A}_t - \alpha_t P_t^B, 0)]\, dt$$

if positive then funded at riskless rate, if negative then funded at funding rate

From Itô's formula for jump diffusion, we get

Eq(3)

$$d\hat{A}_t = \underbrace{\left(\frac{\partial \hat{A}_t}{\partial t} + \frac{1}{2}\sigma_t^2 S_t^2 \frac{\partial^2 \hat{A}_t}{\partial S_t^2}\right) dt + \frac{\partial \hat{A}_t}{\partial S_t} dS_t}_{\text{Black Scholes Equation}} + dJ^B \underbrace{[\hat{A}_t(1,0) - \hat{A}_t(0,0)]}_{\text{change in value if bank defaults}} + dJ^C \underbrace{[\hat{A}_t(0,1) - \hat{A}_t(0,0)]}_{\text{change in value if counterparty defaults}}$$

Set the hedges in Eq(2) as follows

$\Delta_t = -\ \partial \hat{A}_t / \partial S_t$

$\alpha_t = [\hat{A}_t(1,0) - \hat{A}_t(0,0)] / P_t^B$

$\beta_t = [\hat{A}_t(0,1) - \hat{A}_t(0,0)] / P_t^C$

Substituting these hedges and Eq(3) into Eq(2) we get the following Partial Differential Equation where all the stochastic factors are now cancelled

$$\frac{\partial \hat{A}_t}{\partial t} + \frac{1}{2}\sigma_t^2 S_t^2 \frac{\partial^2 \hat{A}_t}{\partial S_t^2} + (q_t - c_t) S_t \frac{\partial\ \hat{A}_t}{\partial S_t} - r\hat{A}_t\ = s_t^F[\hat{A}_t + \hat{A}_t(1,0) - \hat{A}_t(0,0)]\ - \lambda_t^B[\hat{A}_t(1,0) - \hat{A}_t(0,0)]$$
$$- \lambda_t^B[\hat{A}_t(1,0) - \hat{A}_t(0,0)]$$

where

$s_t^F = r_t^F - r_t$ the spread of the funding rate over the risk free rate

$\lambda_t^B = r_t^B - r_t$ can be interpreted as the intensity of default of the bank as represented by the spread on the bank rate

$\lambda_t^C = r_t^C - r_t$ can be interpreted as the intensity of default of the counterparty as represented by the spread on the counterparty rate

The solution to U_t in Eq(1) can be written as

$$U_t = -\ (1 - R_B) \int_t^T \underbrace{\lambda_u^B D_{r_u + \lambda_u^B + \lambda_u^C}(t,u)\ E^{Q}[max\{A(u,S_u),0\}|t\,]du}_{\text{DVA}}$$

$$-\ (1 - R_C) \int_t^T \underbrace{\lambda_u^C D_{r_u + \lambda_u^B + \lambda_u^C}(t,u)\ E^{Q}[max\{A(u,S_u),0\}|t\,]du}_{\text{CVA}}$$

$$- \int_t^T \underbrace{s_u^F D_{r_u + \lambda_u^B + \lambda_u^C}(t,u)\ E^{Q}[max\{A(u,S_u),0\}|t\,]du}_{\text{FVA}}$$

where

$D_x(t,T) = e^{\int_t^T x_u\,du}$ the risky discount factor equivalent to a default density

The above equation can be interpreted as

$$Counterparty\ Credit\ Risk\ Adjustment \ = \ DVA \ + \ CVA \ + \ FVA$$