Analysis
Lesson 21
The Fundamental Theorem of Calculus

This Lesson has two Parts and each has three homework problems. Part I is very challenging and you may wish to review Riemann Integration before starting.

Part I: Proving the Fundamental Theorem of Calculus

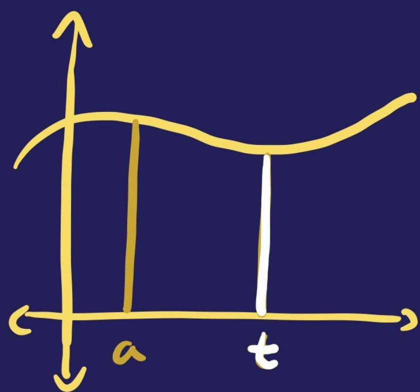
Watch Playlist FTC-1to9

Fundamental Theorem of Calculus:

If $F(t)=\int_a^t f(x)\,dx$ then $F'(t)=f(t)$.

Also needs more hypothesis on the function f

We will find them as we complete the proof.

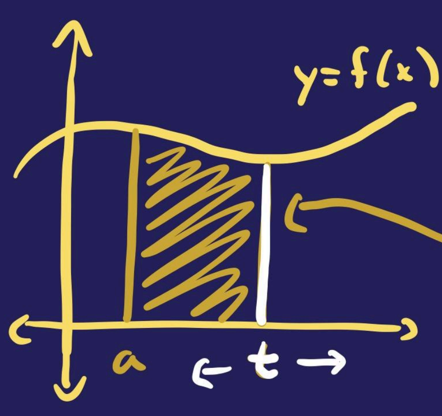


Fundamental Theorem of Calculus:

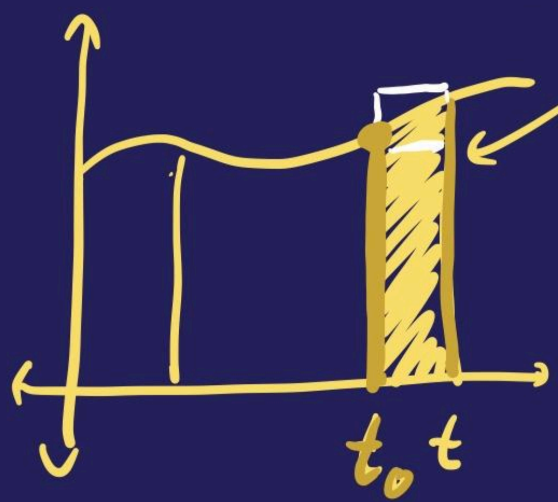
If $F(t)=\int_a^t f(x)\,dx$ then $F'(t)=f(t)$.

If we assume **f** is continuous and positive, then

$\mathcal{F}(t)$ is an Area



$$F'(t_0) = \lim_{t \to t_0} \frac{F(t) - F(t_0)}{t - t_0}$$



$$\text{area} = F(t) - F(t_0)$$

$$\leq \max_{[t_0, t]} f \cdot (t - t_0)$$

$$\leq (f(t_0) + \varepsilon)\,(t - t_0)$$

if $|t - t_0| < \delta_\varepsilon$

$$\text{area} \geq \min_{[t_0, t]} f \cdot (t - t_0)$$

$$\geq (f(t_0) - \varepsilon)\,(t - t_0)$$

$$(f(t_0) + \varepsilon) \geq \frac{F(t) - F(t_0)}{t - t_0} \geq (f(t_0) - \varepsilon)$$

Apply squeeze thm

$t \to t_0$ $\quad \delta_\varepsilon \to$ small as we need for any ε

$$f(t_0) \geq \lim_{t \to t_0^+} \frac{F(t)-F(t_0)}{t-t_0} \geq f(t_0)$$

So $F'(t_0) = f(t_0)$ ★

This proof used

$f > 0$, f continuous

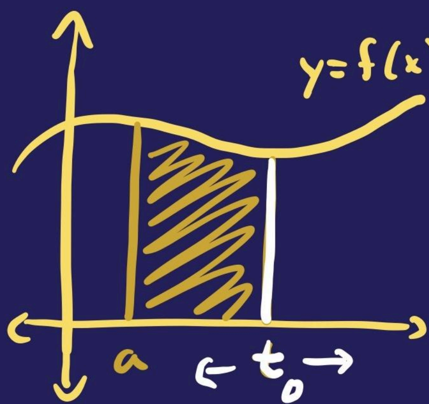
$t > t_0$ so must

repeat for $t < t_0$

to really get ★

Fundamental Theorem of Calculus:

If $F(t)=\int_a^t f(x)\,dx$ then $F'(t)=f(t)$.



What do we need to prove this?

$$F'(t_0)=\lim_{t\to t_0}\frac{F(t)-F(t_0)}{t-t_0}$$

$$F'(t_0) = \lim_{t \to t_0} \frac{F(t) - F(t_0)}{t - t_0}$$

Need

$$F(t) - F(t_0) = \underbrace{\int_a^t f(x)\,dx - \int_a^{t_0} f(x)\,dx = \int_{t_0}^{t} f(x)\,dx}_{\text{Need a theorem that says this}}$$

Thm I

Next

$$F'(t_0) = \lim_{t \to t_0} \frac{1}{(t - t_0)} \underbrace{\int_{t_0}^{t} f(x)\,dx}$$

Need

Thm II

$$\int_{t_0}^{t} f(x)\,dx \leq \max_{[t_0,t]} f \cdot (t-t_0)$$

Also need

Thm III

$$\int_{t_0}^{t} f(x)\,dx \geq \min_{[t_0,t]} f \, (t-t_0)$$

Need: Need f contin at t_0

$$\lim_{t \to t_0} \max_{[t_0,t]} f = \lim_{t \to t_0} \min f$$

then we squeeze and get

$$\min_{[t_0,t]} f \leq \frac{F(t)-F(t_0)}{t-t_0} \leq \max_{[t_0,t]} f$$

$$\leq f(t_0) \leq$$

same value $\Leftarrow f(t_0)$

$F'(t_0)$ has this value

So now prove Thm I II + III hopefully only using f is Riemann Integrable.

$\boxed{HW1}$ Prove that if f is continuous at t_0

then $\lim\limits_{t \to t_0} \max\limits_{[t_0,t]} f(x) = f(t_0)$

and $\lim\limits_{t \to t_0} \min\limits_{[t_0,t]} f(x) = f(t_0)$

Lets do the max together:

pause + try writing $\forall \varepsilon \ldots$

Given:

$\forall \varepsilon > 0 \ \exists \delta'_\varepsilon > 0$ s.t $|x - t_0| < \delta_\varepsilon \Rightarrow |f(x) - f(t_0)| < \varepsilon$

Show:

$\forall \varepsilon > 0 \ \exists \delta_\varepsilon > 0$ s.t $0 < |t - t_0| < \delta_\varepsilon \Rightarrow \left| \max\limits_{[t_0,t]} f(x) - f(t_0) \right| < \varepsilon$

Proof :

① Given any $\varepsilon > 0$ Choose $\delta_\varepsilon = \boxed{\delta_\varepsilon'} > 0$ ① $\delta_\varepsilon' > 0$ given

② Whenever $|t - t_0| < \delta_\varepsilon$

we have $t_0 - \delta_\varepsilon < t < t_0 + \delta_\varepsilon$ ② ——

③ So if $x \in [t_0, t]$ then

$t_0 - \delta_\varepsilon < t_0 \leq x \leq t < t_0 + \delta_\varepsilon$ ③ defn of interval

④ So $|x - t_0| < \delta_\varepsilon$ ④ ——

⑤ $|f(x) - f(t_0)| < \varepsilon$ ⑤ given

⑥ In particular $x = x_{max}$ ⑥ defn max + plug in to step 5

$f(x_{max}) = \max_{[t-t_0]} f(x)$

$\left|\max_{[t-t_0]} f(x) - f(t_0)\right| < \varepsilon$

final $\left|\max_{[t_0, t]} f(x) - f(t_0)\right| < \varepsilon$ final step 6

$\boxed{\text{HW2}}$ If f is Riemann Integrable and $t > t_0$

then $$\int_{t_0}^{t} f(x)\,dx \leq (t - t_0) \max_{[t_0, t]} f(x)$$

and

$$\int_{t_0}^{t} f(x)\,dx \geq (t - t_0) \min_{[t_0, t]} f(x)$$

Lets do the max together:

Proof:

① $$\int_{t_0}^{t} f(x)\,dx = \inf_{P} \text{UpperSum}(P)$$

① Defn of Riem Integral + Given f is Riemann Int.

② $$\int_{t_0}^{t} f(x)\,dx \leq \text{UpperSum}(P)$$ for any partition P

② by defn of inf.

③ $$\int_{t_0}^{t} f(x)\,dx \le \max_{[x_0, x_1]} f(x)\,(x_1 - x_0)$$

don't need a sum because only one rectangle

③ P $x_0 = t_0$

$x_1 = t$

$N = 1$

④ $$\int_{t_0}^{t} f(x)\,dx \le \max_{[t_0, t]} f(x) \cdot (t - t_0)$$

④ Sub in
$x_0 = t_0$
$x_1 = t$.

Do min
Use LowerSum

QED

$\boxed{HW3}$ If f is Riemann Integrable and $t<t_0$

then $\displaystyle\int_{t_0}^{t} f(x)\,dx \overset{?}{\leq} (t-t_0) \overset{\min}{\cancel{\max_{[t_0,t]}}} f(x)$

and

$$\int_{t_0}^{t} f(x)\,dx \overset{?}{\geq} (t-t_0) \overset{\max}{\cancel{\min_{[t_0,t]}}} f(x)$$

$$\int_{t_0}^{t} f(x)\,dx = -\int_{t}^{t_0} f(x)\,dx \quad \text{when } t<t_0$$

In HW 2 $\displaystyle\int_{t}^{t_0} f(x)\,dx \leq \max_{[t,t_0]} f\,(t_0-t)$

$$-\int_{t}^{t_0} f(x)\,dx \geq -\max_{[t,t_0]} f\,(t_0-t)$$

$$\geq -\max_{[t,t_0]} f\,(t_0-t) = \max_{[t,t_0]} f\,(t-t_0)$$

Need

$$F(t)-F(t_0)=\underbrace{\int_a^t f(x)\,dx-\int_a^{t_0} f(x)\,dx=\int_{t_0}^t f(x)\,dx}_{\text{Need a theorem that says this}}$$

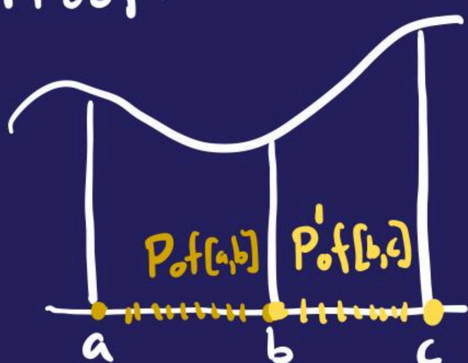
Thm I

$a=a$
$b=t_0$
$c=t$

Thm If $a<b<c$ and f is Riemann Integrable

then $$\int_a^b f(x)\,dx+\int_b^c f(x)\,dx=\int_a^c f(x)\,dx$$

Proof:



$$\int_a^b f(x)\,dx=\inf_{P \text{ of } [a,b]} \text{UpperSum}(P)$$

$$a=x_0<x_1<x_2<\ldots<x_N=b$$

$$\int_b^c f(x)\,dx=\inf_{P' \text{ of } [b,c]} \text{Upper Sum}(P')$$

$$b=x_0'<x_1'<x_2'<\ldots<x_N'=c$$

Suppose we have P of $[a,b]$ and P' of $[b,c]$

we can define P'' of $[a,c]$

$$a=x_0<x_1<x_2<\ldots<x_N=b=x_0'<x_1'<x_2'<\ldots<x_N'=c$$

$$a=x_0<x_1<\ \ldots\ <x_N=x_0'<x_1'<\ \ldots\ x_N'=c$$

$$\int_a^c f(x)\,dx \le \text{Upper Sum}(P'') \quad \text{by defn of inf on } \int_a^c$$

$$= \text{UpperSum}(P) + \text{UpperSum}(P')$$

but we can choose P_ε depending on $\varepsilon > 0$

so that $\text{Upper Sum}(P_\varepsilon) \le \int_a^b f(x)\,dx + \varepsilon$

by defn of inf

also can choose P_ε' depending on $\varepsilon > 0$

so that $\text{Upper Sum}(P_\varepsilon') \le \int_b^c f(x)\,dx + \varepsilon$

We can build P_ε'' out of P_ε and P_ε'

$$\int_a^c f(x)\,dx \le \int_a^b f(x)\,dx + \varepsilon + \int_b^c f(x)\,dx + \varepsilon$$

But we can do this $\forall \varepsilon > 0$

$$\int_a^c f(x)\,dx \le \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$$

For equality use $\ge$ Lower Sums

and the same ideas. QED

So now we have the Fundamental Theorem of Calculus if f is Riemann Integrable and continuous at t_0

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Part II Consequences of the Fundamental Theorem of Calculus

Watch [Playlist FTC-10to12]

Consequences of the Fundamental Theorem of Calculus

If $F(t)=\int_a^t f(x)\,dx$ then $F'(t)=f(t)$.

Suppose you are given $f(x)$ and asked to find $\int_a^b f(x)\,dx$

Look for an antiderivative, F, $F'=f$

but there are many possible $F+C$

$\int f(x)\,dx = F(x)+C$ because $\frac{d}{dx}C=0$.

<u>Thm</u> $\int_a^b f(x)\,dx = F(b)-F(a)$ for any F s.t $F'(x)=f(x)$ $\forall x \in [a,b]$

<u>Proof</u>: ① Let $F(t)=\int_a^t f(x)\,dx$ then $F'(t)=f(t)$ — ① by Fund thm of Calc

② $F(a)=\int_a^a f(x)\,dx = 0$ — ② by $\int_a^b g(x)\,dx=0$ when $a=b$

③ $F(b)=\int_a^b f(x)\,dx$ — ③ by step ① choice of F taking $t=b$.

④ $F(b)-F(a)=\int_a^b f(x)\,dx - 0 = \int_a^b f(x)\,dx$ — ④ by Steps 2+3.

⑤ Suppose we have a second antiderivative H s.t $H'(x)=f(x)$ $\forall x\in[a,b]$

Then $\frac{d}{dx}\left(H(x)-F(x)\right)=H'(x)-F'(x)$ — ⑤ by sums of derivatives

(6) $\frac{d}{dx}\left(H(x)-F(x)\right) = f(x) - f(x) = 0$ $\forall x \in [a,b]$ (6) by steps 1 + 5

(7) $H(x)-F(x) = C$ ← constant (7) Thm: If a function has derivative $=0$ on $[a,b]$ then it is constant on $[a,b]$

(8) $H(x) = F(x) + C$

(9) $H(b)-H(a) = \left(F(b)+C\right) - \left(F(a)+C\right)$ (9) by step 8

(10) $= F(b)-F(a)$ (10) $C-C=0$

(11) $= \int_a^b f(x)\,dx$ (11) by step 4.

QED

$\boxed{\text{HW1}}$ Explain why $\int_a^b x^p\,dx = \frac{b^{p+1}}{p+1} - \frac{a^{p+1}}{p+1}$ for $p \in \mathbb{N}$.

Integration by Parts

comes from the Fundamental Thm of Calculus + the Product Rule

true $\forall x \in [a,b]$

Product Rule $\frac{d}{dx}\left(u(x)v(x)\right) = u'(x)v(x) + u(x)v'(x)$

$$\int_a^b \underbrace{\frac{d}{dx}\left(u(x)v(x)\right)}_{\substack{\text{antiderivative} \\ \text{is } u(x)v(x)}} dx = \int_a^b u'(x)v(x) + u(x)v'(x)\,dx$$

Fund Thm of Calc

Thm: $u(b)v(b) - u(a)v(a) = \int_a^b v(x)\underbrace{u'(x)\,dx}_{du} + \int_a^b u(x)\underbrace{v'(x)\,dx}_{dv}$

Another Notation

$$uv\Big|_a^b = \int_a^b v\,du + \int_a^b u\,dv$$

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

$\boxed{\text{HW 2}}$ Find $\int_0^\pi x\sin(x)\,dx$ using this rule.

U substitution & Chain Rule

Chain Rule:

$$\frac{d}{dx} f(u(x)) = \left.\frac{d}{du} f(u)\right|_{u(x)} \cdot \frac{du}{dx} = f'(u(x))\,u'(x)$$

$$\int_a^b \underbrace{\frac{d}{dx} f(u(x))}_{\text{antiderivative of this is } f(u(x))} dx = \int_a^b f'(u(x))\,u'(x)\,dx$$

$$\underbrace{f(u(b)) - f(u(a))}_{\text{by Fund Thm of Calc}} = \int_a^b f'(u(x))\,u'(x)\,dx$$

u substitution & Chain Rule

Chain Rule:

$$\frac{d}{dx}F(u(x)) = \frac{d}{du}F(u)\Big|_{u(x)} \cdot \frac{du}{dx} = F'(u(x))\,u'(x)$$

$$\int_a^b \underbrace{\frac{d}{dx}F(u(x))}_{\text{antiderivative of this is } f(u(x))}\,dx = \int_a^b F'(u(x))\,u'(x)\,dx$$

$$\underbrace{F(u(b)) - F(u(a))}_{\text{by Fund Thm of Calc}} = \int_a^b F'(u(x))\,u'(x)\,dx$$

$$\int_{u(a)}^{u(b)} F'(t)\,dt = \int_a^b F'(u(x))\,u'(x)\,dx$$

Now set $f(x) = F'(x)$

$t = u$

Thm: $$\int_{u(a)}^{u(b)} f(u)\,du = \int_a^b f(u(x))\,\underbrace{u'(x)\,dx}_{du}$$

u substitution rule

$$\int_{x=0}^{\pi} \cos(2x)\,dx = \int_{u=0}^{2\pi} \cos(u)\,\frac{du}{2} = \ldots$$

$u = 2x$

$du = u'(x)\,dx$

$du = 2\,dx$

Application of u sub.

$du = \frac{du}{dx}\,dx$

$\boxed{HW3}$ Find

$$\int_{0}^{\pi} x \sin(x^2)\,dx$$

If necessary review examples in your Calculus textbook.