Differential Geometry

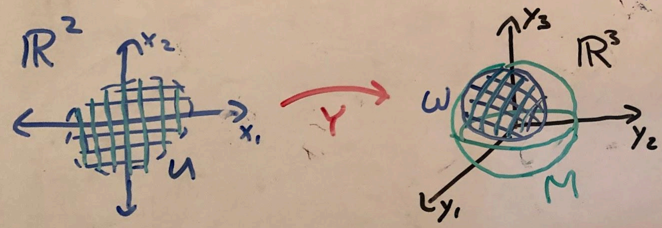
Lesson 21

Smooth Surfaces and Submanifolds

(be sure to go through this entire document watching all the videos)
HW is due on April 27

The **Video** **SurfacePart1** defines a ***Smooth Surface*** as a set in Euclidean Space that has ***coordinate charts or patches*** around every point explaining the next three photos:

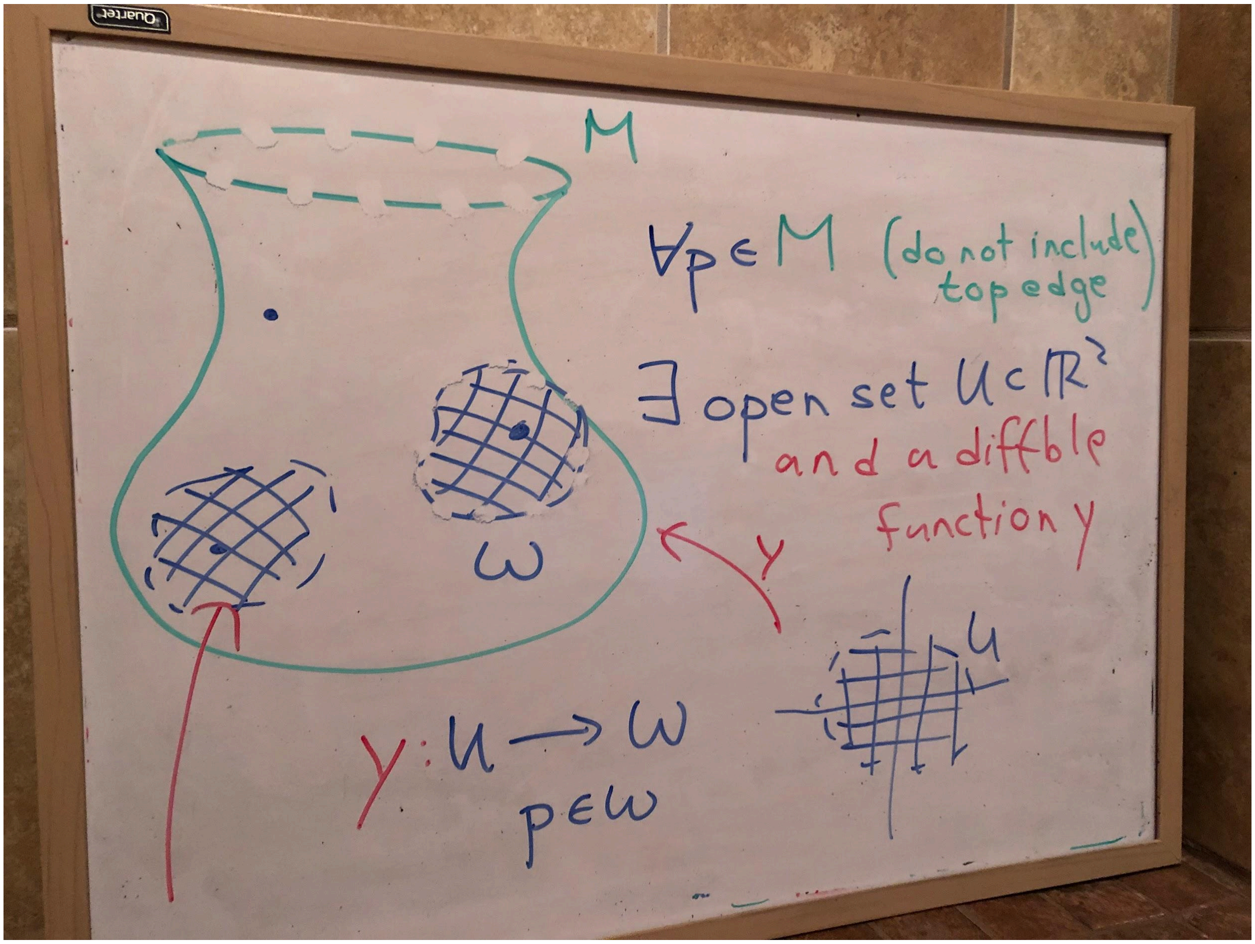
Defn: A smooth surface, $M \subset \mathbb{R}^3$, is a set such that $\forall p \in M$ $\exists$ open set $U \subset \mathbb{R}^2$ smooth function $Y : U \subset \mathbb{R}^2 \to W \subset M$ with $p \in W$ such that the columns of DY_q are linearly independant at every $q \in U$.

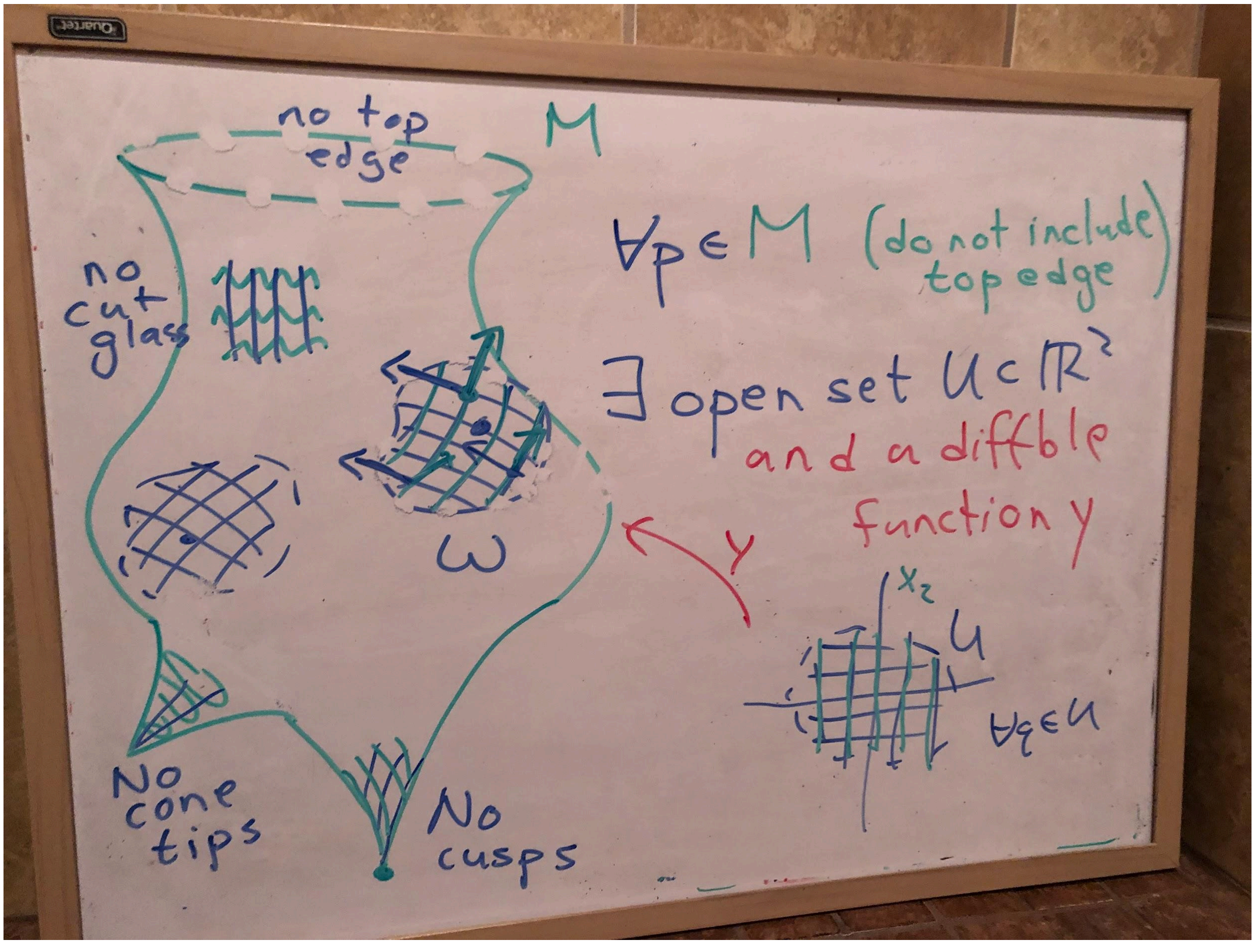


$\vec{Y}$ is called a coordinate chart or patch

$$DY_q = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} \end{pmatrix}$$

the columns are called tangent vectors they must be linearly independant


M
∀p ∈ M (do not include top edge)
∃ open set U ⊂ ℝ²
and a diffble function y
y
ω
U
y : U → ω
p ∈ ω

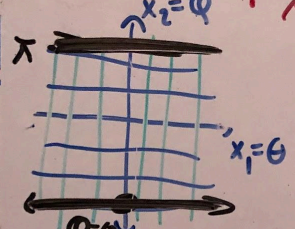


In the **Video** **SurfacePart2** we prove a unit sphere is a surface using spherical coordinates to construct two charts as in the next two photos. The book uses six charts as in the third photo.

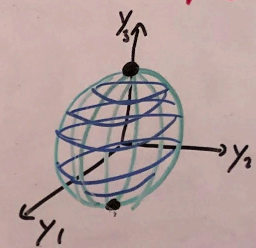
The unit sphere $p_1^2+p_2^2+p_3^2=1$ is a smooth surface.

Pf Use spherical coords to find the first patch

$\rho=1$



$$\vec{y}(\vec{x}) = \begin{pmatrix}\cos\theta\sin\varphi\\ \sin\theta\sin\varphi\\ \cos\varphi\end{pmatrix}$$



The patch cannot include $\varphi=0$ or $\varphi=\pi$

$$y: U \longrightarrow W \subset M^2$$

because $\rho=1$ our answers are in M^2

$$D\vec{y} = \begin{pmatrix} -\sin\theta\sin\varphi & \cos\theta\cos\varphi \\ \cos\theta\sin\varphi & \sin\theta\cos\varphi \\ 0 & -\sin\varphi \end{pmatrix}$$

has linearly indep columns on

$U=(-\infty,\infty)\times(0,\pi)$

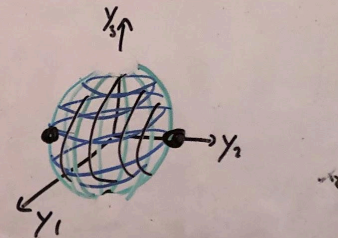
$\varphi=0$ and $\varphi=\pi$ are not good

Can also check cross product is not 0

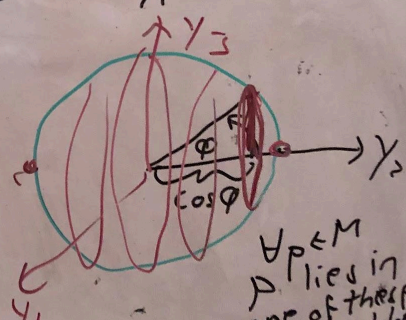
The <u>unit sphere</u> $p_1^2+p_2^2+p_3^2=1$ is a smooth surface.

<u>Pf</u> 2nd patch that covers both poles

Reorder the spherical coords so that the y_2 axis has poles



$$\vec{y}\begin{pmatrix}\theta\\ \varphi\end{pmatrix} = \begin{pmatrix}\cos\theta\sin\varphi\\ \cos\varphi\\ \sin\theta\sin\varphi\end{pmatrix}$$



$U = (-\infty,\infty)\times(0,\pi) \to W \subset M$

$\forall p \in M$ p lies in one of these two patches.

S^2 given by

$$x^1\left(\begin{pmatrix} s \\ t \end{pmatrix}\right) = \begin{pmatrix} \sqrt{1-s^2-t^2} \\ s \\ t \end{pmatrix}, \quad x^2\left(\begin{pmatrix} s \\ t \end{pmatrix}\right) = \begin{pmatrix} -\sqrt{1-s^2-t^2} \\ s \\ t \end{pmatrix},$$

$$y^1\left(\begin{pmatrix} s \\ t \end{pmatrix}\right) = \begin{pmatrix} s \\ \sqrt{1-s^2-t^2} \\ t \end{pmatrix}, \quad y^2\left(\begin{pmatrix} s \\ t \end{pmatrix}\right) = \begin{pmatrix} s \\ -\sqrt{1-s^2-t^2} \\ t \end{pmatrix},$$

$$z^1\left(\begin{pmatrix} s \\ t \end{pmatrix}\right) = \begin{pmatrix} s \\ t \\ \sqrt{1-s^2-t^2} \end{pmatrix}, \quad z^2\left(\begin{pmatrix} s \\ t \end{pmatrix}\right) = \begin{pmatrix} s \\ t \\ -\sqrt{1-s^2-t^2} \end{pmatrix}.$$

Each of these coordinate patches covers an open hemisphere; see Figure 5.2.1. We leave it to the reader to verify that these six maps are actually coordinate patches. ◊

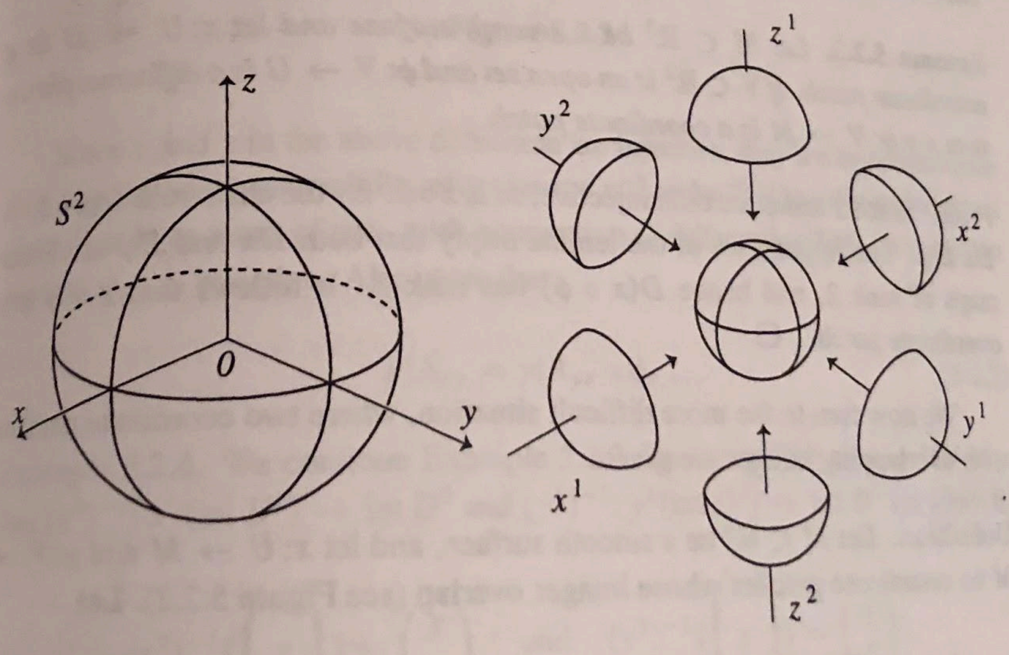
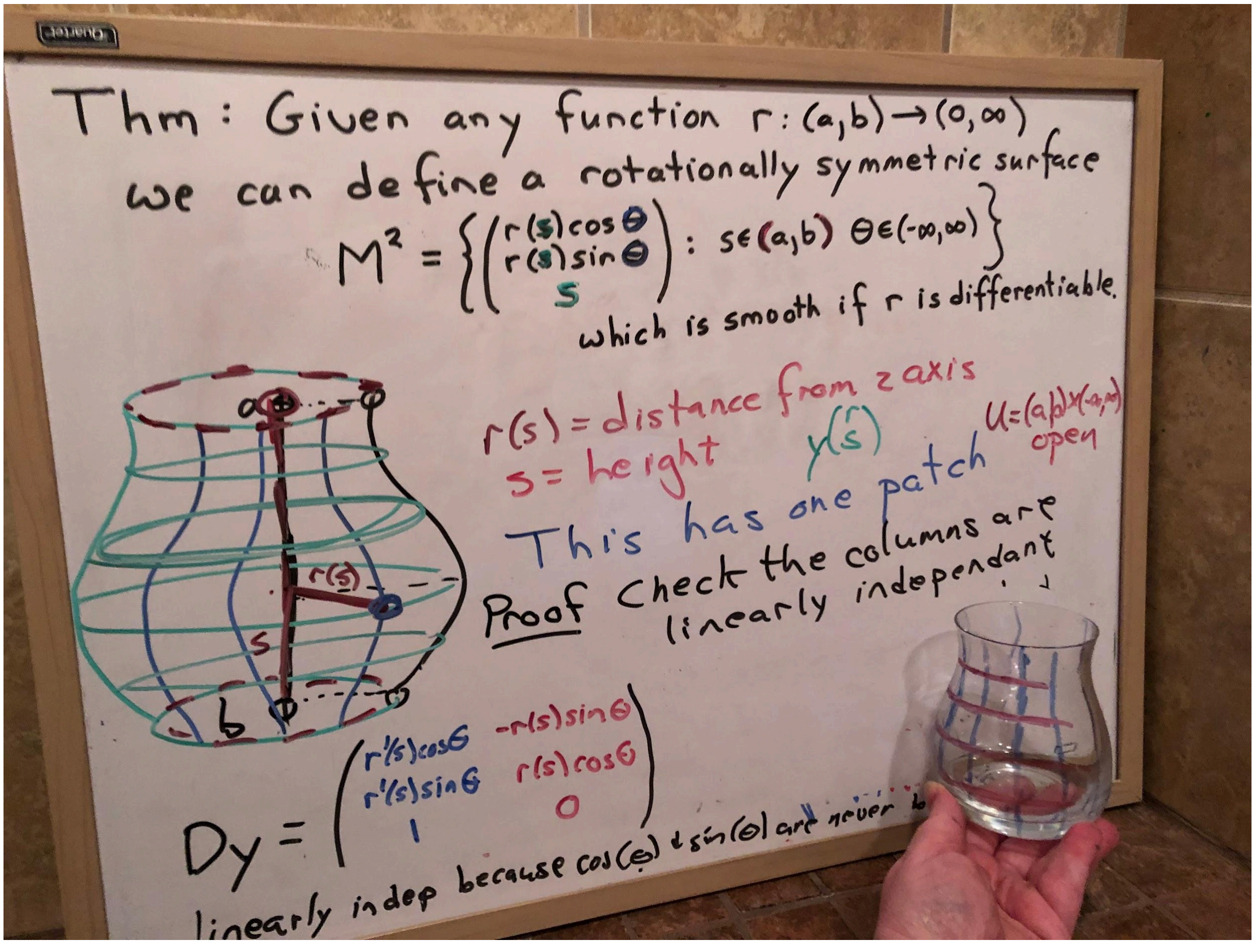


Figure 5.2.1

What is the relation between smooth surfaces, topological surfaces and simplicial surfaces? By definition any smooth surface is a topological surface. It then follows from Theorem 3.4.5 that every compact smooth surface can be triangulated. Is every topological surface also a smooth surface? Whereas not

In the **Video** [SurfacePart3]() we study the ***rotationally symmetric surfaces:***



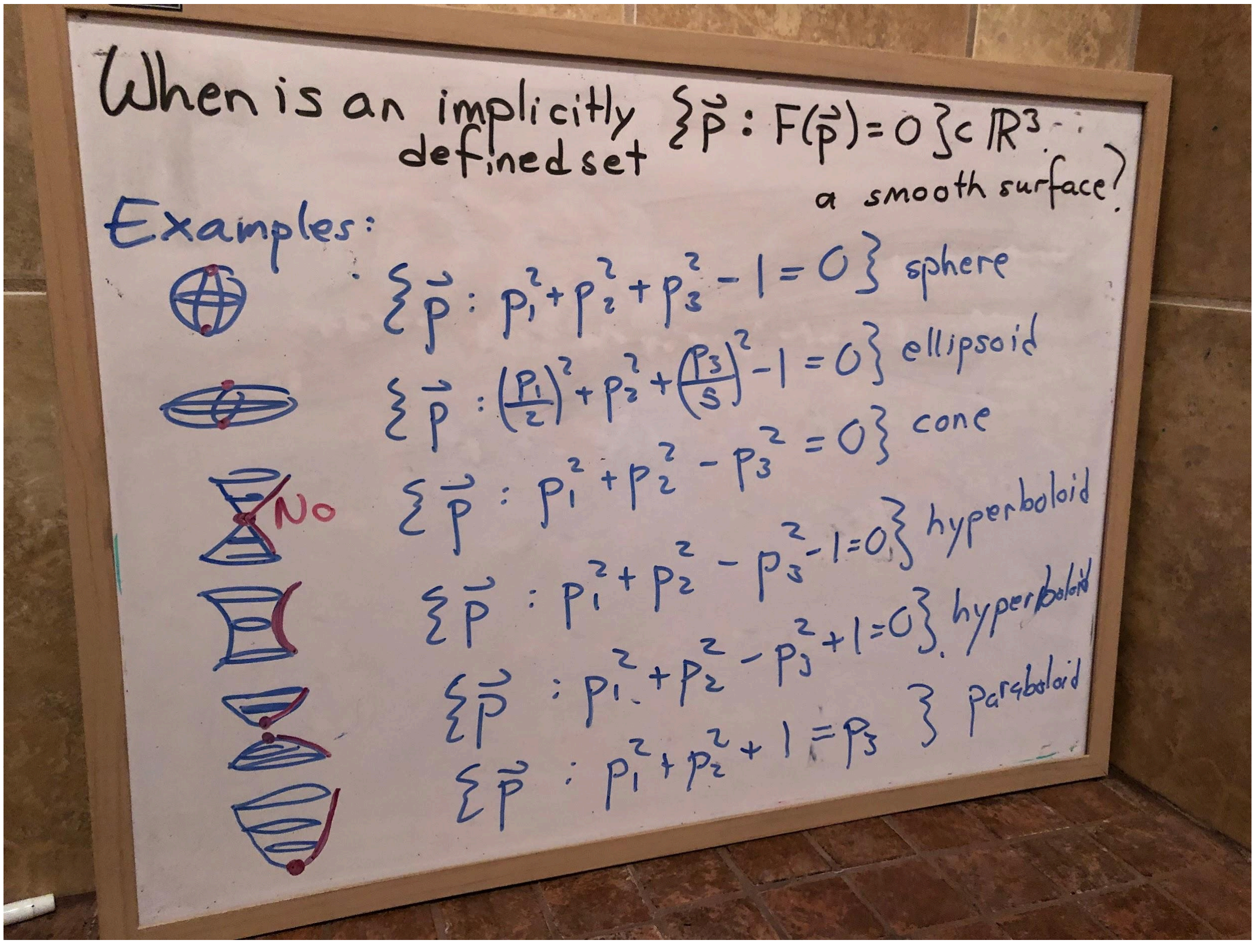
Classwork:

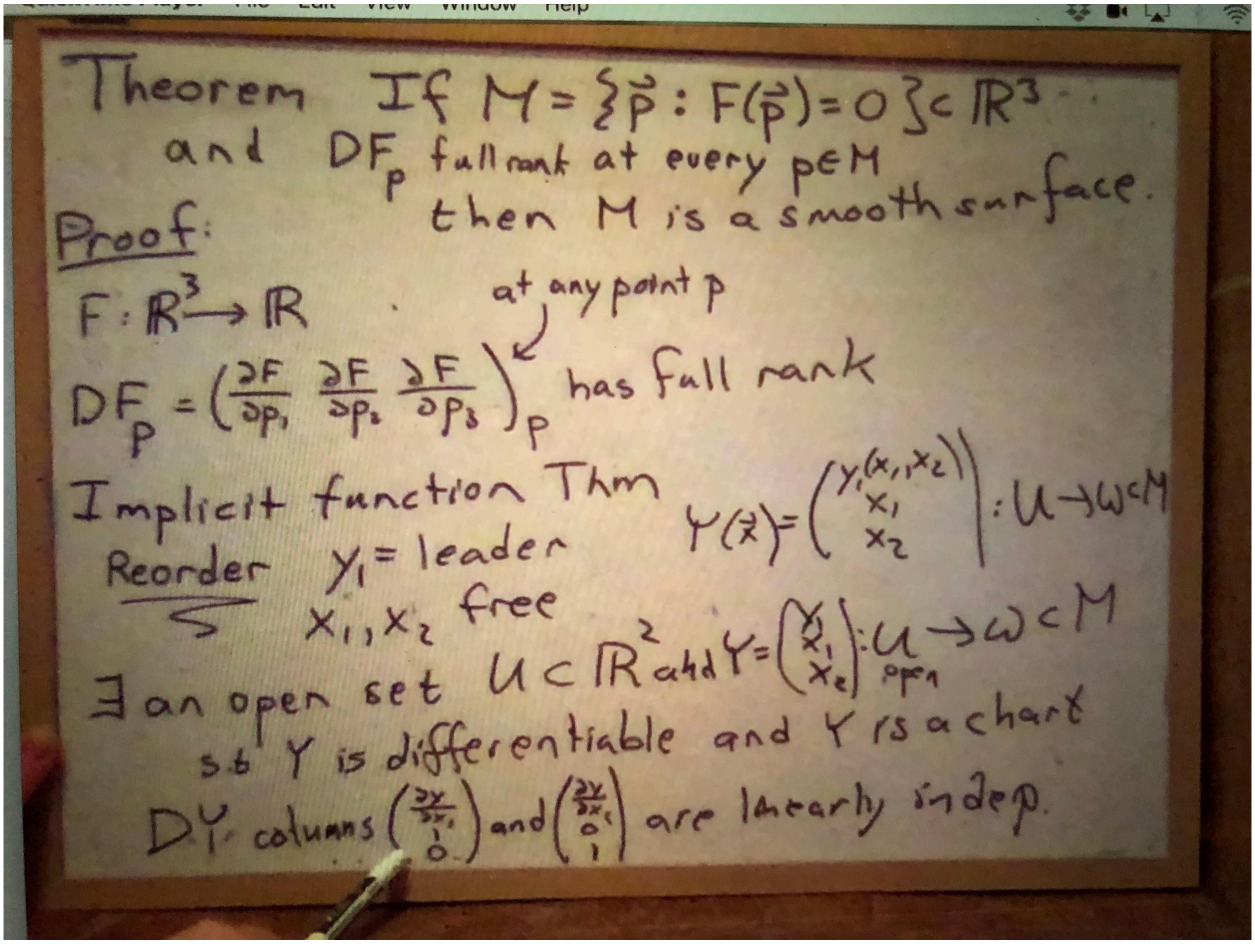
(1) Check the columns of dY in the rotationally symmetric surface are linearly independent by showing their cross product is not 0
(2) Show the columns of dY in the rotationally symmetric surface are perpendicular by showing the dot product is 0.

* Do the following functions define rotationally symmetric surfaces? Plot with MATLAB (skip these if you are behind schedule)

(3) Study r(s) =sqrt(1-s^2) on s=(-1,1)
(4) Study r(s)=s on (-5,5)
(5) Study r(s)=s^2+1 on (-2,2)
(6) Study r(s)=1/s on (0,1)

In Video **SurfacePart4** we consider ***implicitly*** defined sets as in the next two photos:





(7) Try to verify the sphere is a smooth surface using the implicit function theorem above. Answer is below.

(8) Try to verify the cone is a smooth surface or not using the implicit function theorem. Answer is below.

In **Video** **[SurfacePart5]** we explain the answers to (7) and (8)

Theorem If $M = \{\vec{p} : F(\vec{p}) = 0\} \subset \mathbb{R}^3$ and DF_p full rank at every $p \in M$ then M is a smooth surface.

Application:

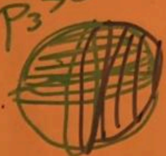
$$M = \{\vec{p} : p_1^2 + p_2^2 + p_3^2 - 1 = 0\} \subset \mathbb{R}^3$$

Find F: $F(p_1, p_2, p_3) = p_1^2 + p_2^2 + p_3^2 - 1$

Find DF: $DF = \left(\frac{\partial F}{\partial p_1}\ \frac{\partial F}{\partial p_2}\ \frac{\partial F}{\partial p_3}\right) = (2p_1\ \ 2p_2\ \ 2p_3)$

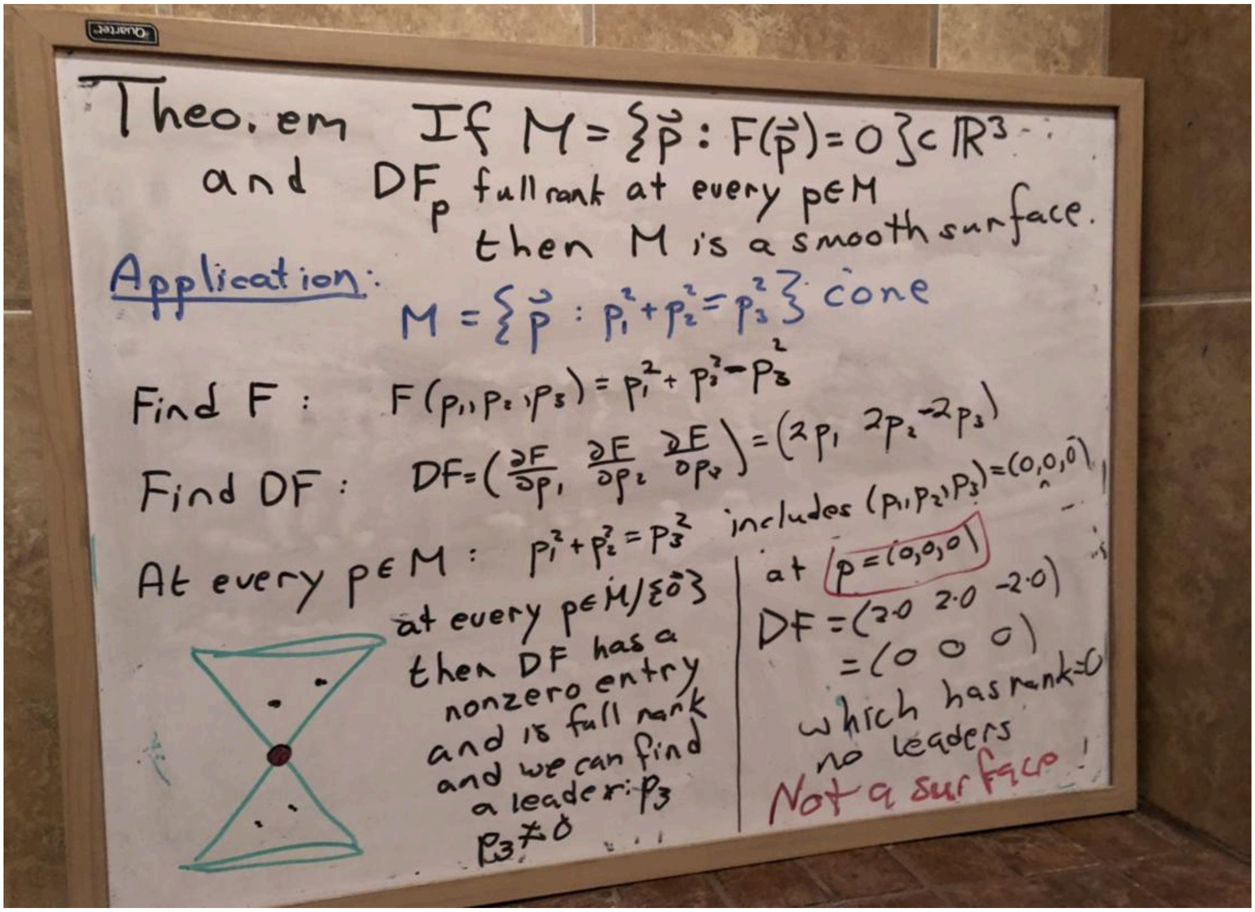
At every $p \in M$: $p_1^2 + p_2^2 + p_3^2 = 1$ so not all three are 0
so at least one $p_i \neq 0$ so one entry of DF is not zero
so DF has full rank

So the sphere has a chart at each p



When $p_3 > 0$ chart has p_3 = leader p_1 and p_2 free

When $p_2 > 0$ chart has p_2 = leader p_1 and p_3 free

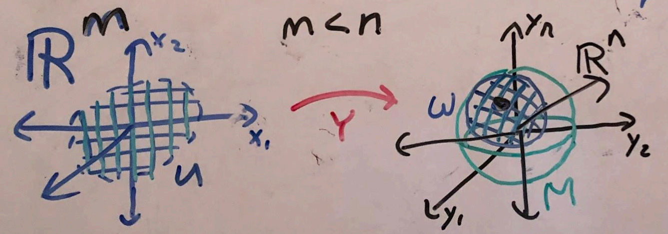


(9-10)* verify the hyperboloids are surfaces using the implicit function theorem and find the chart at point of your choice. You can plot the implicit surface and the chart together using MATLAB to check. (if you are late do your personal surface rather than the hyperboloid)
(11)* verify the paraboloid is a surface using the implicit function theorem and find the chart at a point of your choice. You can plot the implicit surface and the chart together using MATLAB to check (if you are late do the ellipsoid given in the photo above rather than the paraboloid)

Once I have checked your work in (9) and (11) you are ready for Part III of the final. You may send these to me immediately while continuing below to earn more stars for HW.

We continue now in **Video** **SubmanifoldPart1** with the definition of a ***submanifold*** as in the photos below:

Defn: A submanifold $M^m \subset \mathbb{R}^n$ is a set such that $\forall p \in M$ $\exists$ open set $U \subset \mathbb{R}^m$ and a smooth function $Y : U \subset \mathbb{R}^m \to W \subset M$ with $p \in W$ such that the columns of DY_q are linearly independant at every $q \in U$.

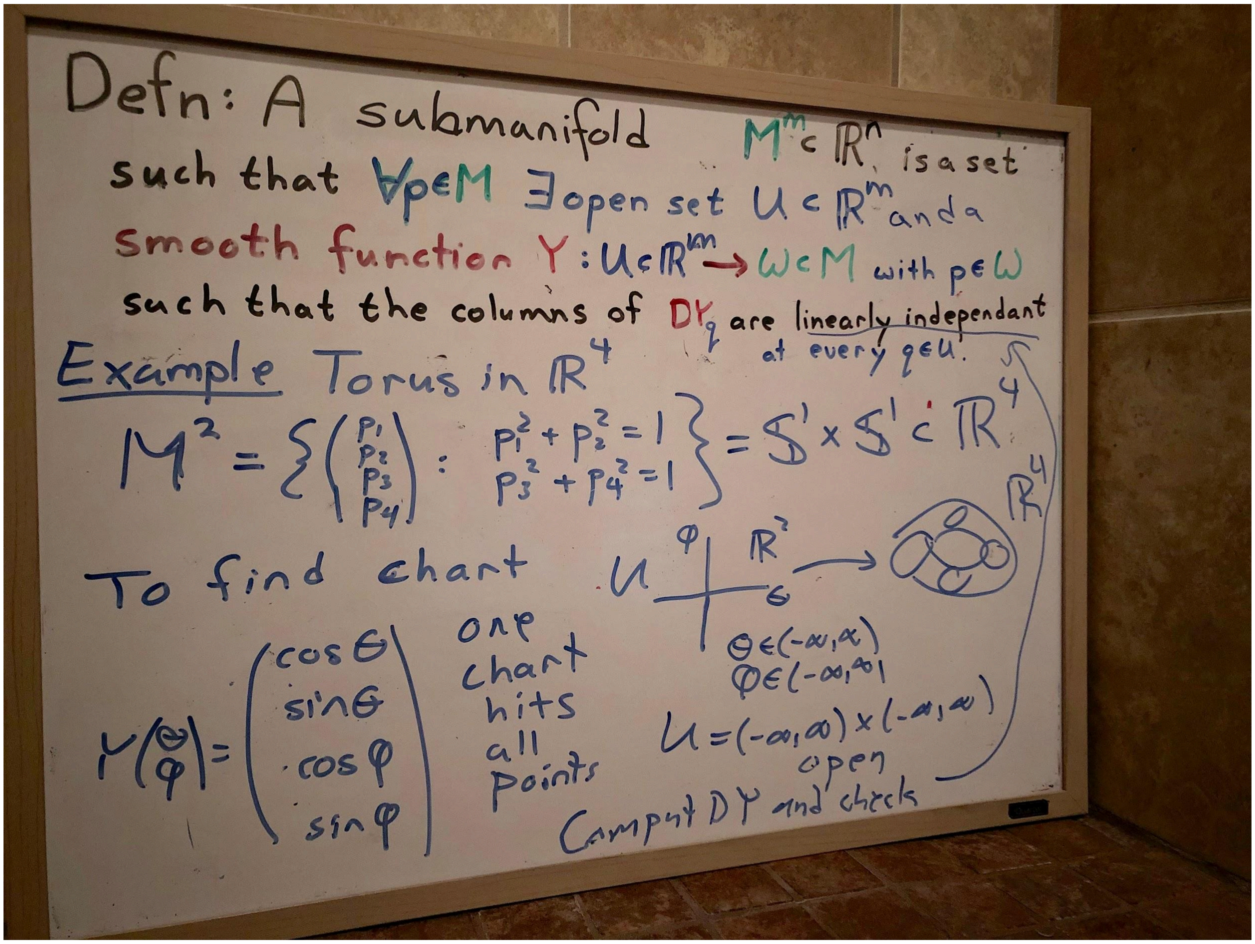


$\vec{Y}$ is called a coordinate chart or patch

$$DY = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_m} \\ \frac{\partial y_2}{\partial x_1} & & & \\ \vdots & & & \\ \frac{\partial y_n}{\partial x_1} & & & \frac{\partial y_n}{\partial x_m} \end{pmatrix}$$

has Full rank

the columns are called tangent vectors they must be linearly independant



(12)* Finish verifying that the chart of the torus in 4D space is a chart by finding DY and verifying the columns are linearly independent..

In **Video** **[SubmanifoldPart2]** we apply the implicit function theorem in higher dimensions to prove a set is a submanifold as in the photos:

Defn: A submanifold $M^m \subset \mathbb{R}^n$ is a set such that $\forall p \in M$ $\exists$ open set $U \subset \mathbb{R}^m$ and a smooth function $Y : U \subset \mathbb{R}^m \to W \subset M$ with $p \in W$ such that the columns of DY_q are linearly independant at every $q \in U$.

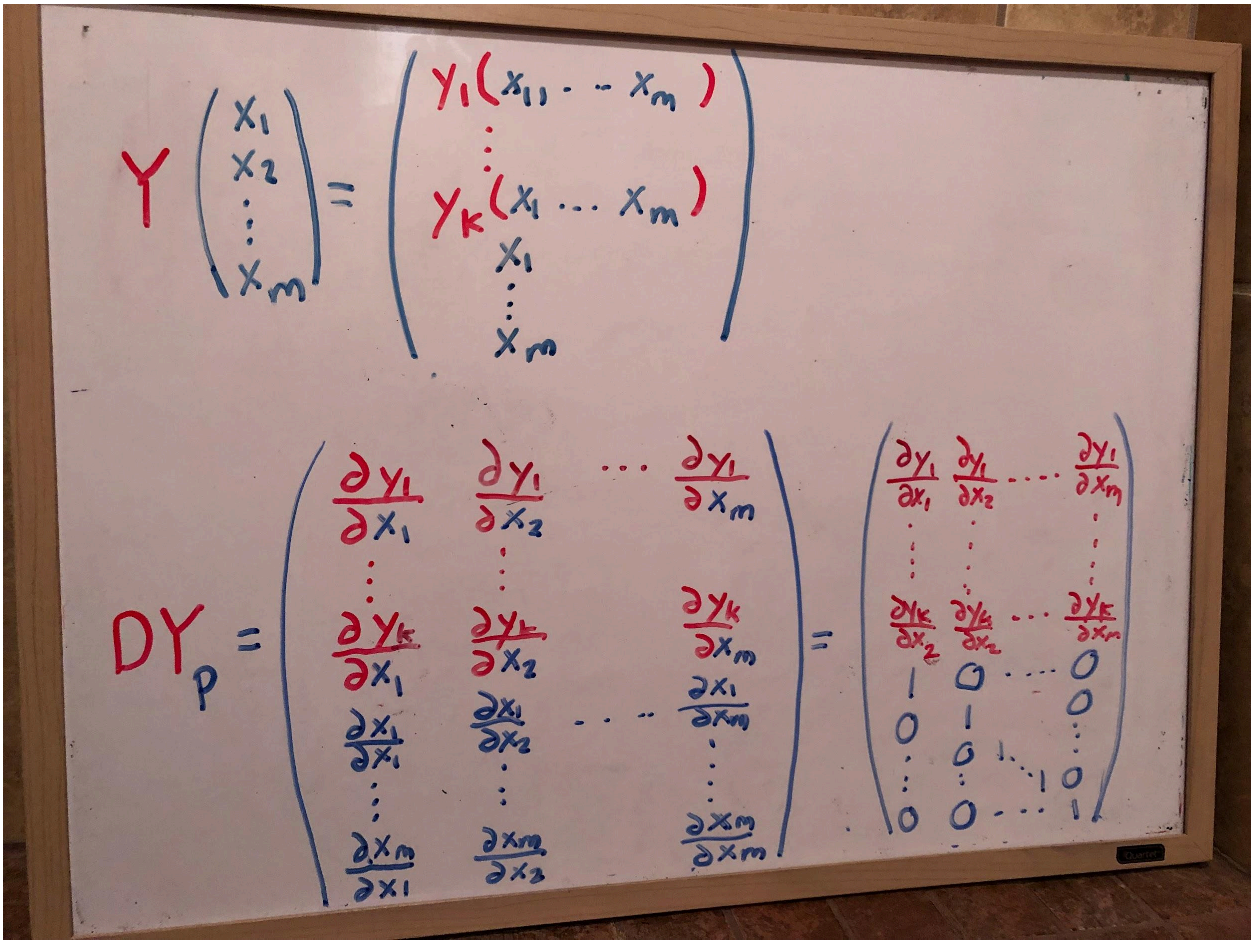
Implicit Function Thm:

If $M = \{\vec{p} : \vec{F}(\vec{p}) = \vec{0}\} \subset \mathbb{R}^n$ where $F : \mathbb{R}^n \to \mathbb{R}^k$ has DF_p full rank $= k$ at every $p \in M$ then M^m is a submanifold $m = n - k$

our chart:

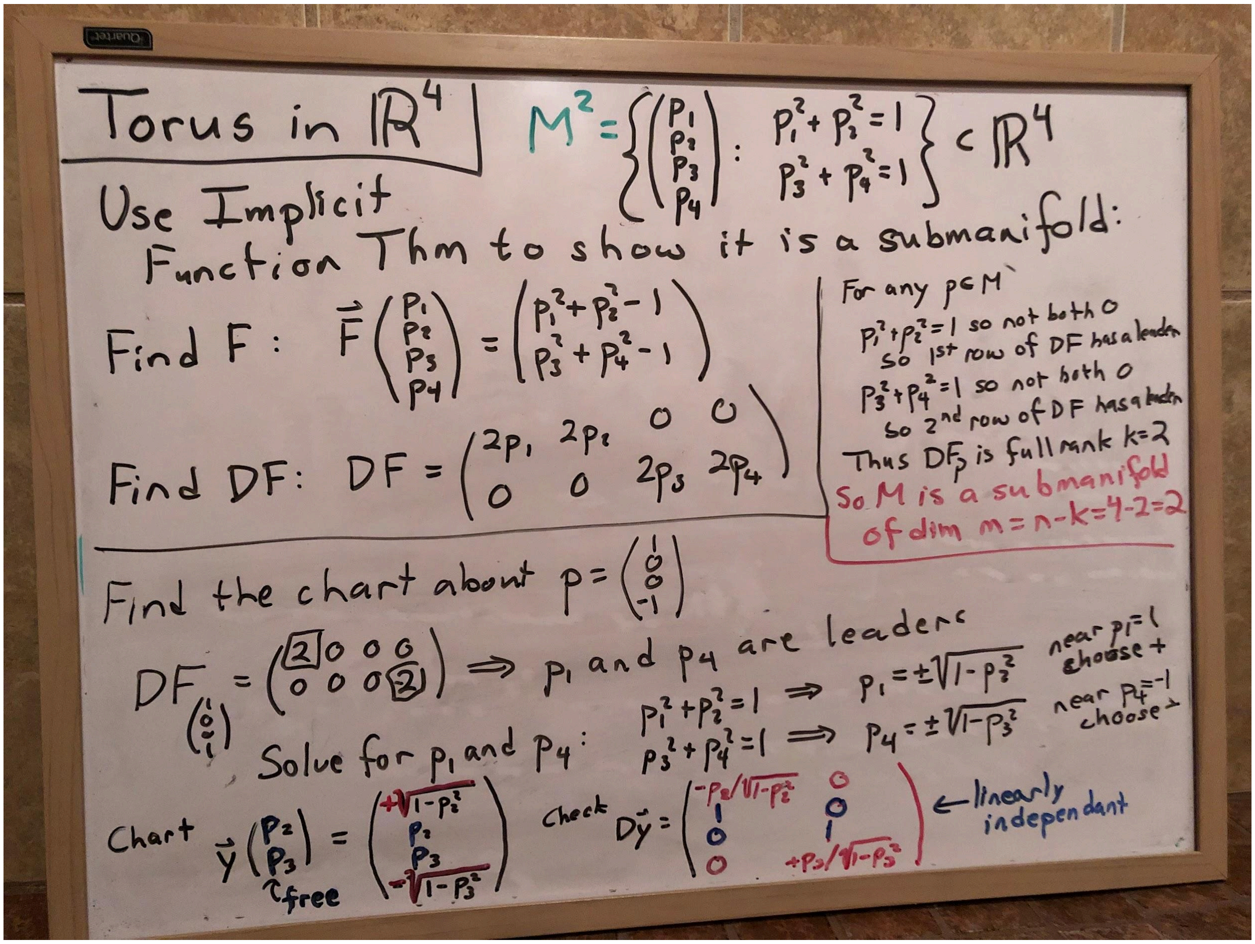
$$Y\begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} Y_1(x_1, \ldots, x_m) \\ \vdots \\ Y_k(x_1, \ldots, x_m) \\ x_1 \\ \vdots \\ x_m \end{pmatrix}$$

$$DY_q = \begin{pmatrix} \frac{\partial Y_1}{\partial x_1} & \frac{\partial Y_1}{\partial x_2} & \cdots & \frac{\partial Y_1}{\partial x_m} \\ \vdots & \vdots & & \vdots \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$ has rank m

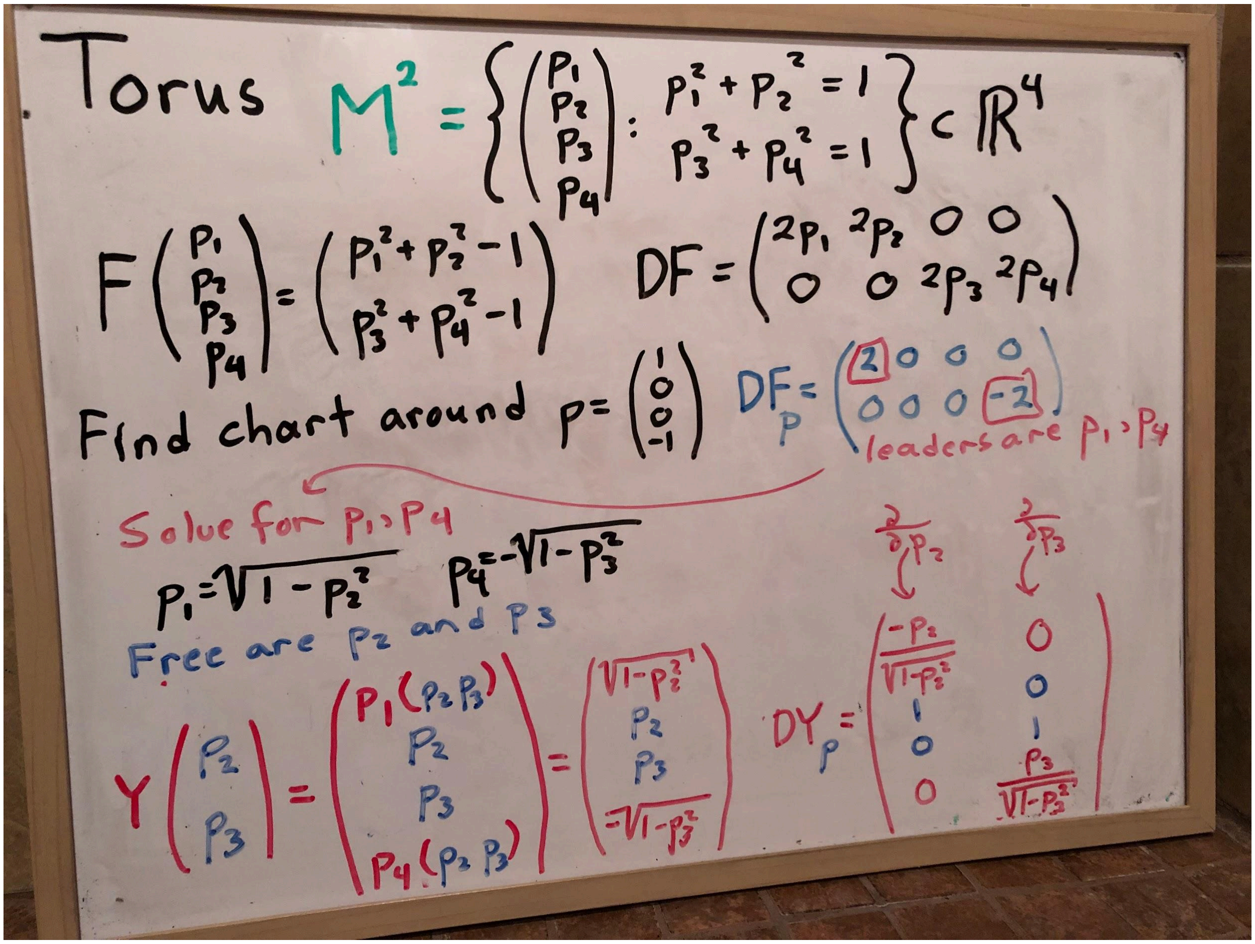


$$Y\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} y_1(x_1, \dots x_m) \\ \vdots \\ y_k(x_1 \dots x_m) \\ x_1 \\ \vdots \\ x_m \end{pmatrix}$$

$$DY_p = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_m} \\ \vdots & \vdots & & \\ \frac{\partial y_k}{\partial x_1} & \frac{\partial y_k}{\partial x_2} & & \frac{\partial y_k}{\partial x_m} \\ \frac{\partial x_1}{\partial x_1} & \frac{\partial x_1}{\partial x_2} & \cdots & \frac{\partial x_1}{\partial x_m} \\ \vdots & \vdots & & \vdots \\ \frac{\partial x_m}{\partial x_1} & \frac{\partial x_m}{\partial x_2} & & \frac{\partial x_m}{\partial x_m} \end{pmatrix} = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_m} \\ \vdots & \vdots & & \vdots \\ \frac{\partial y_k}{\partial x_2} & \frac{\partial y_k}{\partial x_2} & \cdots & \frac{\partial y_k}{\partial x_m} \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$



There is an error in Dy at the end of the video so here is a correction as a photo:



(13)* Use the implicit function theorem to show the torus in R^4 is a submanifold as in the final photo but this time find the chart Y about another point p=(0,-1,1,0) and check the columns of DY are linearly independent. (if you are late use p=(-1,0,1,0) in the same torus)

The homework is the starred classwork and is due on April 27.

Solutions to the original HW are [here](#).