# Fuzzy Cantor set

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The Cantor set is a good example of an elementary fractal. The object first used to demonstrate fractal dimensions is actually the Cantor set. The process of generating this fractal is very simple. The set is generated by the iteration of a single operation on a line of unit length. In each iteration, the middle third from each lines segment of the previous set is simply removed. As the number of iterations increases, the number of separate line segments tends to infinity while the length of each segment approaches zero. Based on the construction of the Cantor set, we generalized this one-dimensional idea to a length other than 1/3, excluding the degenerate cases of 0 and 1, in the case of Sugeno fuzzy set.

In set theory as Cantor defined and Zermelo and Fraenkel axiomatized, an object is either a member of a set or not. In *fuzzy set theory* this condition was relaxed by Lotfi A. Zadeh so an object has a *degree of membership* in a set, a number between 0 and 1. For example, the degree of membership of a person in the set of "tall people" is more flexible than a simple yes or no answer and can be a real number such as 0.70.

The fuzzy theory is an appropriate tool for describing those phenomena that occur in the interaction of particles and the forces of nature because this theory provides a strict mathematical framework in which vague conceptual phenomena can be precisely studied. Therefore, fuzzy theory can be merged with quantum theory for describing its concepts which are difficult to explain by conventional logic.
The goal of this paper is to help to close the gap between quantum theory concepts and the connections with string theory for nucleons, protons and neutrons having quarks as fundamental components and leptons (electrons muons , neutrinos etc…) .

## 1 . About Fuzzy Sets Theory

A fuzzy sunset A of a classical set U is characterized by its membership function

$\varphi_A : U \to [0,1]$

If $\varphi_A$ Is the set $\{0,1\}$, then we have a crisp subset. [1-3]

**Definition of $\alpha$ -level**

Let A be a subset of $U$ and $\alpha \in [0,1]$, then the $\alpha$ level of the fuzzy subset A is the classical subset of $U$ that is defined as

$[A]_\alpha = \{x \in U : \varphi_A(x) \geq \alpha\}$, for $\alpha \in [0.1]$

We are interested in on the fuzzy subsets $A$ of $\Re$ so their $\alpha$ levels are given by

$[A]_\alpha = [a^-{}_\alpha, a^+{}_\alpha]$, $a^-{}_\alpha \leq a^+{}_\alpha$, for each $\alpha \in (0,1]$. We use the symbol $\Re_\alpha$ to denote the class of these fuzzy subsets

**Definition of Fuzzy measure,**

Let $\Lambda$ be a sigma $\sigma$ - algebra of a classical set $\Omega$ omega. A map $\mu : \Lambda \to [0,\infty)$ is called a fuzzy measure such satisfies:

a) $\mu(\varnothing) = 0$ and b) if $A, B \in \Lambda$ and $A \subseteq B$ then $\mu(A) \leq \mu(A)$

The definition of measure of Sugeno (1974) [4] considered the boundary conditions $\mu(\Omega) = 1$ in a). that is a normalization of the fuzzy measure.

**Definition of Lebesgue measure,**

Let $A \in \Re_\alpha$ and $\alpha \in [0,1]$. The usual Lebesgue $\mu(A)$ of the $\alpha$ -level of $A$ is given by

$\mu([A]_\alpha) = a_\alpha^+ - a_\alpha^-$

The usual Lebesgue measure is a fuzzy measure (Roman, Flores) [5,6].

**Definition of Sugeno integral**

Let $\mu$ be a fuzzy measure on $(\Re, \Sigma)$. If $f \in \mathrm{F}^{\mu}(\Re)$ and $A \in \Sigma$, then the Sugeno integral of $f$ on $A$ with respect to the fuzzy normalized measure $\mu$. The Sugeno integral is given by

$$\int_A f d\mu = \vee_{\alpha \geq 0} \left[ \alpha \wedge \mu(A \cap \{f \geq \alpha\}) \right], \quad A \in \Sigma,$$

Where $\vee$, $\wedge$ denote the operations sup and inf on $[0, \infty]$, respectively. If $A = \Re$ then

$$\int_{\Re} f d\mu = \int f d\mu = \vee_{\alpha \geq 0} \left[ \alpha \wedge \mu\{f \geq \alpha\} \right] \qquad (4)$$

**Remark**. Consider the distribution function $F$ associated to $f$:
$F(\alpha) = \mu(A \cap \{f \geq \alpha\})$, then, due to the prepositions $\mu(A \cap \{f \geq \alpha\}) \geq \alpha$ and $\mu(A \cap \{f \geq \alpha\}) \leq \alpha$ we have

$$F(\alpha) = \int_A f d\mu = \alpha \qquad (5)$$

Thus, from computational point of view the fuzzy integral can be calculated solving the equation $F(\alpha) = \alpha$.

## 2. **The fuzzy cantor set**

In connection with our problem, is important to observe that, physically a point particle in space closely resembles a fuzzy point in Zadeh fuzzy set theory [7]. In the usual set theory, a point either belongs or does not belong to a subset. But according to Zadeh' fuzzy set theory a point can be a member of a subset with probabilistic or uncertain feature. a fuzzy point $p_f$ in the set $\mathbf{Z}$ is a fuzzy subset with a membership function: $\mu_{p_f}(x) = y$ for $x = x_0$ or $x = 0$

otherwise, where $0 < x < 1$, where a fuzzy point $p_f$ is said to have support $x_0$ and $x$ values. The point in the set **Z** at which $\mu_{p_f}(x) > 0$ constitute the support of the fuzzy point.

Here, we need a function $f(x)$ with $f(0) = 0$, $f(1) = 0$ and $f_{\max}(x) = 1$ so we choose $f(x) = 4x(1-x)$; this function fully satisfies the postulates of the fuzzy theory indicated on this section 1.

In this context we define the function $f(x) = \varphi_A : \Omega \to [0,1]$ given by $f(x) = 4x(1-x)$; $f(x)$ which must be a membership of a fuzzy subset $F$ of $\mathbb{R}$ whose $\alpha-$ levels are given by

$$[F]_\alpha = \{x \in \mathbb{R} : 4x(1-x) \geq \alpha\} = \left[\frac{1-\sqrt{1-\alpha}}{2}, \frac{1+\sqrt{1-\alpha}}{2}\right]$$

If $\mu$ is the usual Lebesgue measure on $\Omega = \mathbb{R}$, then the level function $F(\alpha)$ is

$$F(\alpha) = \mu([F]_\alpha) = \frac{1+\sqrt{1-\alpha}}{2} - \frac{1-\sqrt{1-\alpha}}{2} = \sqrt{1-\alpha}$$

Thus, the Sugeno integral is

$$\int_\Omega f d\mu = \sup_{\alpha \in [0,1]} \left[\alpha \wedge \sqrt{1-\alpha}\right]$$

Since $F(\alpha)$ has a decreasing function part, we have

$$\int_\Omega f d\mu = \alpha = F(\alpha) = \frac{\sqrt{5}-1}{2} = 0.61803 \qquad (6)$$

Which gives the fixed point of $F(\alpha)$. This numerical result between $[0,1]$ is the most likely for Heisenberg's principle under the fuzzy approach

To construct this set (denoted by $C^S$), we begin with the interval $[0,1]$ and remove the open set $(\alpha^2,\alpha)$, from the closed interval $[0,1]$. The set of points that remain after this first step will be called $C_1^S$, that is, $C_1^S = [0,\alpha^2] \cup [\alpha,1]$. In the second step, we remove the middle of the two segments of $C_1^S$, be what remains after the first two steps etc... Repeating this process, the limiting set $C_n^S$, called by us the Cantor-Sugeno set, so we have

$$\alpha(1-\alpha)+2\alpha^3(1-\alpha)+2^2\alpha^5(1-\alpha)+2^3\alpha^7(1-\alpha)+...$$

That is

$$\alpha(1-\alpha)\left[1+2\alpha^2+2^2\alpha^4+...\right]=\alpha(1-\alpha)\sum_n(1+2^n\alpha^{2n})=S$$

$$\lim_{n\to\infty} S=\alpha(1-\alpha)/(1-2\alpha^2)\to 1 \text{ when } n\to\infty$$

That is, this set that we can call Sugeno – Cantor set, $C^S$, also, as the original Cantor set $C$, $C^S$ is not empty and it is uncountable.

Let $C^S = \bigcap_{n=1}^{\infty} C_n^S$

Where

$C_0^S = [0,1]$, $C_1^S = [0,\alpha^2] \cup [\alpha,1]$, ……..

And

$$C_0^S > C_1^S > C_2^S .........C_n^S = (2\alpha^2)^n$$

$C^S$ is null. To show this we choose any $\varepsilon > 0$, and $n \ni, 2(\alpha^2)^n < \varepsilon, n = \left|\dfrac{\ln\varepsilon}{\ln 2(\alpha^2)^n}\right|$

$C^S$ is uncountable

Each $x \in |0,1|$ can be expressed as follows

$x = \sum_{n=1}^{\infty} b_n (\alpha^2)^n$ where $b_n = \{0,1,2...\}$ and $x = (0, b_1, b_2, b_3...)_{\alpha^2}$

$1 = \sum_{n=1}^{\infty} 2(\alpha^2)^n = (0,22222....)_{\alpha^2}$

$C_1^S$ does not have $x^{\acute{}s}$ with $b_1 = 1$ or $b_2 = 1$ so $x \in C^s$ iff $\forall b_i \in \{0,2\}$

We compute the fractal dimension of the Cantor Sugeno middle- $\alpha(1-\alpha)$ set or Hausdorff dimension of $C^S$ ,

$$DimC^S = \lim_{n\to\infty} \ln 2^n / \ln\left(1/\alpha^2\right)^n = \ln 2 / \ln \alpha^2 = 0.72$$

Which be compared with the original Cantor set with Hausdorff dimensions of instead of ln2/ln3 = 0.6309

The standard In measure theory, a branch of mathematics, is the **Lebesgue measure**, is the way of assigning a measure to subsets of *k* dimensional Euclidean space. For k = 1, 2, or 3, it coincides with measure of length, area, or volume respectively.

Now we consider the value of $\alpha$ with respect the value 1/3 of the Cantor set. In the context of the Lebesque measure defined as $M_L^S = 1 - a$ , where

$$M_L^S = 1 - a = 1 - \frac{\alpha(1-\alpha)}{1/3}$$

Here, $\alpha(1-\alpha) = 0.6180 \times 0.382 = 0.236$ and 1/3 is the interval removed for the first time divided by 1/3 , so that $a = 0.708$ $M_L^S = 1 - a \approx 0.3$ .

This fuzzy set is between the Cantor set and the Smith–Volterra–Cantor set is topologically equivalent to the middle-thirds Cantor set.

Also, the quantum wave functions $\psi(x)$ of subatomic particles (as leptons; electrons muons, baryons like the quark's family with spin 1/2, intergalactic massive photons or primordial black hole $[8,9]$ ), are normalized like $\int \psi(x)\psi*(x)dx = 1$.

When we construct a Cantor set, whether deterministic or fuzzy, we end up with two Cantor sets, the zero set and the empty set, the former consists of infinite points, and the latter with space -time with Compton wavelength [], These two Cantor sets are extremely important for understanding the chiral standard theory of particles []. The zero set represents the quantum particle and its Hausdorff dimension is (0.28) . The empty set models the quantum wave but it also models quantum space-time itself [], its Hausdorff dimension is 0.72.

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## Chiral non local Maxwell Equations and Lorentz gauge

Theory predicts that the Higgs boson lasts only a short time and decays into various types of elementary particles. Until now, their decays into different gauge bosons (elementary particles that carry force) had been correctly established. The other family of particles, the fermions, make up matter. According to the standard model of particle physics, fermions (such as tau particles, electrons, muons, and quarks) acquire mass in the same way as bosons, through the Brout-Englert-Higgs mechanism. Therefore, the Higgs boson could decay directly into bosons or fermions.

$$\vec{E} = \frac{1}{\varepsilon_0}\left(\vec{D} + \beta\nabla\times\vec{D}\right)$$
$$\vec{B} = \mu_0\left(\vec{H} + \beta\nabla\times\vec{H}\right)$$
$$\vec{B} = \nabla\times\left(\vec{A} + \beta\nabla\times\vec{A}\right) = \nabla\times\vec{F}$$
$$\vec{B} = \nabla\times\vec{F} \rightarrow \nabla\times\vec{A} = \mu_0\vec{H}$$
$$\vec{E} \equiv -\frac{\partial}{\partial t}\vec{F} - \nabla V$$

where

$$\vec{F} = \vec{A} + \beta\nabla\times\vec{A}$$

$$\nabla\times\vec{E} = -i\omega\vec{B} = -i\omega\nabla\times\vec{F}$$
$$\nabla\times\vec{H} = i\omega\vec{D} + \vec{J}$$

$$\nabla\times\nabla\times\vec{H} = i\omega\nabla\times\vec{D} + \nabla\times\vec{J}$$

$$\frac{\vec{B}}{\mu_0} - \vec{H} = \beta\nabla\times\vec{H}$$
$$\frac{\nabla\times\vec{F}}{\mu_0\beta} - \frac{\nabla\times\vec{A}}{\mu_0\beta} = \nabla\times\vec{H}$$
$$\frac{\nabla\times\vec{F} - \nabla\times\vec{A}}{\mu_0\beta} = \nabla\times\vec{H}$$

$$\nabla\times\left(\frac{\nabla\times\vec{A}}{\mu_0}\right) = \nabla\times\vec{H} = i\omega\vec{H} = i\omega\vec{D} + \vec{J}$$
$$\nabla\left(\nabla\cdot\vec{A}\right) - \nabla^2\vec{A} =$$

$$\frac{\nabla\times\vec{F}-\nabla\times\vec{A}}{\mu_0\beta}=\nabla\times\vec{H}=i\omega\vec{D}+\vec{J}$$

$$\nabla\times\left(\frac{\nabla\times\vec{F}-\nabla\times\vec{A}}{\mu_0\beta}\right)=i\omega\nabla\times\vec{D}+\nabla\times\vec{J}$$

$$\nabla\times\left(\nabla\times\vec{F}\right)-\nabla\times\left(\nabla\times\vec{A}\right)=i\omega\mu_0\beta\nabla\times\vec{D}+\mu_0\beta\nabla\times\vec{J}$$

$$-i\omega\vec{F}-\nabla V=\frac{1}{\varepsilon_0}\left(\vec{D}+\beta\nabla\times\vec{D}\right)$$

$$-i\omega\vec{F}\varepsilon_0-\varepsilon_0\nabla V=\vec{D}+B\nabla\times\vec{D}$$

$$-i\omega\vec{F}\varepsilon_0-\varepsilon_0\nabla V=\frac{1}{\mu_0 i\omega}\left(\nabla\times\nabla\times\vec{A}\right)-\frac{\vec{J}}{i\omega}+\beta\left(\frac{\nabla\times\nabla\times\vec{F}-\nabla\times\nabla\times\vec{A}}{\mu_0 i\omega\beta}\right)-\frac{\mu_0\beta\beta\nabla\times\vec{J}}{\mu_0 i\omega\beta}$$

$$-i\omega\varepsilon_0\vec{F}-\varepsilon_0\nabla\vec{V}=\frac{1}{i\omega\mu_0}\left(\nabla\times\nabla\times\vec{A}\right)-\frac{\vec{J}}{i\omega}+\left(\frac{\nabla\times\nabla\times\vec{F}-\nabla\times\nabla\times\vec{A}}{\mu_0 i\omega}\right)-\frac{\beta\nabla\times\vec{J}}{i\omega}$$

$$-i\omega\varepsilon_0\vec{F}-\varepsilon_0\nabla\vec{V}=-\left(\frac{\vec{J}+\beta\nabla\times\vec{J}}{i\omega}\right)+\frac{\nabla\times\nabla\times\vec{F}}{\mu_0 i\omega}$$

$$\omega^2\mu_0\varepsilon_0\vec{F}-i\omega\mu_0\varepsilon_0\nabla V=-\mu_0\left(\vec{J}+\beta\nabla\times\vec{J}\right)+\nabla\times\nabla\times\vec{F}$$

$$\nabla\times\nabla\times\vec{F}=\nabla\left(\nabla\cdot\vec{F}\right)-\nabla^2\vec{F}$$

$$-\nabla\left(\nabla\cdot\vec{F}\right)+\nabla^2\vec{F}+\omega^2\mu_0\varepsilon_0\vec{F}-i\omega\mu_0\varepsilon_0\nabla V=-\mu_0\left(\vec{J}+\beta\nabla\times\vec{J}\right)$$

$$-\left(\nabla\left(\nabla\cdot\vec{F}+i\omega\mu_0\varepsilon_0 V\right)\right),$$

$$\nabla\cdot\vec{F}+i\omega\mu_0\varepsilon_0 V=0 \quad \text{(Lorentz gauge)}$$

The wave equations for F and V are

$$\nabla^2\vec{F}+\omega^2\mu_0\varepsilon_0\vec{F}=-\mu_0\left(\vec{J}+\beta\nabla\times\vec{J}\right)$$

$$\nabla^2 V+\omega^2\mu_0\varepsilon_0 V=-\rho/\varepsilon_0$$

If $\left(\vec{J}+\beta\nabla\times\vec{J}\right)=0$

$$\nabla^2\vec{F}+\omega^2\mu_0\varepsilon_0\vec{F}\Rightarrow\omega^2/c^2=k^2$$

If $\nabla \times \vec{D} = -\frac{1}{2\beta}\vec{D}, \quad \nabla \times \vec{H} = -\frac{1}{2\beta}\vec{H}, \quad \Rightarrow \quad \nabla \times \vec{E} = -\frac{1}{2\beta}\vec{E}, \quad \nabla \times \vec{B} = -\frac{1}{2\beta}\vec{B},$

$\Rightarrow \vec{E} \,\square\, \vec{B} \,\square\, \vec{H} \,\square\, \vec{D}$

$$\nabla \times \vec{F} = -\frac{1}{2\beta}\vec{F},$$

Here we see the Lorentz equation

$$\nabla \cdot \vec{F} + i\omega\mu_0\varepsilon_0 V = 0$$

## **Non local Maxwell Equations with** $\vec{E} = i\vec{B}$ and the chiral electron

$$\nabla \times \vec{E} = -\frac{1}{cdt}\left(1 + T\nabla \times\right)\vec{B}$$

$$\nabla \times \vec{B} = \frac{1}{cdt}\left(1 + T\nabla \times\right)\vec{E}$$

$$\nabla \times \nabla \times \vec{E} = -\frac{\partial}{c\partial t}\left(1 + T\nabla \times\right)\nabla \times \vec{B}$$

$$\nabla \times \nabla \times \vec{B} = \frac{\partial}{c\partial t}\left(1 + T\nabla \times\right)\nabla \times \vec{E}$$

$$\nabla \times \nabla \times \vec{E} = -\frac{\partial}{c\partial t}\left(1 + T\nabla \times\right)\frac{\partial}{c\partial t}\left(1 + T\nabla \times\right)\vec{E}$$

$$\left(1 + \frac{T^2}{c^2}\frac{\partial^2}{\partial t^2}\right)\nabla \times \nabla \times \vec{E} = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2}(1 + 2T\nabla \times)\vec{E}$$

$$\frac{T^2}{c^2}\frac{\partial^2}{\partial t^2} = -1$$

$$\nabla \times \vec{E} = -\frac{1}{2T}\vec{E}$$

$$\nabla \times \vec{B} = -\frac{1}{2T}\vec{B}$$

If $\frac{\partial}{\partial t} = i\omega$ then

$$\vec{E} = \frac{T}{c}\frac{\partial}{\partial t}\vec{B} \to \frac{T}{c} i\omega \vec{B} \to \frac{\omega T}{c} i\vec{B} = i\vec{B}$$

In terms of

$\vec{A}$ and $\Phi$

$$\nabla \times \vec{B} = -\frac{1}{2T}\vec{B} = \nabla \times \nabla \times \vec{A} = -\frac{1}{2T}\nabla \times \vec{A}$$

$$\nabla \times \vec{E} = -\frac{1}{2T}\vec{E} = \nabla \times \left(-\frac{\partial}{\partial t}\vec{A} - \nabla\Phi\right) = -\frac{1}{2T}\left(-\frac{\partial}{\partial t}\vec{A} - \nabla\Phi\right)$$

From here we obtain the gauge Lorentz and the wave equation for $\vec{A}$

$$\nabla \cdot \vec{A} - \frac{1}{c^2}\frac{\partial}{\partial t}\Phi = 0$$

And

$$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A} = 0$$

Here we can obtain the electron equation derived from the electromagnetic field. There exists a physical equivalency between Dirac and Maxwell theories which can be stated as follows. It is well known that Lorentz' equation is the Lorentz invariant formed by taking the scalar product of the four-gradient and the electromagnetic four-potential,

$$\left(\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right) \cdot (\Phi, \vec{A}) = \nabla \cdot \vec{A} - \frac{1}{c^2}\frac{\partial}{\partial t}\Phi = 0$$

The scalar and vector potential ca be written in the form of carrier wave expansions under the chiral approach

$$\Phi = \Phi(\vec{r}, t)e^{i\omega\sigma_1 t}, \qquad \vec{A} = \sigma_3 \vec{\sigma}\Phi$$

Where we use the Pauli matrices. Here the zitterbewegung motion of the electron is absent so the resultant equation is

$$(\sigma_0 E - \sigma_3 c\vec{\sigma} \cdot \vec{p})\Phi = mc^2 \sigma_1 \Phi$$

An electromagnetic contribution to the mass of the electron due to the quantum radiation field associated with its motion is a well-known concept in QED. Indeed the carrier-wave frequency of the electron's four-potential [] is equal to $mc^2/\hbar$, which is the high frequency cut off for the quantum radiation field assumed in QED atomic structure calculations. The present derivation of Dirac's equation suggests that the total mass of the electron is electromagnetic in nature. This result is consistent with a previous result in which the charge of the electron was derived from Maxwell's equations.

## Velocity Gauge of electromagnetic theory and the dark energy

Here, it is shown that the Lorentz and coulomb gauges are limiting cases of the velocity gauge. The free -space wave equations for the potential fields from the Maxwell'equations are:

$$\nabla^2\Phi + \frac{1}{c^2}\frac{\partial}{\partial t}\nabla\cdot\vec{A} = 0 \tag{1}$$

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A} = \nabla(\nabla\cdot\vec{A}) + \frac{1}{c^2}\frac{\partial}{\partial t}\nabla\Phi \tag{2}$$

In order to separate these coupled equations a gauge must be chosen. The Lorentz gauge is

$$\nabla\cdot\vec{A} - \frac{1}{c^2}\frac{\partial}{\partial t}\Phi = 0$$

And the Coulomb gauge is

$$\nabla\cdot\vec{A} = 0$$

Here we define the velocity gauge

$$\nabla\cdot\vec{A} - \frac{1}{u^2}\frac{\partial}{\partial t}\Phi = 0$$

If $u = c$ we have the Lorentz gauge and if $u \to \infty$, the Coulomb gauge is obtained. The wave equations (1) and (2) are generalized to

$$\nabla^2\Phi + \frac{1}{u^2}\frac{\partial^2}{\partial t^2}\Phi = 0 \qquad (3)$$

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A} = \left(\frac{1}{c^2} - \frac{1}{u^2}\right)\frac{\partial}{\partial t}\nabla\Phi \qquad (4)$$

Here we can see that the propagation speed of the vector potential is $c$ while the propagation is $u$ .

Taking the gradient of (3) and the partial time derivative of (4) we get

$$\nabla^2\nabla\Phi + \frac{1}{u^2}\frac{\partial^2}{\partial t^2}\nabla\Phi = 0 \qquad (5)$$

$$\nabla^2\frac{\partial}{\partial t}\vec{A} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\frac{\partial}{\partial t}\vec{A} = \left(\frac{1}{c^2} - \frac{1}{u^2}\right)\frac{\partial^2}{\partial t^2}\nabla\Phi \qquad (6)$$

Adding equations (5) and (6) we have

$$\nabla^2\left[\frac{\partial}{\partial t}\vec{A} + \nabla\Phi\right] - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left[\frac{\partial}{\partial t}\vec{A} + \nabla\Phi\right] = 0 \qquad (7)$$

Note that all terms containing $u > c$ disappear. Now using $\vec{E} = -\frac{\partial}{\partial t}\vec{A} - \nabla\Phi$ in (7) We obtain

$$\nabla^2\vec{E} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{E} = 0$$

The wave equation for the electric field, showing propagation at speed c.

The scalar potential $\Phi$ with $u > c$ , has produced a vector potential as required physically by the measured electric field $\vec{E}$ propagating at c. It is easy to show that the magnetic field propagates at c. Taking the curl of (4), the wave equation for $\vec{A}$ in free space

$$\nabla^2(\nabla\times\vec{A}) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\nabla\times\vec{A} = \left(\frac{1}{c^2} - \frac{1}{u^2}\right)\frac{\partial}{\partial t}\nabla\times\nabla\Phi$$

But $\vec{B} = \nabla\times\vec{A}$ , and $\nabla\times\nabla\Phi = 0$ , so the wave equation for $\vec{B}$ is

$$\nabla^2\vec{B} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{B} = 0$$