1.

$$f(x) \equiv x^3 - (a+b+1)x^2 + (a+b+ab)x - ab$$

a. Use the remainder theorem to show that $(x-1)$ is a factor of $f(x)$

$f(1) = 1 - (a+b+1) + (a+b+ab) - ab = 0$

$\therefore (x-1)$ is a factor of $f(x)$

(2)

b. Find $f(a)$

$$\begin{aligned} f(a) &= a^3 - (a+b+1)a^2 + (a+b+ab)a - ab \\ &= a^3 - a^3 - a^2b - a^2 + a^2 + ab + a^2b - ab \\ &= 0 \end{aligned}$$

(2)

c. Write down the factors of $f(x)$.

$f(x) = (x-1)(x-a)(x-b)$

(2)

2.

$$4p^2 - q^2 = 25$$

Prove by contradiction that there are no pairs of positive integers *p* and *q* for which this is true.

Assume there are at least one pair of positive integers *p* and *q* such that $4p^2 - q^2 = 25$ is true.

$$\begin{aligned} 4p^2 - q^2 &= 25 \\ (2p+q)(2p-q) &= 25 \end{aligned}$$

case 1

$$\begin{aligned} (2p+q)(2p-q) &= 25 \times 1 \\ 2p - q &= 1 \\ q &= 1 - 2p \\ 2p + q &= 25 \\ 2p + 1 - 2p &\neq 25 \\ \\ 2p - q &= 25 \\ q &= 2p - 25 \\ 2p + 2p - 25 &= 25 \\ p &= \frac{25}{2} \text{ (not integers)} \end{aligned}$$

case 2

$$\begin{aligned} (2p+q)(2p-q) &= 5 \times 5 \\ 2p - q &= 5 \\ q &= 2p - 5 \\ \\ 2p + q &= 5 \\ 2p + 2p - 5 &= 5 \\ p &= \frac{5}{2} \text{ (not integers)} \end{aligned}$$

This contradicts the assumption, therefore there are no pairs of positive integers *p* and *q* for which $4p^2 - q^2 = 25$ is true.

(5)

3.

Solve in the range $0 \leqslant \theta \leqslant 360^\circ$

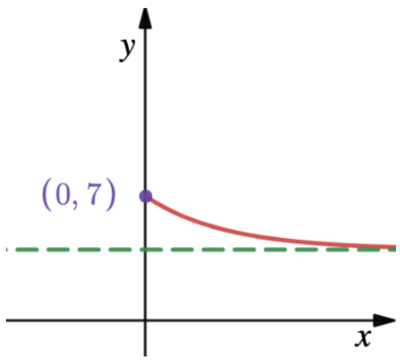
$$2\sec\theta = \tan\theta - \cot\theta$$
$$\frac{2}{\cos\theta} = \frac{\sin\theta}{\cos\theta} - \frac{\cos\theta}{\sin\theta}$$
$$2\sin\theta = \sin^2\theta - \cos^2\theta$$
$$2\sin\theta = \sin^2\theta - \left(1 - \sin^2\theta\right)$$
$$2\sin^2\theta - 2\sin\theta - 1 = 0$$
$$\sin\theta = \frac{2 \pm \sqrt{2^2 - 4(2)(-1)}}{2(2)}$$
$$\sin\theta \neq \frac{1+\sqrt{3}}{2}$$
$$\sin\theta = \frac{1-\sqrt{3}}{2}$$
$$\theta = -21.5,\ 201.5,\ +360n$$
$$\theta = 201.5,\ 338.5^\circ$$

(4)

4.

$$\mathrm{f}(t) = 3(0.8)^t + 4,\ t \geqslant 0$$

a. Sketch $y = \mathrm{f}(t)$ showing the equation of the asymptote and the value of the *y* intercept.



(2)

b. State the range of $\mathrm{f}(t)$

$$4 < \mathrm{f}(t) \leqslant 7$$

(2)

c. Solve $\mathrm{f}(t) = 5$ giving your answer to 2 decimal places.

$$3(0.8)^t + 4 = 5$$
$$(0.8)^t = \frac{1}{3}$$
$$t\ln 0.8 = \ln\frac{1}{3}$$
$$t = \frac{-\ln 3}{\ln 0.8}$$
$$t = 4.92\ (2\text{dp})$$

(3)

d. Find the value of t when $f'(t) = -\frac{1}{10}$ giving your answer correct to 2 decimal places.

$$\begin{aligned}3\ln(0.8)(0.8)^t &= -\frac{1}{10}\\ (0.8)^t &= -\frac{1}{30}\ln(0.8)\\ t\ln 0.8 &= \ln\left[-\frac{1}{30}\ln(0.8)\right]\\ t &= 21.96\ (2\text{dp})\end{aligned}$$

(3)

5.

In an industrial process the amount of pollution P caused by burning c tonnes of coal per hour is modelled by the equation

$$P = \frac{c^2+12}{3\sqrt{c}-9},\ c > 9$$

a. Show that the minimum value of P occurs when c satisfies the equation

$$c - 4\sqrt{c} + \frac{4}{c} = 0$$

$$\begin{aligned}\frac{\mathrm{d}P}{\mathrm{d}c} &= \frac{(2c)\left(3\sqrt{c}-9\right) - \left(c^2+12\right)\left(\frac{3}{2\sqrt{c}}\right)}{\left(3\sqrt{c}-9\right)^2} = 0\\ 0 &= (2c)\left(3\sqrt{c}-9\right) - \left(c^2+12\right)\left(\frac{3}{2\sqrt{c}}\right)\\ (2c)\left(3\sqrt{c}-9\right) &= \left(c^2+12\right)\left(\frac{3}{2\sqrt{c}}\right)\\ 2c\sqrt{c} - 6c &= \frac{c^2}{2\sqrt{c}} + \frac{6}{\sqrt{c}}\\ 2c - 6\sqrt{c} &= \frac{c}{2} + \frac{6}{c}\\ 4c - 12\sqrt{c} &= c + \frac{12}{c}\\ 3c - 12\sqrt{c} - \frac{12}{c} &= 0\\ c - 4\sqrt{c} - \frac{4}{c} &= 0\end{aligned}$$

(3)

b. Show that a root of this equation lies in the interval

$$10 < c < 25$$

$$\begin{aligned}&\mathrm{f}(c) = c - 4\sqrt{c} - \frac{4}{c}\\ &\mathrm{f}(10) = -3.049\\ &\mathrm{f}(25) = 4.84\end{aligned}$$

Change signs and $\mathrm{f}(c)$ is continuous in the interval, therefore a turning point lies in the interval.

(3)

c. By taking an initial estimate of the root of this equation $c_0 = 25$

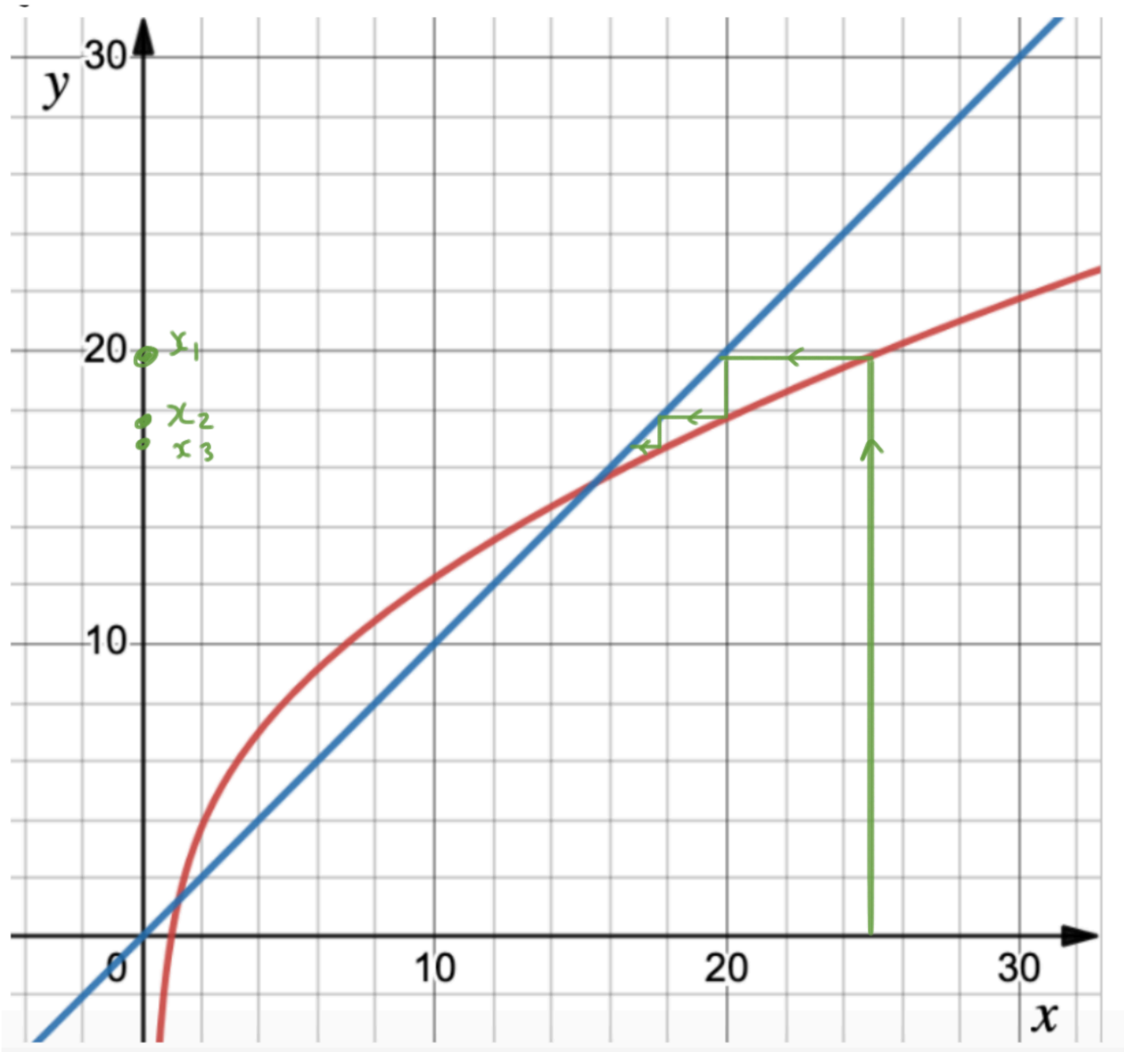
Use the iterative procedure $c_{n+1} = 4\sqrt{c_n} - \dfrac{4}{c_n}$

to find further estimates c_1, c_2 and c_3 giving your answers correct to 3 decimal places.

$c_1 = 19.84$
$c_2 = 17.615$
$c_3 = 16.561$

(2)

d. Illustrate the convergence of their estimates in the graph on the insert sheet.



(2)

6.

a. Calculate the sum of all the even numbers from 2 to 100 inclusive,

$$2+4+6+...+100$$

$$\text{sum} = \frac{50}{2}(2+100) = 2550$$

(3)

b. In the arithmetic series

$$k+2k+3k+...+100$$

k is a positive integer and *k* is a factor of 100.

i. Find, in terms of *k*, an expression for the number of terms in this series.

$$\begin{aligned} u_n &= a+(n-1)d \\ 100 &= k+(n-1)k \\ \frac{100-k}{k} &= n-1 \\ n &= \frac{100-k}{k}+1 \\ n &= \frac{100}{k} \end{aligned}$$

(2)

ii. Show that the sum of this series is

$$50+\frac{5000}{k}$$

$$\begin{aligned} \text{sum} &= \frac{n}{2}[2a+(n-1)d] \\ &= \frac{100}{2k}\left[2k+\left(\frac{100}{k}-1\right)k\right] \\ &= \frac{50}{k}\left(2k+\frac{100-k}{k}(k)\right) \\ &= \frac{50}{k}(k+100) \\ &= 50+\frac{5000}{k} \end{aligned}$$

(4)

c. Find, in terms of *k*, the 50th term of the arithmetic sequence

$$(2k+1),\ (4k+4),\ (6k+7),\ \ldots,$$

giving your answer in its simplest form.

$$\begin{aligned} d &= 4k+4-(2k+1) = 2k+3 \\ a &= 2k+1 \\ U_n &= a+(n-1)d \\ U_{50} &= 2k+1+49(2k+3) \\ &= 100k+148 \end{aligned}$$

(2)

7.

Evaluate

a.

$$\int_0^{\frac{\pi}{6}} 2\sin^2 3x \,\mathrm{d}x$$

$$\begin{aligned}\cos 2A &= 1 - 2\sin^2 A \\ 2\sin^2 A &= 1 - \cos 2A\end{aligned}$$

$$\begin{aligned}I &= \int_0^{\frac{\pi}{6}} 2\sin^2 3x \,\mathrm{d}x \\ &= \int_0^{\frac{\pi}{6}} (1 - \cos 6x)\mathrm{d}x \\ &= \left[x - \frac{1}{6}\sin 6x\right]_0^{\frac{\pi}{6}} \\ &= \frac{\pi}{6} - 0 - (0) \\ &= \frac{\pi}{6}\end{aligned}$$

(3)

b.

$$\int_0^1 x\mathrm{e}^{x^2+1}\mathrm{d}x$$

$$\begin{aligned}\int_0^1 x\mathrm{e}^{x^2+1}\mathrm{d}x &= \frac{1}{2}\int_0^1 (2x)\left(\mathrm{e}^{x^2+1}\right)\mathrm{d}x \\ &= \frac{1}{2}\left[\mathrm{e}^{x^2+1}\right]_0^1 \\ &= \frac{1}{2}\left(\mathrm{e}^2 - \mathrm{e}^1\right)\end{aligned}$$

(3)

8.

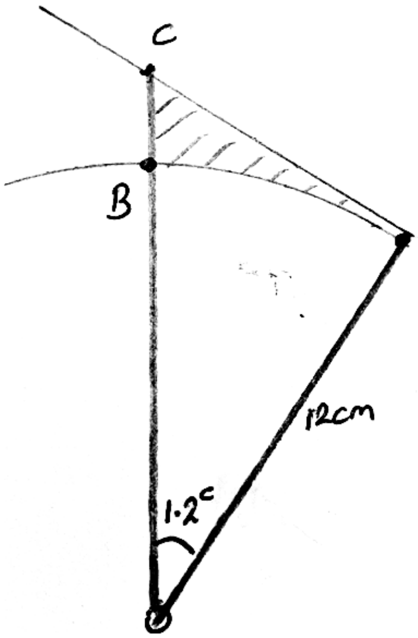


fig 1

OAB is a sector of a circle centre O, radius 12 cm and angle 1.2^c.
AC is a tangent to the circle.
Find the perimeter of ABC.

$$\begin{aligned}\tan 1.2 &= \frac{AC}{12} \\ AC &= 30.866 \\ \\ \cos 1.2 &= \frac{12}{OC} \\ OC &= 33.116 \\ &= \\ \text{perimeter} &= AC + \text{arc } AB + (OC - OB) \\ &= 30.866 + 12(1.2) + 33.116 - 12 \\ &= 66.4\,\text{cm (3sf)}\end{aligned}$$

(4)

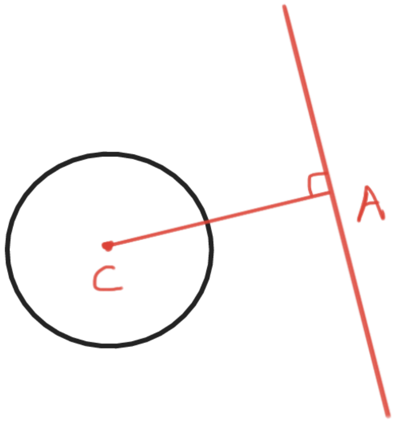
9.

Find the shortest distance between the circle

$$(x-3)^2 + (y-2)^2 = 9$$

and the line

$$3x + 2y + 13 = 0$$



$$3x + 2y + 13 = 0$$

$$y = -\frac{3}{2}x - \frac{13}{2}$$

$$\text{gradient of AC} = \frac{2}{3}$$

$$\text{equation of line AC}$$

$$y - 2 = \frac{2}{3}(x-3)$$

$$\text{solving for A}$$

$$-\frac{3}{2}x - \frac{13}{2} - 2 = \frac{2}{3}x - 2$$

$$-\frac{13}{2} = \frac{13}{6}x$$

$$x = -3$$

$$y = -2$$

$$A = (-3, -2)$$

$$AC = \sqrt{(3+3)^2 + (2+2)^2} = \sqrt{52}$$

$$\text{shortest distance} = \sqrt{52} - 3$$

$$= 2\sqrt{13} - 3$$

(9)

10.

a. Use the binomial expansion to find constants *A* and *B* such that

$(x+h)^n \equiv x^n + Ax^{n-1} + Bx^{n-2} + \ldots$

giving your answers in terms of *n* and *h*.

$$(x+h)^n = x^n + \binom{n}{1}x^{n-1}h + \binom{n}{2}x^{n-2}h^2$$

$$\begin{aligned} A &= \binom{n}{1}h \\ &= \frac{n!}{1!(n-1)!}h \\ &= nh \\ B &= \binom{n}{2}h^2 \\ &= \frac{n!}{1!(n-2)!}h^2 \\ &= n(n-1)h^2 \end{aligned}$$

(3)

b. Use differentiation from first principles to show that

$\frac{\mathrm{d}}{\mathrm{d}x}(x^n) = nx^{n-1}$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}x^n &= \lim_{h\to 0}\frac{(x+h)^n - x^n}{h} \\ &= \lim_{h\to 0}\frac{x^n + nhx^{n-1} + n(n-1)h^2x^{n-1} + \ldots - x^n}{h} \\ &= \lim_{h\to 0}\left(nx^{n-1} + n(n-1)hx^{n-1} + \ldots\right) \\ &= nx^{n-1} \end{aligned}$$

(3)

11.

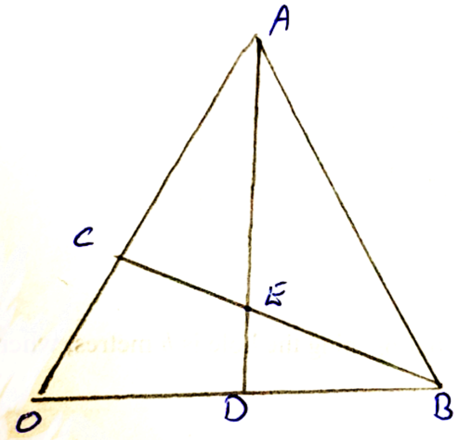


fig 2

In fig 2 $\overrightarrow{OA} = 3\mathbf{a}$, $\overrightarrow{OB} = 2\mathbf{b}$, $\overrightarrow{OB} = 2\overrightarrow{OD}$, $\overrightarrow{OA} = 3\overrightarrow{OC}$, $\overrightarrow{DE} = p\overrightarrow{DA}$ and $\overrightarrow{CE} = q\overrightarrow{CB}$

a. Find in terms of $\mathbf{a}$ and $\mathbf{b}$ expressions for

i. $\overrightarrow{DA}$

$$\begin{aligned}\overrightarrow{DA} &= \overrightarrow{DO} + \overrightarrow{OA} \\ &= -\mathbf{b} + 3\mathbf{a}\end{aligned}$$

ii. $\overrightarrow{CB}$

$$\begin{aligned}\overrightarrow{CB} &= \overrightarrow{CO} + \overrightarrow{OB} \\ &= -\mathbf{a} + 2\mathbf{b}\end{aligned}$$

(3)

b. Show that $\overrightarrow{OE} = 3p\mathbf{a} + (1-p)\mathbf{b}$

$$\begin{aligned}\overrightarrow{OE} &= \overrightarrow{OD} + \overrightarrow{DE} \\ &= \mathbf{b} + p(-\mathbf{b} + 3\mathbf{a}) \\ &= 3p\mathbf{a} + (1-p)\mathbf{b}\end{aligned}$$

(2)

c. Find another expression for $\overrightarrow{OE}$ in terms of $\mathbf{a}$, $\mathbf{b}$ and q

$$\begin{aligned}\overrightarrow{OE} &= \overrightarrow{OC} + \overrightarrow{CE} \\ &= \mathbf{a} + q(-\mathbf{a} + 2\mathbf{b}) \\ &= (1-q)\mathbf{a} + 2q\mathbf{b}\end{aligned}$$

(2)

d. Find the values of p and q.

$$3p = 1 - q \quad [1]$$
$$1 - p = 2q \quad [2]$$
sub [2] into [1]
$$3(1-2q) = 1 - q$$
$$3 - 6q = 1 - q$$
$$2 = 5q$$
$$q = \frac{2}{5}$$
$$p = \frac{1}{5}$$

(2)

12.

$$e^{2x}\frac{dy}{dx} = x + k \qquad \text{where } k \text{ is a constant}$$

a. Find the solution to this differential equation in the form $y = f(x)$ given that $y = 1$ when $x = 0$

$$e^{2x}\frac{dy}{dx} = x + k$$
$$\int dy = \int e^{2x}(x+k)dx$$
$$y = \frac{1}{2}e^{2x}(x+k) - \int\left(\frac{1}{2}e^{2x}\right)dx$$
$$y = \frac{1}{2}e^{2x}(x+k) - \frac{1}{4}e^{2x} + c$$
$$1 = \frac{1}{2}e^{0}(k) - \frac{1}{4}e^{0} + c$$
$$c = \frac{5}{4} - \frac{k}{2}$$
$$y = \frac{1}{2}e^{2x}(x+k) - \frac{1}{4}e^{2x} + \frac{5}{4} - \frac{k}{2}$$

(8)

b. Given that the curve $y = f(x)$ has a turning point when $x = 6$, find the value of k.

$$e^{2x}\frac{dy}{dx} = x + k$$
$$e^{2(6)}(0) = 6 + k$$
$$k = -6$$

(1)

13.

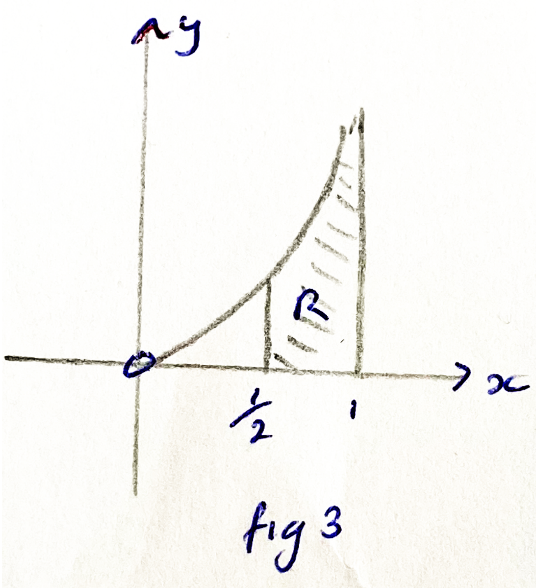


Fig 3 shows part of the curve $y = \sin^{-1} x$.

The region R is bounded by the curve, the $x-$ axis and the lines $x = \frac{1}{2}$ and $x = 1$

a. Complete the table below

x	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	1
$\sin^{-1} x$	0.5236	0.7297	0.9851	1.5708

(1)

b. Use the trapezium rule with 4 ordinates to find an estimate for the size of area R giving your answer correct to 3 decimal places.

$$R = \frac{1}{2}\left(\frac{1}{6}\right)[0.5236 + 1.5708 + 2(0.7297 + 0.9851)]$$
$$= 0.460 \text{ (3sf)}$$

(3)

c. Use the substitution $x = \sin u$ to evaluate exactly

$$\int_{\frac{1}{2}}^{1} \sin^{-1} x \, \mathrm{d}x$$

expression for $\mathrm{d}x$

$$x = \sin u$$
$$\frac{\mathrm{d}x}{\mathrm{d}u} = \cos u$$
$$\mathrm{d}x = \cos u \, \mathrm{d}u$$

$$\sin^{-1} x = u$$

changing limit

$$1 = \sin u$$
$$u = \frac{\pi}{2}$$
$$\frac{1}{2} = \sin u$$
$$u = \frac{\pi}{6}$$

$$I = \int_0^1 \sin^{-1} x \, \mathrm{d}x$$
$$I = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} u \cos u \, \mathrm{d}u$$
$$I = [(u)(\sin u)]_{\frac{\pi}{6}}^{\frac{\pi}{2}} - \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (1)\,(\sin u)\mathrm{d}u$$
$$= \frac{\pi}{2}\sin\frac{\pi}{2} - \frac{\pi}{6}\sin\frac{\pi}{6} + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin u \, \mathrm{d}u$$
$$= \frac{\pi}{2} - \frac{\pi}{12} - [\cos u]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$
$$= \frac{5\pi}{12} - \left(0 - \frac{\sqrt{3}}{2}\right)$$
$$= \frac{5\pi}{12} - \frac{\sqrt{3}}{2}$$

(6)

d. Explain why the trapezium rule gives an overestimate for the value of area *R*.

The curve is bending downward

(1)

Insert for Question 5