A Set of Documents and Mathematics PBAT Problems

Dear Consortium Math Teachers,

We are all privileged to be part of the Consortium, where we have the opportunity to teach mathematics with great imagination and share its beauty and power with our students. We have the opportunity to present mathematics so that students can come to have a new emotion and appreciation for this wonderful discipline. We also have the opportunity to challenge our students in new and exciting ways, helping them begin to realize they are capable of thinking deeply about mathematics.

Reuben Hersh, the American mathematician, stated, “Solving problems and making up new ones is the essence of mathematical life.” In the consortium, we believe that placing problem-solving at the center of our work can help our students become nuanced mathematical thinkers and develop a genuine understanding of mathematics. That is why our PBAT work is based on asking our students to solve non-routine problems.

Our PBAT work in mathematics is important. We have taken the lead in this country by centering our work around problem-solving and engaging our students in meaningful, transformative experiences. This allows them to recognize their capabilities in achieving things they never thought possible. This PBAT work represents the culmination of each Consortium school’s math program, where students can share their creativity and understanding of mathematics.

We are enclosing three documents and PBAT problems.

1. **Quality and Criteria for a Mathematics PBAT**
   This document represents the ideas of teachers from nearly every consortium school about the qualities and criteria for a mathematics PBAT. In five sessions, we critically examined our PBAT work and discussed how we can ensure that the work students do on a math PBAT is both authentic and transformative. The PBAT experience in mathematics should be one that students never forget, as it offers them the opportunity to use their minds in deep and new ways that foster genuine pride. This document should become central to all schools as they plan their PBAT initiatives.

1. **Non-Routine Problems Defined**
   This document was written in response to teachers' desire to understand the meaning of non-routine problems. The language of “non-routine” comes from NCTM and provides students opportunities to do more authentic work in mathematics. This document should be looked at closely by all math departments in the consortium. Non-routine problems should be integral to every student's experience in mathematics, starting in ninth grade.

2. **Consortium Mathematics Rubric**
   This rubric was created six years ago by the mathematics teachers in the consortium. The five domains are based on the NCTM process standards, which focus on student mathematical thinking.

3. **PBAT Problems**
   As of 2025, there are 62 non-routine problems for you to look at, play with, and think about how you might utilize them with your students. Each problem has undergone a vetting process. Either the problem has been used successfully with students, fostering high-level thinking and understanding, or it was vetted by a group of 10 teachers who reviewed the problem using a vetting protocol.

   We thank the following schools for sharing their PBAT tasks for this document: ICE, Lab School for Finance and Technology, Bronx Collaborative, International H.S. at LaGuardia, Leaders High School, Beacon, Bronx Lab, and Lyons.

   The PBAT problems have been broken into 3 categories:

   > **For the strongest students**: This includes problems that are rich and complex as given, or can be extended to become rich and complex, requiring sophisticated mathematical thinking. They offer opportunities for extensions, allowing students at this level to be challenged appropriately.
   >
   > **For middle students:** This includes rich problems that require nuanced mathematical thinking about mathematical ideas from Algebra 1 and possibly Algebra 2 or fundamental ideas in number theory. These problems should allow for extensions so that students can be appropriately challenged.
   >
   > **For students who struggle in math:** These tasks are designed for students who may struggle with mathematics and mathematical thinking for various reasons, including gaps in their math education. They involve rich problems that encourage mathematical thinking about fundamental concepts in algebra and number theory. There are opportunities for extensions, allowing students at this level to be challenged appropriately.

In the Consortium, we have a diverse group of 38 schools. Our student populations exhibit both similarities and differences. We all have students who arrive at our schools with positive experiences in mathematics, and we all have many students whose past experiences have led them to dislike or even fear this subject. Part of our mission is to help students rethink what mathematics is and what they are capable of doing. Problem-solving can be a means of making that happen. Non-routine problem-solving can be a liberating experience for many students, as the open-ended nature of these problems allows them to explore various approaches without feeling constrained by the “correct way of doing something.”

Some Suggestions on Preparing to Use the Problems

a. **Get to know your students as well as possible.** What are your students’ attitudes toward mathematics? How does each student’s attitude impact their willingness to engage with challenging problems? How can I help my students see mathematics in a more positive way?
b. **Get to know your students as mathematical thinkers.** What do you do daily to get to know your students as mathematical thinkers? How can I use non-routine problems as interim assessments? How can I use journaling to learn more about my students as mathematical thinkers?
c. **Get to know each PBAT problem as well as possible**. What are the different ways a student might think about this problem? What makes this problem rich? What questions could I pose to students to help deepen their thinking about the problem? There are a lot of Problems in the Non-Routine PBAT Problem Set now. So, if you’re looking for where to start, here’s 19 PBAT Problems Everyone Needs to Play With.
d. **Ensure that students choose or are matched with a problem that is appropriately challenging.** We want students to have choices, but we want them to select from a set of problems that are suitably challenging based on who they are as mathematical thinkers. How will you do that?

Many questions may arise while students are working on their problems. Please feel free to contact Jonathan or AJ for support. We hope that all students will have a truly valuable experience working on their non-routine problems.

Sincerely,

Dr. Jonathan Katz
Mathematics Specialist
New York Performance Standards Consortium

Quality and Criteria for a Mathematics PBAT

Introduction

A performance-based assessment task (PBAT) should be a non-routine, complex problem scenario that allows for multiple avenues of problem-solving. A PBAT is a final task in which a student demonstrates their ability to think and reason mathematically. Before this PBAT, all students should have had many experiences throughout their high school careers in which they grappled with non-routine, complex, problematic scenarios (interim assessments) so that they are not surprised by the PBAT's non-routine nature. The PBAT may or may not be an open-ended task with multiple "correct" solutions or a single solution; nevertheless, the task must be appropriately challenging, mathematically rich, and able to be approached in multiple ways. The PBAT assignment may be given to students by the teacher or co-selected with students.

A student's PBAT work should reflect the highest level of mathematical understanding and thinking that the student has demonstrated throughout their years at the school. Thus, the content embedded in the PBAT should vary according to the student. Some students have shown a strong ability to think and reason mathematically; therefore, the PBAT they work on should reflect that depth of understanding. Other students approach this work with considerable trepidation and years of difficulty making sense of mathematics, so the PBAT they work on should be challenging yet suitable for their level of understanding. All tasks should aim to promote student autonomy and independent mathematical thinking.

Non-routine problems are not those routinely found in textbooks. A non-routine problem places students in an unfamiliar context, compelling them to tackle a complex issue using various mathematical ideas. A complex non-routine problem demands significant thought and time, encompassing strategizing, reasoning, making connections, making sense of ideas, generalizing, and justifying one's process and solution through rigorous mathematical analysis. It also requires perseverance. A student may encounter a wide range of emotions, including frustration, excitement, confusion, joy, and pride. Over time, the problem should gain meaning for the problem-solver as they come to own and understand it.

The PBAT requires students to write a sophisticated mathematical document that details the journey they experienced, including their processes and the solution to the non-routine problem. Students will then need to defend their work in front of a mathematically knowledgeable panel. During this oral defense, students will be expected to engage in a mathematical conversation involving an on-demand problem arising from the PBAT itself, expanding on their understanding. The on-demand problem should not be a routine exercise disconnected from the student's work. The comprehensive paper and the oral defense are deeply interconnected but are scored as separate components of the PBAT.

The PBAT task must align with the five dimensions in the Consortium Math rubric. These five dimensions reflect the aspects of the thinking process that students undergo when solving a problem. They have a long history dating back to their presentation by NCTM in 1989. When creating a PBAT task, these dimensions need to be at the forefront of our minds to ensure we demand deep mathematical thinking and reasoning from our students.

In the following sections, we will concentrate on the areas of our work that teachers deemed most significant to reflect on regarding the math PBAT work. These sections include: independent mathematical thinking, student metacognition, scripting and scaffolding, and student choice and differentiation.

Independent Mathematical Thinking

An independent thinker is aware that multiple strategies exist for solving a problem. They should be able to connect different problems to their own mathematical experiences and relate conceptual ideas and procedural methods presented in their PBAT. An independent thinker demonstrates a strong willingness to persevere and understands the importance of defending and explaining one's reasoning. Overall, an independent thinker takes ownership of both the process and the product when working on a PBAT.

A teacher's role is to provide guidance and encouragement, ask appropriate questions, celebrate small successes, build trust among students, and assess readiness. The teacher's role is not to give answers or offer "too much" help. The thinking must belong to the student, not the teacher.

Revision is part of the writing process; however, it can raise some issues. The main goal of revision is to ensure that the product remains truly the student's work throughout the process. When a teacher asks a student to make revisions, the student needs to articulate what needs to be revised, why it needs to be revised, and how they intend to make the revisions. We must ensure that the revision process is based on the student's thinking and includes their original ideas.

If we create a math program designed to foster independent mathematical thinking, we enhance our prospects of influencing how our students grapple with the PBAT. In this math program, students can expect to be challenged to think mathematically each day. They will cultivate independence if we offer them numerous low-stakes problem-solving experiences throughout their high school math courses. These experiences should be varied, including non-routine contextual problem-solving. Students must be appropriately challenged on a continual basis, with opportunities to reflect on their growth as problem solvers and thinkers.

Student Metacognition

Alan Schoenfeld, one of the leading thinkers on mathematical problem solving for over thirty years, has emphasized the importance of students critically examining their own work as they solve problems. Since understanding one's thought processes and developing the ability to self-regulate during problem solving are crucial for any student’s growth as a problem solver, it is vital for students to discuss their thinking while grappling with the PBAT. What might that look like in student work?

Student metacognition can be articulated throughout the paper, when appropriate. Students might discuss:

- What approach to the problematic situation did you take?
- Why did you take that approach?
- How did you know your actions made sense and would lead you to a solution?
- Did you need to rethink your approach? Why? What did you do?
- How did you become convinced that your thinking about the problem was correct and that your solution made sense?

Scaffolding and Scripting

It is appropriate to provide different levels of scaffolding to students as needed. However, our goal is to minimize scaffolding to maximize the students’ independent thinking. Scaffolding can be beneficial if its purpose is to enhance student thinking. Unfortunately, it can also have the opposite effect, forcing students to think in a restricted manner, thus interfering with their original thinking and reasoning. We should view ourselves as facilitators, engaging with a student’s thinking process only when necessary. If it appears that a student does not understand the mathematics needed to complete a PBAT task, this signals that they must pause on the PBAT and review the content in a different context.

Scripting directs students through each step of a problem using either statements or guiding questions. This narrows the focus and hinders students from thinking independently. Students should have time to think for themselves and select their strategy or approach to solving a problem, even if their reasoning is flawed or inefficient.

Scripting should not be used during PBAT work. When a teacher believes that scripting is necessary to complete a PBAT, it indicates that the PBAT problem is inappropriate at this time.

Scripting might be useful in Pre-PBAT math classes where students learn to develop as mathematical thinkers. This approach could model the types of questions students need to ask themselves as they grow as problem-solvers and mathematical thinkers. However, students may become dependent on scripts, which limits their development as mathematical thinkers; therefore, we must be cautious in our use of scripts during the early years of high school. As students progress in their high school careers, the use of scripting should be reduced and eventually eliminated from their mathematics work.

Student Choice and Differentiation

Student choice is crucial for enhancing engagement, fostering personal connections, and promoting accountability. Allowing student choice also creates additional opportunities for critical thinking and original work.

We recognize that opportunities for student choice must be structured and differentiated. How can we ensure that we provide students with a meaningful mathematics experience? How can we make certain that the tasks each student undertakes are suitable for them at this time? We should keep this statement from George Polya in mind as we work with students.

If he challenges the curiosity of his students by setting their problems proportionate to their knowledge and helps them to solve their problems with stimulating questions, he may give them a taste for, and some means of, independent thinking.

To ensure rigor in PBAT assignments that are also unique for each type of student and learner, teachers must clearly articulate their mathematical expectations for each student. These expectations should align with the Consortium Math Rubric. In the Consortium, we have the ability to know our students well. Teachers can challenge each student beyond the basic set of expected understandings and skills to achieve their highest level of mathematical thinking.

Closing Remarks

The expectations outlined in this document may seem challenging for a teacher. We are asking a great deal from our students, urging them to explore new realms of mathematical thinking. We express our belief that our students can achieve this, and evidence shows that it can lead to powerful and exciting results. Therefore, we conclude with this question, which should guide us in our work:

- How can I ensure that each of my students has a truly memorable experience with their math PBAT that helps them become deeper mathematical thinkers?

Non-Routine Problems Defined

The New York Performance Standards Consortium places non-routine problems at the center of our work in mathematics. We state in our guiding document in mathematics, "Quality and Criteria for a Math" PBAT:

> A performance-based assessment task (PBAT) should be a summative, non-routine, complex problem-solving scenario that allows for multiple approaches. A PBAT serves as a final graduation task where students demonstrate their ability to think and reason mathematically. It should reflect a high level of mathematical understanding and reasoning.

So, let us begin by talking about what we mean by a non-routine problem. What is it, and why do we want students to work with such problems?

Non-routine problems are “cognitively non-trivial task[s]; [where] the solver does not already know a method [for solving]" (Selden, A., Selden, J., Hauk, S., & Mason, A., 2000, p. 129). This contrasts with a routine problem that can be solved mechanically by someone with past experience working with similar situations (Nancarrow, 2004). Thus, a problem is classified as routine or non-routine not by its structure or content but by the solver's previous experiences.

Since no obvious formulas or procedures exist for solving non-routine problems, solvers must apply learned concepts, facts, and strategies in new ways. Solving them takes creativity and originality, allowing for the unique ways each student thinks to be expressed in their written work (http://mathelogical.in/non-routine-mathematics/). The emphasis on creativity and originality underscores why non-routine problems are a valuable graduation criterion, paralleling the independent thinking necessary for higher-level mathematics.

Here is a well-known example of a non-routine problem. We recommend using it as a classroom lesson. It is called The Locker Problem.

There are 1000 lockers lined up numbered 1 to 1000 and 1000 students. The lockers are all closed. The first student, Jasmine, walks by and opens all the lockers. Then the second student, Al, walks by and goes to every second locker starting at #2 and closes it. Then Mary walks by and goes to every third locker starting at #3 closing the opened lockers and opening the closed lockers. The 4th student walks by and goes to every fourth locker starting at #4 closing the opened lockers and opening the closed lockers. This routine goes on until student 1000, Elvina, goes to locker #1000 and either closes it or opens it. After this is finished, which lockers will be open? Why?

Why is this a non-routine problem?

It does not have an immediately apparent strategy for solving it. Someone attempting to solve it will need to “play” with the problem to discover a strategy that might lead them down the right path. It requires originality and creativity to find a solution. A person working on the problem needs to be creative, perhaps drawing on the mathematical thinking they might have experienced before. A problem solver might ask themselves, “Can I simplify this problem? Will it help me make sense of what is going on here?” This problem allows for multiple avenues for solving it, where the unique way a person thinks will be expressed by the approach and strategy used.

Why use non-routine problems?

- It serves as a way to develop a **deeper mathematical understanding**, which encompasses both conceptual and procedural aspects. (See: Problem Solving Curriculum). Concepts and procedures are cultivated through thinking about contextual non-routine problems.
- It serves as a means of developing mathematical thinking. Students will begin to realize that math **allows them to think creatively with greater freedom than they previously believed.** For many students, this is exciting. Most students perceive mathematics as rigid, formulaic, and disconnected from art, imagination, and themselves. Engaging with non-routine problems during their math experience starts to alter that misconception.
- Encouraging students to engage in non-routine problem solving can **facilitate the transition from specific to general thinking**; in other words, it enhances their ability to think in more abstract ways. For example, if a student gets to engage with three very different contextual problems, each highlighting a distinct aspect of linearity, they can begin to analyze these situations and develop a deeper, more nuanced understanding of the uniqueness of linear functions (See: Problem Solving Curriculum).
- The importance of non-routine problem-solving in mathematics stems from the belief that mathematics is primarily about reasoning, not memorization. Problem-solving **allows students to develop understanding and explain the processes they use to arrive at solutions, rather than simply recalling and applying a set of procedures.**
- Students realize **they are capable of thinking and doing mathematics** by engaging with non-routine problems. They start to shift their perspectives on what mathematics is and how they think of it.

New York Performance Standards Consortium
Performance Assessment: Mathematics

Student ______________________________

Project Title (e.g. Mathematical Modeling, The Can Project): ______________________________

Project Topic (e.g. Linear programming, Volume -surface area optimization): ______________________________

Circle One: **Written** **Oral**
Circle One: **Teacher** **External Evaluator**

Evaluator (Print name)______________________________

Overall Holistic Evaluation ______________ **Signature** ______________________________ **Date**______________

09/2016

Performance Indicators	Outstanding	Good	Competent	Needs Revision
Problem Solving	Selects appropriate and efficient strategies to solve non-routine problems. Provides in-depth analysis of strategies Executes conceptually sound mathematical procedures accurately.	Selects appropriate and efficient strategies to solve non-routine problems. Provides some analysis of strategies Executes conceptually sound mathematical procedures with minor computational errors.	Selects appropriate, but inefficient, strategies to solve non-routine problems, and executes conceptually sound mathematical procedures with minor computational errors. or Selects appropriate and efficient strategies to solve non-routine problems but executes mathematical procedures with minor conceptual and computational errors.	Selects an inappropriate strategy or Makes major conceptual errors or procedural errors.
Reasoning & Proof	Makes valid conceptual/theoretical argument(s) and mathematically justifies it logically and thoroughly.	Makes valid conceptual/theoretical argument(s) and mathematically justifies it logically.	Makes argument(s) and justifies most mathematical statements accurately.	Makes arguments but does not justify mathematical statements accurately.
Communication	Always uses mathematical language and notations accurately. Always clearly explains mathematical thinking in an organized and detailed way.	Mostly uses mathematical language and notations accurately. Mostly clearly explains mathematical thinking in an organized and detailed way.	Sometimes uses mathematical language and notations accurately. Sometimes clearly explains mathematical thinking in an organized and detailed way.	Limited use of mathematical language and notation in an accurate manner. Rarely clearly explains mathematical thinking in an organized and detailed way.
Connections	Demonstrates an in-depth understanding of the relationships between mathematical concepts, procedures, and/or strategies.	Demonstrates an understanding of the relationships between mathematical concepts, procedures, and/or strategies.	Demonstrates a limited understanding of the relationships between mathematical concepts, procedures, and/or strategies.	Does not demonstrate understanding of the relationships between mathematical concepts, procedures, and/or strategies.
Representation	Creates an accurate and sophisticated mathematical representation(s), inherent to the task, to solve problems or portray solutions.	Creates an accurate mathematical representation(s), inherent to the task, to solve problems or portray solutions.	Creates an accurate mathematical representation(s), inherent to the task, to solve problems or portray solutions, but may be imprecise or contain minor errors.	Does not create an accurate mathematical representation, inherent to the task, to solve problems or portray solutions.

62 Mathematics Problems for PBATS

(1) For Strongest Students

1. A Walk to the Door (Zeno's Paradox)
2. Last Artist Standing
3. Sums of Consecutive Counting Numbers (with proof)
4. The Peg Game
5. Triangles and More Triangles
6. Squares and Rectangles on a 10 by 10 grid
7. What is the unit's digit?
8. The String Problem
9. Making Connections
10. The Magic of Math with Extension
11. The Magical Game with Extension
12. An Array from 1 to 9
13. An Array from 1-26
14. What's Happening?
15. Crosswalks
16. On and Off
17. Destroying Squares
18. Multitude of Triangles
19. Decreasing Numbers
20. Painted Cubes
21. Cross the Network
22. Chip Stacking
23. Segments of Triangles
24. The Three Piles
25. Cops & Bookies
26. Even or Odd Sum
27. One or Two

(2) For Middle Students

1. Moving the Stack
2. The Water Jug Problem
3. The Magic of Math
4. The Magical Game
5. Counting with Shapes (Figurate numbers)
6. Getting Ahead
7. Great Pyramid Problem
8. Growing Dots and Growing Objects
9. McNuggets
10. Squares
11. The Game of 27

12. The Twelve Days of Christmas with Handshake Problem Extension
13. The Number Bracelets
14. What is the Remainder?
15. Take Three from Five
16. Geoboards
17. Gio-A Student Creation
18. A Mathematician's Favorite Problem
19. Zeros
20. On Reflection
21. Unknown Dumbbells
22. I Take, You Take
23. 1001 Coins
24. Weight Problem
25. Decimal Expansion

(3) For Students who struggle in math / who have had an interrupted or limited math education

1. Jose's Rabbit
2. Cops and Robbers
3. How to Meet New Friends and Triangular Numbers
4. Indiana Jones and the Mystery of the Pyramid's Treasure
5. Odds and Evens
6. The Race
7. The Spiral Alphabet
8. Powers of 2
9. Trains
10. The Three Problems

Notes on All Tasks

For the strongest students	
1. A Walk to the Door (based on Zeno's Paradox)	An interplay between the finite and the infinite-Zeno would be happy.
2. Last Artist Standing (based on The Josephus Problem)	Patterns within patterns and the uniqueness of the powers of two.
3. Sums of Consecutive Counting Numbers (with proof)	The uniqueness of the powers of two and how they differ from other even numbers.
4. The Peg Game	Visual and numeric patterns explain the way to win a game
5. Triangles and More Triangles (Fractals--Sierpinski's Triangle)	The pattern of fractals can be visually and mathematically beautiful with many mathematical possibilities.
6. Squares and rectangles on a 10 by 10 grid	Mathematical patterns can simplify an overwhelming situation.
7. What is the unit's digit?	Patterns help you to understand a seemingly crazy computation.
8. The String Problem	A wild problem that looks at functional relationships based on factors with a unique relationship between geometry and number theory.
9. Making Connections	What makes math powerful/beautiful are its many deep connections. Here is an opportunity for students to explore that idea.
10. The Magic of Math (with extension)	Math is magical and this problem becomes much more interesting and challenging when students grapple with the extension.
11. The Magical Game (with extension)	A math game that becomes much more interesting and challenging when students grapple with the extension. We changed the name from The Game of 15 to the new name because too much information about The Game of 15 exists online.
12. An Array from 1 to 9	*Math as a science of patterns* comes alive in this problem
13. An Array from 1-26	How does a student make sense of this fascinating problem? How does a student think within a finite system?
14. What's Happening?	Students explorations: what do your students think about? What conjectures do they make and what can they prove?
15. Crosswalks	This is the Seven Bridges of Konigsberg.
16. On and Off	Challenging…Interesting… It is similar to the Tower of Hanoi but with its own twist.

17. **Destroying Squares**	Problems within problems…How many squares are on a grid and how do you *unsquare* them?
18. **Multitude of Triangles**	How can a student use visual thinking along with algebraic thinking? Patterns are crucial and beautiful in this problem.
19. **Decreasing Numbers**	This is a combinatorics problem. If your student hasn't thought about combinations this can be challenging but doable. How do your students think about this problem? How do they bring order and sense to this situation?
20. Painted Cubes	Surprisingly challenging, worthy of your engagement.
21. Cross the Network	Riff off of the 7 Bridges of Konigsberg
22. Chip Stacking	This is an incredibly difficult problem, but once the student begins to crack it, it really opens up into a fascinating study of the Tower of Hanoi.
23. Segments of Triangles	This problem really gets to the question "What really makes a triangle?". The myriad of ways that lines can make triangles makes this problem very interesting
24. The Three Piles	This is the game of Nim (https://en.wikipedia.org/wiki/Nim) but with a specific case of 3, 5, and 7.
25. Cops & Bookies	An expanded version of the 7 Bridges of Konigsberg. Students will ultimately need to solve the original 7 Bridges problem to fully solve this one.
26. Even or Odd	This problem is a rare guest in the math classroom and more nuanced than you think ;)
27. One or Two	While it has similarities to the Game of 27, it is much more challenging and yet still enjoyable.
For Middle Students	
1. Moving the Stack (known as the Tower Of Hanoi)	Patterns and functions explain a puzzling situation. What happens if you change the rules?
2. The Water Jug Problem	A famous problem raising issues about number relationships. How far can students take this problem?
3. The Magic of Math	Math is magical. Why? What questions arise for the student? How far can they take it?
4. The Magical Game	What do the students think about this situation? Where do they take it? We changed the name from The Game of 15 to the new name because too much information about The Game of 15 exists online.
5. Counting with Shapes (based on Figurate Numbers)	The numerical connectedness between geometric representations – triangular and hexagonal numbers.
6. Getting Ahead	A systems problem that can be challenging for some students. It is related to *The Race*.

7. **Great Pyramid Problem**	How does a student think about a situation with creativity and an understanding of the patterns of mathematics?
8. **Growing Dots and Growing Objects**	A comparison between two visual functions. What meaning can a student make out of them?
9. **McNuggets**	Thinking about number relationships in a weird situation.
10. Squares	A pattern where the finite meets the infinite. How does a student make sense of this?
11. The Game of 27	Using arithmetic ideas to strategize and then generalize.
12. **The Twelve Days of Christmas (with Handshake Problem)**	Students need to understand the problem, create worthy questions that they can attempt to answer. Extension asks students to make connections to the handshake problem.
13. The Number Bracelets	A great problem that many students have thought about. It will bring new insights to a mathematical explorer.
14. What is the remainder?	Patterns help us to understand so much about the structures of mathematics.
15. Take Three from Five	Investigating and proving an idea in number theory
16. Geoboards	Squares on a geoboard.What patterns do your students see? Don't forget the rotated squares. What generalizations can they make?
17. Gio-A Student Creation	This problem is based on an ICE student's way of thinking. How does one of your students make sense out of it?
18. A Mathematician's Favorite Problem	A wonderful problem that can go deeper and deeper. Polya talked about it in *Solving It*.
20. Zeros	Patterns and number sense meet in this problem. Don't be fooled by the pattern as I was.
21. On Reflection	Different levels of students can enter into this problem.
22. Unknown Dumbbells	If you make sure that this is appropriately challenging for a student it will be interesting to see how the student thinks about it.
23. 1001 Coins	Students will have to work backwards to fully grasp the nuance of this problem, though working on a smaller problem can be a useful strategy too.
24. Weight Problem	Thinking in base 2 will help you.
25. Decimal Expansion	Intriguing patterns within decimals.

For Students who Struggle in Math	
1. Jose's Rabbit	Number relationships and the patterns they create including Fibonacci or can we think about this problem using combinatorics?
2. Cops and Robbers	Making sense of a situation using ideas of linearity. It is similar to the Race with smaller numbers. How might you extend if a student needs to be further challenged?
3. **How to Meet New Friends and Triangular Numbers**	This wonderful problem and its relationship to triangular numbers.
4. Indiana Jones and the Mystery of the Pyramid's Treasure	Pascal's Triangle and its many patterns. This problem has more prompts than others because many struggling students might find it difficult to see many patterns in the triangle.
5. Odds and Evens	What is the sum of the first 100 consecutive odd numbers? Even numbers? Counting numbers? See the wonder of patterns making for interesting connections.
6. **The Race**	A wonderful problem that can be thought about in different ways including arithmetically. How are the different ways connected?
7. The Spiral Alphabet	Can I make sense of the patterns created by the spiral and create generalizations from them?
8. Powers of 2	Powers of Two are fascinating. We see them all over mathematics. What does your student think about them?
9. Trains	Partitions Ramanaujan would be happy :)
10. The 3 Problems	This group of 3 problems look at a basic arithmetic idea.

Section 1:

For the Strongest Students

Top

A Walk to the Door

You are going to walk to the door. In your first move, you walk halfway to the door. In your second move, you walk halfway from your new spot. In your third move, you walk halfway from the newest spot…

How many steps will it take you to reach the door if you keep repeating the process of walking halfway?

Prove you are correct mathematically?

Top

Last Artist Standing

One thousand Artists stood in a very large circle. Each, starting with the Jennifer Lopez look-alike, wore a sign on his or her back with a numeral from 1 to 1000 in a clockwise sequence. They began counting off. The Jennifer Lopez look-alike said, “One in,” and remained in the circle. The 7”6’ basketball player to her left said, “Two out,” and left the circle. The rap singer next in sequence said, “Three,” and remained in the circle. The United States Senator who was next in sequence said, “Four out,” and left the circle.

So it continued with each person sporting an odd numeral stating the numeral, saying “in,” and remaining in the circle and with every person wearing an even numeral leaving the circle.

It was easy to visualize who remained in the circle when the count off again reached the Jennifer Lopez look-alike. Since the person before her said, “One thousand out,” and left the circle, she now said, “One in,” and stayed in the circle. Continuing this sequence the rap singer said, “Three out,” and left the circle. This process would keep going on and on until only one person was left in the circle.
Which number is the last artist standing?

Top

Sums of Consecutive Counting Numbers

Which counting numbers cannot be written as a sum of consecutive counting numbers? (Note: Consecutive can be two or more consecutive counting numbers.) Prove that your solution is always true.

Top

The Peg Game

Directions:

1. You can move one chip at a time.
2. A chip can only move forward (not backward).
3. A chip can move into an empty square OR jump over a chip of the other color.
4. A chip may not jump over a same-colored chip.

Starting Board:

Y	Y	Y	Y	Y		R	R	R	R	R

Ending Board:

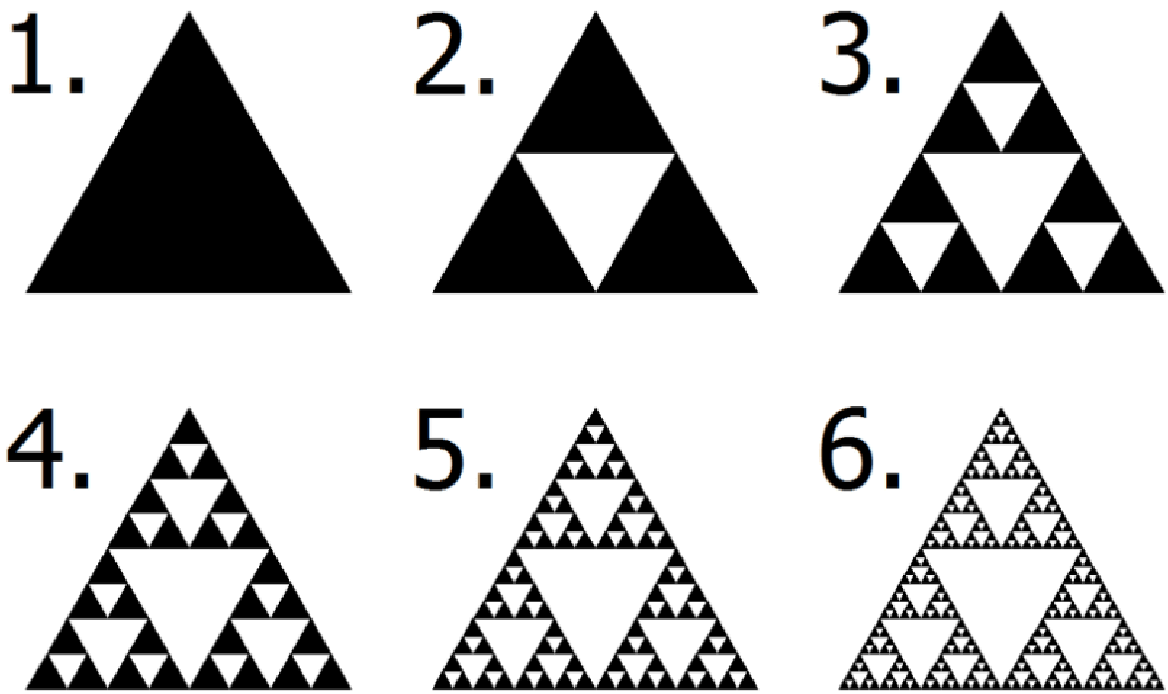
R	R	R	R	R		Y	Y	Y	Y	Y

Your Task: Describe mathematically what is happening with as much detail as you can. Include in your discussion the minimum amount of moves needed to win the game

Top

Triangles and More Triangles

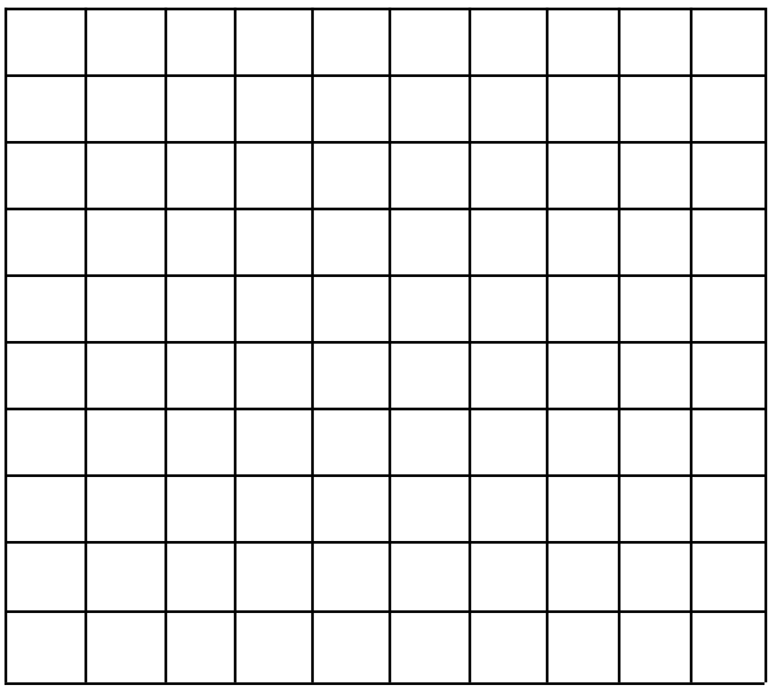
A beautiful pattern is shown below. Your job is to make as much sense out of it as possible. **What questions would you like to answer about this pattern?** Use your greatest mathematical mind to think about this pattern.



Top

Squares and Rectangles on a 10 by 10 Grid

Using the following grid discuss how you would find how many squares are on a 10 by 10 grid. What did you find out? Can you express a way of showing how many squares are on any sized grid?



After you have completed the first question now you should work to find how many rectangles are on a 10 by 10 grid.

Top

What is the unit's digit?

What is the digit in the units place of the sum of

$$1^1 + 2^2 + 3^3 + 4^4 + \ldots + 99^{99} + 100^{100}?$$

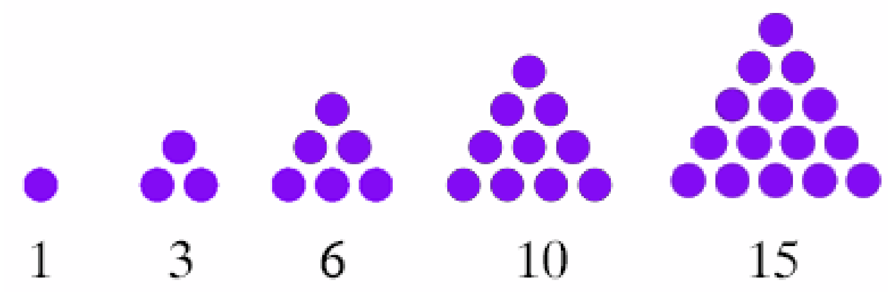
[Top](#)

Making Connections

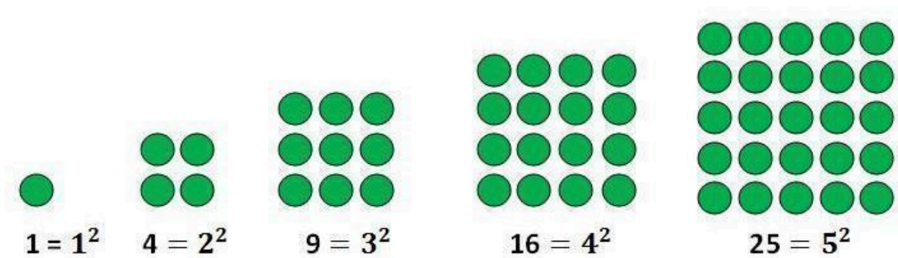
How are triangular numbers, square numbers, hexagonal numbers, rectangular numbers and cubic numbers related?

Mathematical Connections can be powerful, surprising, even beautiful. Some can be very straightforward and other connections can be hidden. What can you find?

Triangular Numbers



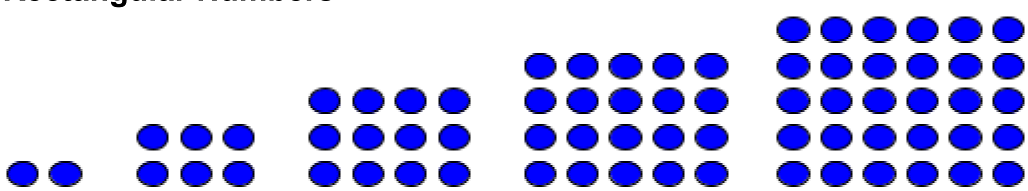
Square Numbers



Cubic Numbers

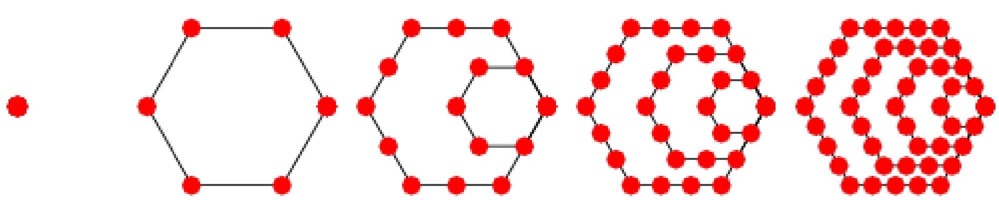
$1 = 1^3$ $8 = 2^3$ $27 = 3^3$ $64 = 4^3$ $125 = 5^3$

Rectangular Numbers



2 6 12 20 30

Hexagonal Numbers



1 6 15 28 45

Top

The String Problem

A string is stretched corner to corner on a floor tiled with square tiles. If the floor is 28 tiles long and 35 tiles wide, how many tiles does the string cover?

(Note: A string does not cover a tile if it only *touches* the vertex of that tile.)

Can you generalize for any rectangular room (x tiles long by y tiles wide) whose floor is covered with square tiles?

What new questions can you ask about this problem?

Top

The Magic of Math (with Extension)

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25

Take a look at the table…
Choose five numbers.. No number can be in the same row or column… Add the numbers up…Repeat a few times…What do you see? What is happening? Why is it happening?

What further investigations would you like to make? Discuss with your teacher/mentor.

Extension:
Rearrange the numbers on the grid so that the sums of each vertical set of numbers, each horizontal set of numbers and each diagonal set of numbers are equal.

Repeat the same process for a 4 by 4 grid. Analyze the results.

Top

The Magical Game (with Extension)

There are 9 cards numbered 1 to 9. The cards are visible to the players. The goal of the game is to pick three cards that add up to 15.

Rules…
Two players take turns picking one card at a time until all the cards are picked or a player has won the game. I

Questions to think about:
1) Does it matter who goes first? Why?
2) Can you figure out a way to win the game? What strategies would you use to make this happen?
3) Can you play where no one gets to 15?

Extension:

Using a 3 by 3 grid and numbers 1 to 9 place a number in each box so that the sums of each vertical, horizontal and diagonal set of numbers are equal.

How does this game relate to a 3 by 3 Square you created? How does that help to understand what is going on in the game?

What would happen with a 4 by 4 grid using the numbers 1 to 16? What would the game look like? What would the grid look like? What would the strategies look like to try and win the game now?

What would happen in an *n by n* Grid?

[Top](#)

An Array From 1 to 9

An array consists of consecutive positive numbers from 1 to 9 that repeat. The first row has one digit. The second row has two digits. The third row has four digits. Each row has double the number of digits as the previous one.

1

2, 3

4, 5, 6, 7

8, 9, 1, 2, 3, 4, 5, 6

What is the 2020^{th} number in the 2020^{th} row?

What is the sum of the numbers in the 2020^{th} row?

What is the nth number in the nth row?

What is the sum of the numbers in the nth row?

Top

An Array from 1 – 26

An array consists of the positive integers from 1 to 26. The first row has 1 digit, the second row has 2 digits, the third row has 3 digits, etc. After 26, the digits repeat again from 1 to 26.

The first four rows of an array are

1

2, 3

4, 5, 6

7, 8, 9, 10

What is the first number in the 2020th row?

What is the sum of all the numbers in the 2020th row?

What is the first number in the nth row?

What is the sum of all the numbers in the nth row?

Top

What's Happening?

Take any two digit number, reverse its digits, and subtract the smaller number from the larger. For example, $42-24=18$.

Try some examples of your own. What's happening? **What conjecture(s) would you like to make? Try to prove that your conjecture(s) is always true**.

What if you do the procedure with a 3 digit number? 4? 5? *N*? Does your conjecture hold up? **Try to prove your conjecture for all numbers.**

Top

Crosswalks

In the old city of Gnoop, there are seven crosswalks **(a-g**) that go across the river in various locations in the city.

1. Can a person plan a walk in the city so that they will cross each crosswalk exactly once? In other words, can a person plan a walk in the city to visit areas **A, B, C, and D** by crossing each crosswalk exactly once?
2. Suppose that crosswalk b collapsed. Can a person plan a walk so that they will start at D and cross each of the crosswalks exactly once? What if that person starts at B?
3. Suppose that the town planners want to build a new crosswalk to replace the one that collapsed. Instead of building it in the same location as before, they want to situate it in such a place where, after it is completed, it will be possible for a person starting anywhere in town to plan a walk that will cross each crosswalk exactly once and return to the starting point. Where should the new crosswalk be built?

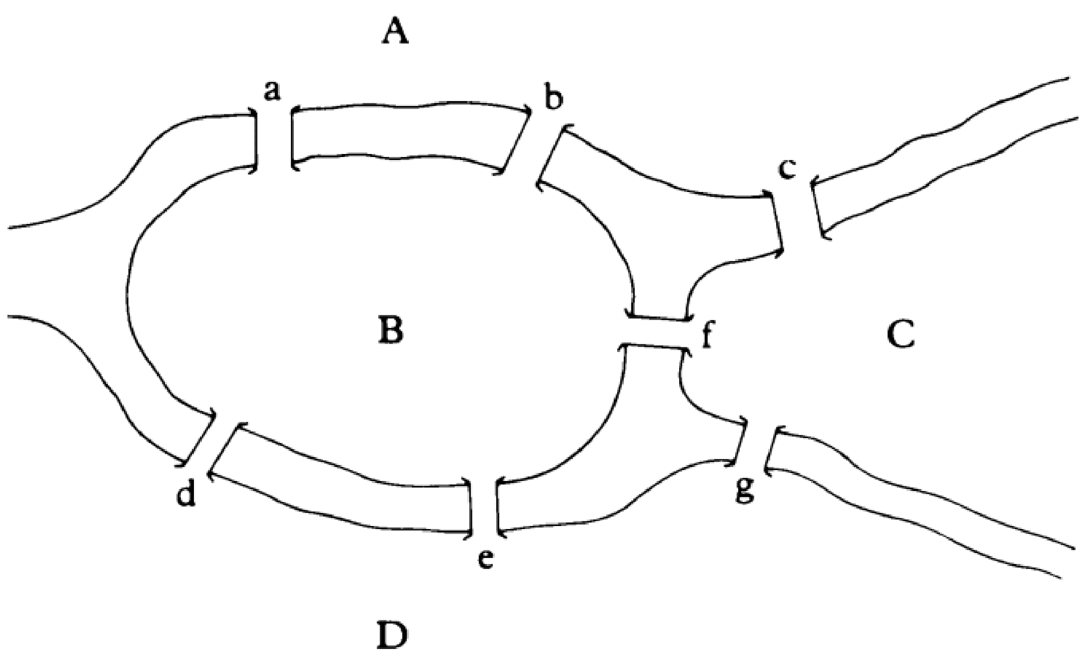


FIGURE 6.1

Note to Teacher:

Question 1 is the main question. Questions 2 and 3 are there to help support student thinking about Question 1. You might just give the problem with Question 1 and support students where needed with the other two questions.

Top

On and Off

To prevent tampering by unauthorized individuals, a row of switches at a defense installation is wired so that, unless the following rules are followed in manipulating the switches, an alarm will be activated:

1. The switch on the right may be turned on or off at will.
2. Any other switch may be turned on or off only if the switch to its immediate right is on and all other switches to its right are off.

What is the smallest number of moves in which such a row of switches, which are all on, may be turned off without activating the alarm if:

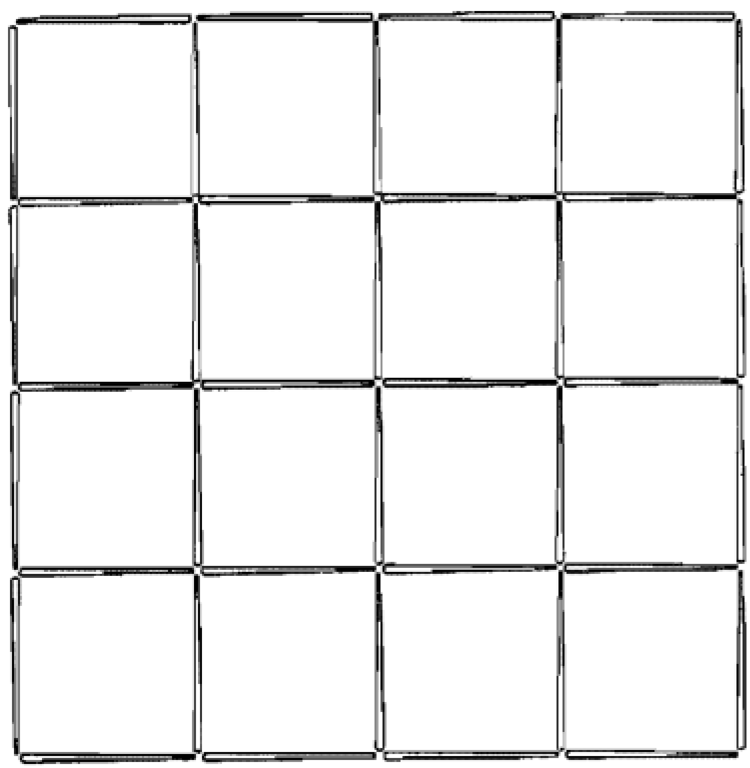
a. There are three switches in the row?
b. There are four switches in the row?
c. There are five switches in the row?
d. There are six switches in the row?
e. There are *n* switches in the row (*n* is odd)
f. If there are *n* switches in a row (*n* is even)

Top

Destroying Squares

Forty toothpicks are arranged as shown in the figure below to form the skeleton of an ***order-4*** checker-board. The problem is to remove the smallest number of toothpicks that will break the perimeter of every square. "Every square" means not just the 16 small ones. There are many other squares you need to eliminate. How many squares do you need to destroy? What is the minimum number of toothpicks you need to remove? Can you go a step further and state a simple proof that the answer is indeed minimum?

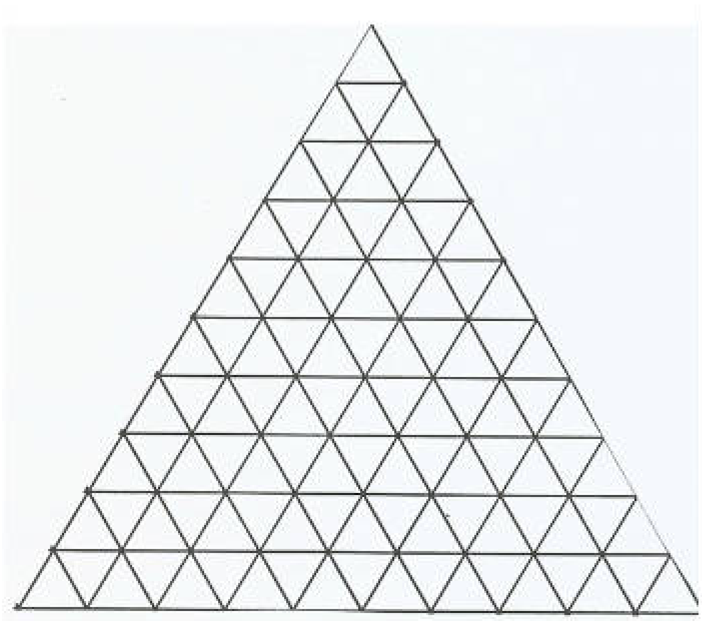
This far from exhausts the puzzle's depth. The obvious next step is to investigate square boards of other sizes. The mathematician is not likely to be content until they have a formula that gives the minimum number of toothpicks that need to be removed for any given size.



Top

Multitude of Triangles

1. How many triangles are there in the diagram below?



2. What new question(s) would you like to think about? You can talk to your teacher about the new question(s).

Top

Decreasing Numbers

A number is called a decreasing number if it has two or more digits and each digit is less than the digit to its left. For example 7421, 964310, and 53 are decreasing numbers. How many decreasing numbers are there?

Top

Painted Cubes

Jo made a cube from some smaller cubes. She painted some of the faces of the large cube and then took it apart again.

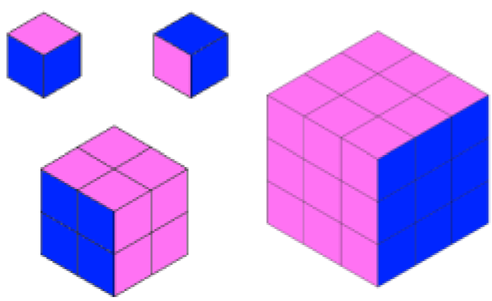
She counted her cubes and noticed that 45 cubes had no paint on them at all.

Can you determine how many small cubes Jo used to make her large cube, and which faces she painted?

Dan made a cube the same size as Jo's large cube, and also painted some of the faces.

How many unpainted cubes might Dan have ended up with?

Image



Now explore the number of unpainted cubes for some other sizes of cube. Here are some questions you might like to consider:

- If the number of small cubes along each edge is , can you find expressions for the number of unpainted cubes when you paint 1, 2, 3, 4... faces?
- The number of unpainted cubes can always be expressed as the product of three factors. What can you say about these factors?
- There is only one way to end up with 45 unpainted cubes. Are there any numbers of cubes you could end up with in more than one way?
- How can you convince yourself that it is impossible to end up with 50 unpainted cubes?

Top

Cross the Network

One of the oldest topological puzzles, familiar to many schoolchildren, consists of drawing a continuous line across the closed network shown in Figure 8.7 so that the line crosses each of the 16 segments of the network only once. The curved line shown here does not solve the puzzle because it leaves one segment uncrossed. No "trick" solutions are allowed, such as passing the line through a vertex or along one of the segments, folding the paper, and so on.

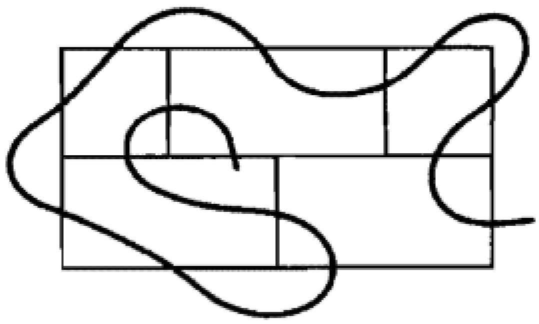


Figure 8.7

Can each line cross each of the 16 segments of the network only once?

Can it be done on a sphere?

Can it be done on a torus?

Top

Chip Stacking

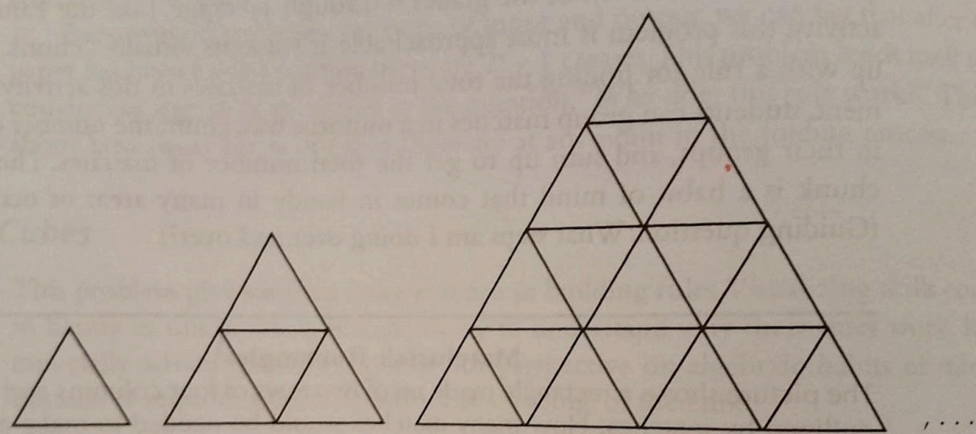
n chips are organized into some number of stacks, of arbitrary height, on a table. Every minute, one chip is removed from each stack and those collected are used to create a new stack. The stacks are then arranged in order from tallest to shortest.

What eventually happens? Why?

Top

Segments of Triangles

Consider this sequence of diagrams:



The first diagram is made up of three line segments of equal length. You can think of the second diagram as made up of nine segments of that same length, and so on.

1. Find the number of segments of the given length that would be needed for the tenth diagram in this sequence.
2. Explain how you would find the number of segments of the given length that would be needed for the 100th diagram in this sequence.
3. How would you find the number of segments of the given length for any diagram in this sequence?

Top

The Three Piles

The game of Three Piles begins with three piles containing 3, 5, and 7 sticks. The rules of play are as follows:

1. Each player takes a turn, taking as many sticks from any pile they wish.
2. You can only take sticks from one pile per turn, but you can take up to the remaining number of sticks in that pile.
3. The player who takes the last stick is the winner.

Is it best to go first or second? What is the winning strategy?

Top

Cops & Bookies

Each of the 22 vertices on the graph in Figure 6.60 represents a shop that serves as a front for a bookie (illegal gambling). The edges joining the vertices represent the only streets connecting these shops. The police have placed stakeouts outside each shop with orders to arrest anyone passing the same shop twice.

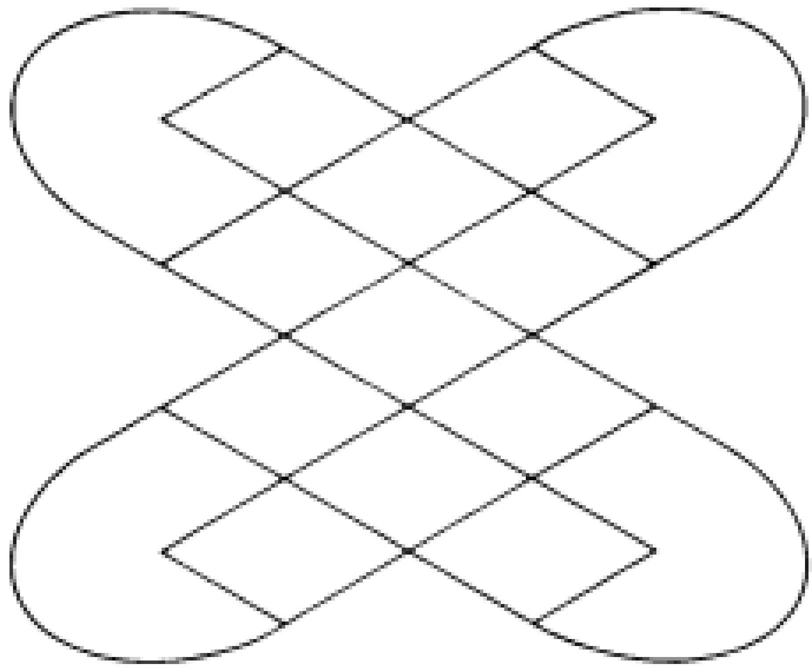


FIGURE 6.60

Can you make all the rounds of all the shops without being arrested?

Top

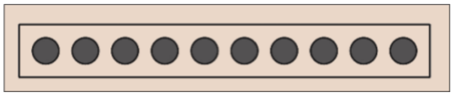
Even or Odd Sum

Problems in Algebra for Teachers by Alexander Karp & Julia Viro

Will the sum of 2,018 consecutive numbers be odd or even?

Top

One or Two



Materials:

- 10 pennies
- Game Template

Instructions:

- Decide a first and second player.
- On their first turn, the first player is allowed to take only one penny.
- Players then alternate turns and can take one penny or two adjacent pennies.
- The player who takes the last penny wins.

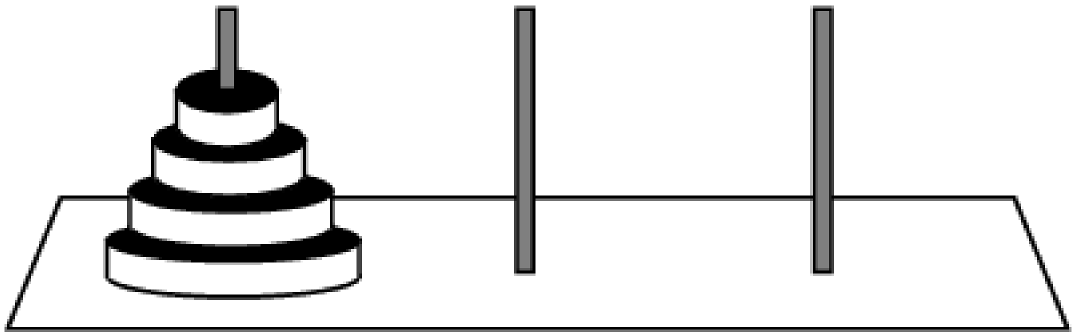
Can you come up with a strategy that will make you always win?

Section 2:

For Middle Students

Top

Moving the Stack



Given a stack of *n* disks arranged from largest on the bottom to smallest on top placed on a rod, together with two empty rods, what is the minimum number of moves required to move the stack from one rod to another, where moves are allowed only if they place smaller disks on top of larger disks?

Top

The Water Jug Problem

In the movie Die Hard 2 there is a climactic scene:
[Zeus Carver and John McClane (the good guys) run up to an open briefcase on a fountain left by
Simon Gruber (the bad guy).]

[Phone call transcript between John, Zeus, and Simon.]

Simon, "I trust you see the message. (The bomb) It has a proximity circuit, so please don't run."

John McClane [the good guy], "Yeah, I got it we're not going to run. How do you turn it (the bomb timer) off?"

Simon, "On the fountain there should be two jugs. You see them; a 5 gallon and a 3 gallon? Fill one of the jugs with exactly 4 gallons and place it on the scale and the timer will stop. You must be precise; one ounce more or less will result in detonation."

How will John and Zeus get exactly 4 gallons into one of the jugs?

Now that you have worked on the problem you now are going to be challenged to figure out **if this problem can work for any set of numbers or for only a particular set of numbers**. Use your best mathematical mind to try to solve this complex problem.

Top

The Magic of Math

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25

Take a look at the table…
Choose five numbers.. No number can be in the same row or column… Add the numbers up…Repeat a few times…What do you see? What is happening? Why is it happening?

What further investigations would you like to make? Discuss with your teacher/mentor.

Top

The Magical Game

There are 9 cards numbered 1 to 9. The cards are visible to the players. The goal of the game is to pick three cards that add up to 15.

Rules…
Two players take turns picking one card at a time until all the cards are picked or a player has won the game. I

Questions to think about:
1) Does it matter who goes first? Why?
2) Can you figure out a way to win the game? What strategies would you use to make this happen?
3) Can you play where no one gets to 15?

What further investigations would you like to make? Discuss with your teacher/mentor.

Top

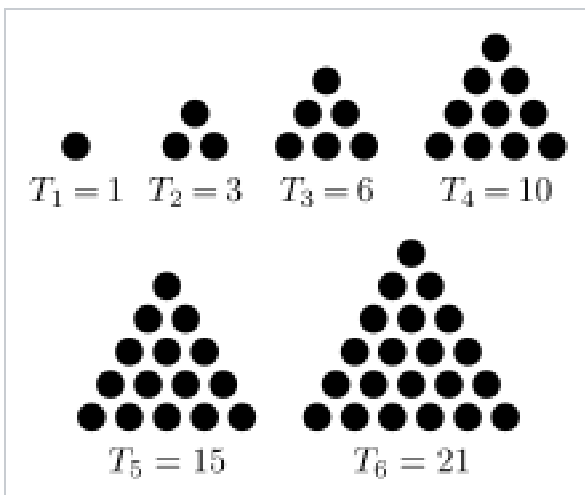
Counting with Shapes

Figurate numbers are **numbers** that can be represented by a regular geometrical arrangement or sequence of evenly spaced points. **Figurate numbers** are most commonly expressed in the form of regular triangles, squares, pentagons, hexagons, etc.

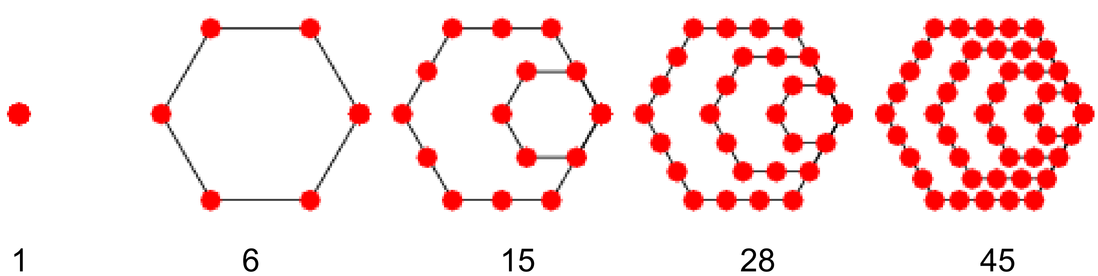
Your task is to explain/describe these two sequences as well you can using your highest level of mathematics.

Now think about these questions:

> **What is going on in these two sequences?**
> **How are these two sequences related?**
> **What new question would you like to investigate?**



The first six triangular numbers drawings



The first five hexagonal numbers and drawings

Top

Getting Ahead

Sue Flay opened a MacDonald’s on White Plains Road and Cassa Role opened a Burger King across the street. Both had to borrow money to open their fast food franchises.

After 500 customers, Sue was still $4000 in debt. By the time she had served 3000 customers, she was ahead by $1000.

After 2000 customers, Cassa Role still owed $6000 to the bank. However, after 4500 customers, she was ahead by $1500.

Create two questions that you can answer from the given information then explain in detail how you would go about answering the questions.

Top

Great Pyramid Problem

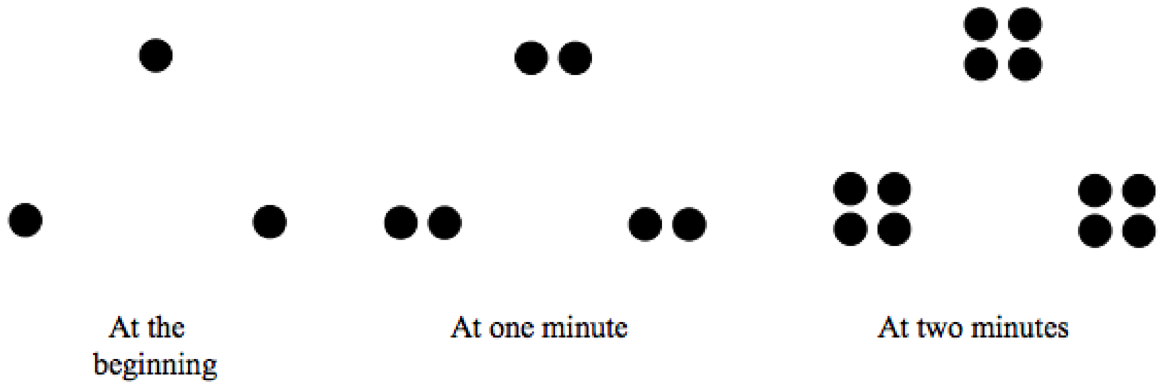
A very creative grocer was stacking oranges one day, She decided to stack them in a triangular pyramid. There was one orange in the top layer, three oranges in the second layer, six oranges in the third layer, and so on. Each layer except the top formed an equilateral triangle. The grocer kept building this for an hour and was proud of her final product.

You are going to use your best mathematical mind to create a set of questions that you would like to think about. As you create these questions you will meet with your teacher to decide if these are good questions that will challenge you. Enjoy thinking about this problem.

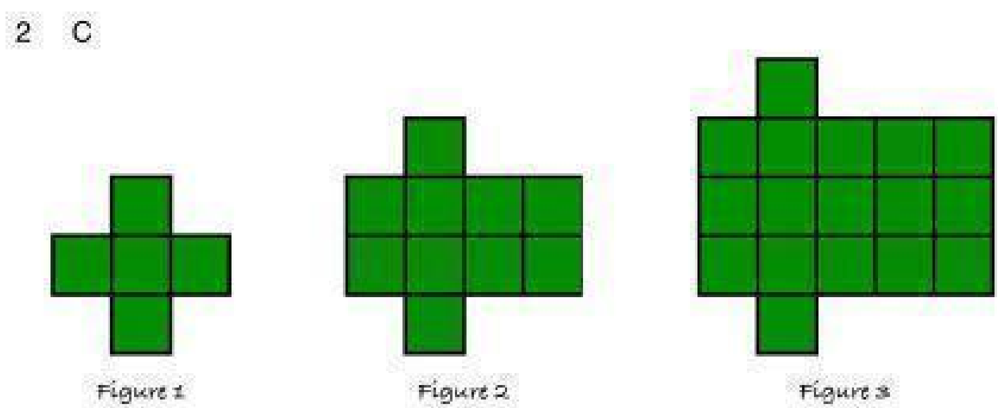
Top

Growing Dots and Growing Objects

Growing Dots



Growing Objects



Take a look at the two patterns…Think about what is happening in each one…Write about everything you can and have discovered.

What do you want to find out about these patterns? Use your best mind to create two challenging questions that you will then try to answer. Share your questions with your teacher before you move further in your deep investigation.

Top

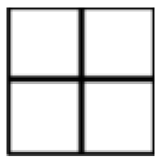
McNuggets

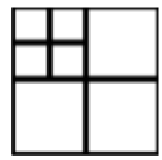
You are at a chicken restaurant and you want to buy chicken Mn\cNuggets. At this restaurant you can **only buy** McNuggets in boxes of 8 Mcnuggets, 10 Mcnuggets and 11 Mcnuggets. What is the greatest amount of McNuggets that can't be purchased at this restaurant? Prove that your answer is correct.

Top

Squares

Can you divide a square into any particular number of smaller squares? This may depend on exactly how many smaller squares you want. The first diagram below shows that any square can be divided into 4 smaller squares. The second diagram shows that any square can be divided into 7 smaller squares.





Notice that the smaller squares don't have to be the same size as each other, but keep in mind that the smaller portions must all be squares, not simply rectangles.

What numbers of smaller squares are possible? What numbers of smaller squares are impossible to create?

Top

The Game of 27

You are going to play a game with the following rules:

- You will be given 27 chips
- There are exactly two players
- The two players will alternate turns
- At each turn, a player removes 1, 2, 3, or 4 chips from the pile.
- The game ends when all the counters have been removed.
- The player who takes the last chip wins the game.

What's the strategy to ensure that a player can win every time he/she plays the game?

Explain the total experience (all the different ideas you tried) you had trying to figure out how to always win the game. Why will your strategy always work?

If you feel you have fully answered the first question then begin to think about this new question: What would be the strategy to always win the game if the player who takes the last chip loses the game?

[Top](#)

The Twelve Days of Christmas

You are going to read this famous poem and use your greatest mathematical mind to create a set of questions you can answer. As you create these questions you will meet with your teacher to decide if these are good questions that will challenge you. Enjoy thinking about this problem.

On the first day of Christmas, my true love gave to me
    A partridge in a pear tree
On the second day of Christmas, my true love gave to me
    Two turtle doves and a partridge in a pear tree
On the third day of Christmas, my true love gave to me
    Three French hens
    Two turtle doves and a partridge in a pear tree
On the fourth day of Christmas, my true love gave to me
    Four calling birds, three French hens
    Two turtle doves and a partridge in a pear tree
On the fifth day of Christmas, my true love gave to me
    Five golden rings, four calling birds, three French hens
    Two turtle doves and a partridge in a pear tree
On the sixth day of Christmas, my true love gave to me
    Six geese a-layin', five golden rings, four calling birds
    Three French hens, two turtle doves and a partridge in a pear tree
On the seventh day of Christmas, my true love gave to me
    Seven swans a-swimmin', six geese a-layin', five golden rings
    Four calling birds, three French hens, two turtle doves
    And a partridge in a pear tree
On the eighth day of Christmas, my true love gave to me
    Eight maids a-milkin', seven swans a-swimmin', Six geese a-layin'
    Five golden rings, four calling birds, three French hens
    Two turtle doves and a partridge in a pear tree
On the ninth day of Christmas, my true love gave to me
    Nine lords a-leapin', eight maids a-milkin', seven swans a-swimmin'
    Six geese a-layin', five golden rings, four calling birds, three French hens
    Two turtle doves and a partridge in a pear tree
On the tenth day of Christmas, my true love gave to me
    Ten ladies dancin', nine lords a-leapin', eight maids a-milkin'
    Seven swans a-swimmin', six geese a-layin', five golden rings
    Four calling birds, three French hens
    Two turtle doves and a partridge in a pear tree
On the eleventh day of Christmas, my true love gave to me

Eleven pipers pipin', ten ladies dancin', nine lords a-leapin'
Eight maids a-milkin', seven swans a-swimmin', Six geese a-layin'
Five golden rings, four calling birds, three French hens
Two turtle doves and a partridge in a pear tree
On the twelfth day of Christmas, my true love gave to me
Twelve drummers drummin', eleven pipers pipin', ten ladies dancin'
Nine lords a-leapin', eight maids milkin', seven swans a-swimmin'
Six geese a-layin' five golden rings, four calling birds, three French hens
Two turtle doves and a partridge in a pear tree

Extension:

After you have thought about the Christmas Poem we would like you to think about the Handshake Problem.

If 5 people meet and shake each other's hand exactly once how many handshakes would there be? What would happen if there were 10 people? Can you find a way to answer the question for any number of people?

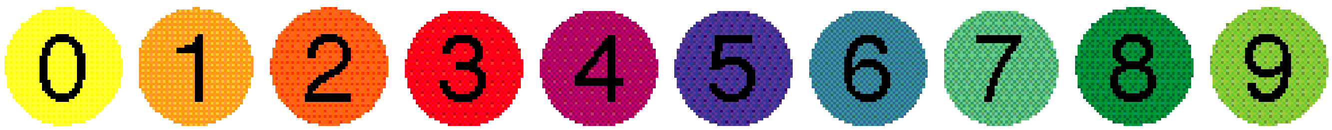
Think about this new problem and ask yourself how is it similar and how is it different from the Twelve Days of Christmas?

Top

The Number Bracelets

This is a game that lots of kids are playing (or so I've heard). You only need to be able to add whole numbers to play it, but there are interesting variations and extensions for people who like to think about mathematical patterns (6th graders, high school students, math majors, graduate students, ...)

Imagine that you have lots of beads, numbered from 0 through 9, as many as you want of each kind.

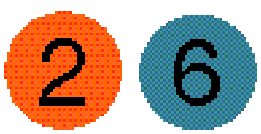


Here are the rules for making a number bracelet:

- Pick a first and a second bead. They can have the same number.
- To get the third bead, add the numbers on the first and second beads. If the sum is more than 9, just use the last (ones) digit of the sum.
  To get the next bead, add the numbers on the last two beads you used, and use only the ones digit. So to get the fourth bead, add the numbers on the second and third beads, and use the ones digit.
  Keep going until you get back to the first and second beads, in that order.
- How long (or short) a bracelet can you make?

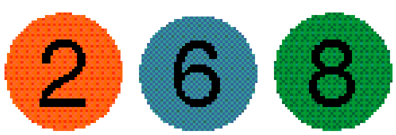
Example.

Choose 2 and 6 for the first and second beads:

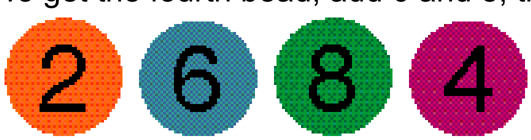


.

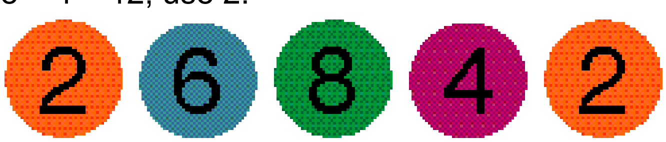
The third bead is 2 + 6 = 8 :



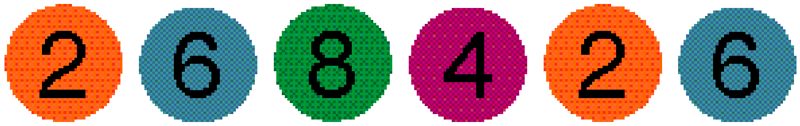
To get the fourth bead, add 6 and 8, then use only the ones digit: 6 + 8 = 14; use 4:



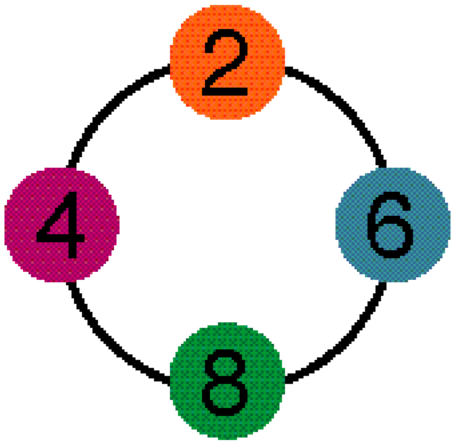
8 + 4 = 12; use 2:



4 + 2 = 6:



But the last two beads are the same as the first two, so instead of making a long string, use 2, 6, 8, and 4 in a loop, or bracelet.



Play with this game…
What questions would you like to pose and try to answer?
What discoveries have you made?

[Note to Teacher: Here are some good questions you might want to suggest to students if they struggle coming up with their own questions.

- How long (or short) a number bracelet can you make?
- Will a number bracelet always loop back to the beginning, or can you have a string of beads that never repeats?
- How many different starting pairs of beads are there?
- How many different number bracelets are there?
- If you start with the same two beads, but in the opposite order, do you get the same bracelet? Do you get the same bracelet in reverse?

Extensions:
1) Change the rule…The original number bracelets game used the Fibonacci sequence rule: add the last two numbers to get the next one. Try a different rule, such as adding twice the second number to get the next number, or add the three previous numbers to get the next number. Use your imagination.]

Top

What is the Remainder

1. What is the remainder when $3^{666,666}$ is divided by 7?

2. What ideas did you wonder about as you worked on this problem? Is there any new idea or question you would like to investigate? Go ahead and try. Talk to your teacher about this.

Top

Take Three from Five

Can you come up with a set of five whole numbers with this one rule: *No three of those numbers gives a sum that is a multiple of three*? You can use both consecutive and non-consecutive numbers.

If you can't find a set of five whole numbers where it's impossible to choose three of them that add up to a multiple of three, **prove that it's impossible**.

What new ideas and questions arise for you that you would like to investigate?Talk with your teacher about adding a new investigation.

Top

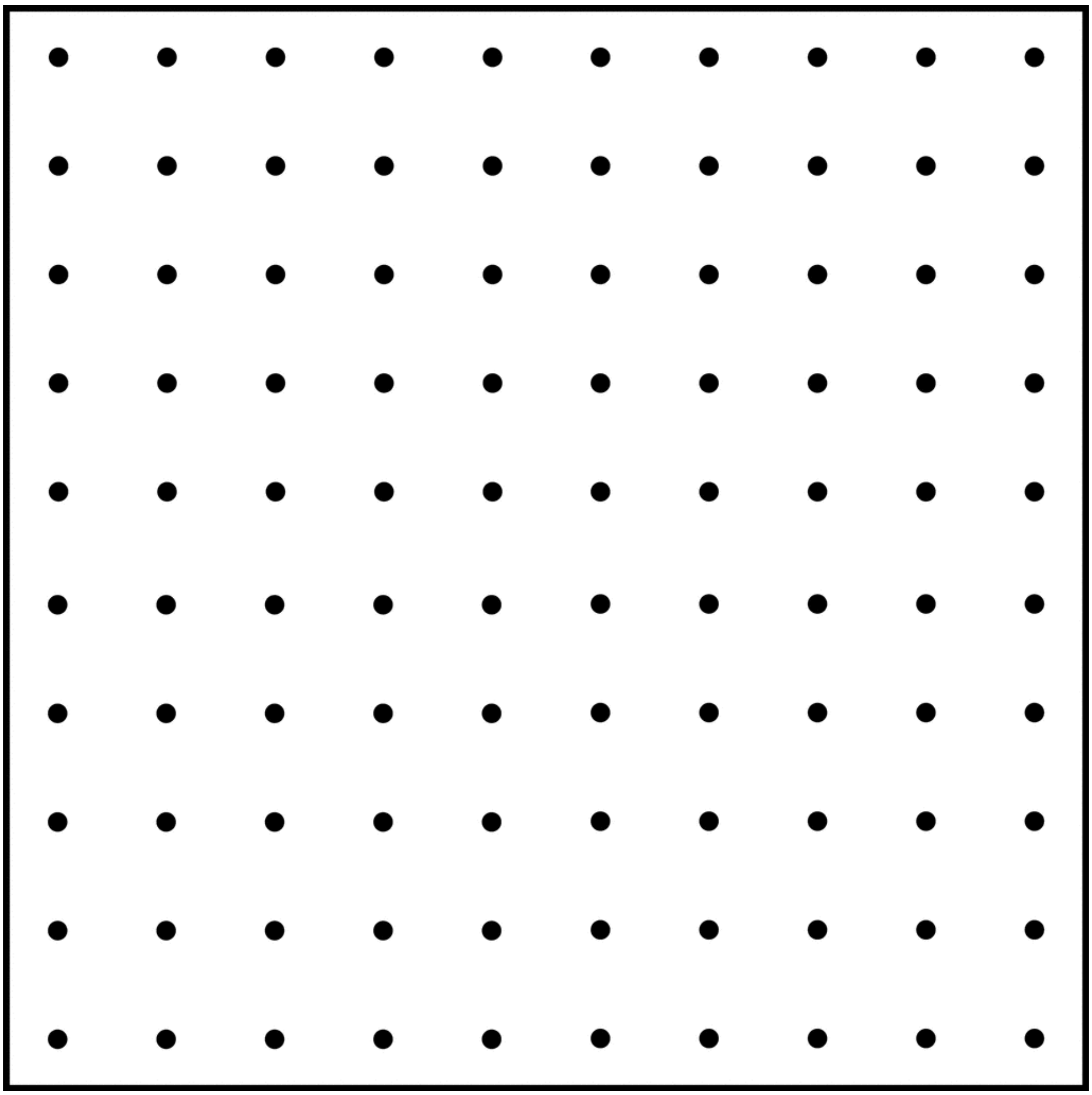
Geoboards

You are going to be given a 10 by 10 geoboard.

How many squares can you create on this geoboard?

What would happen if it was 15 by 15?

If it becomes bigger and bigger can you explain how you can find the number of squares for any *n* by *n* geoboard?



Top

GIO - A Student Creation

Gio comes up with his own process for mentally multiplying the numbers 23 and

52:

*I notice that 23 = 1 + 2 + 4 + 16. So, to multiply (1 + 2 + 4 + 16) * 52, I can use these steps:*

1 52

2 104

4 208

~~8 416~~

16 832

In your own words, explain Gio's process. What steps did he use, and why did they work? What should be his last step to figure out the final product of 23 and 52?

What are the conditions under which Gio's process will work? (i.e. Will it work for any two numbers? If not, what must be true about the numbers?) Prove your answer.

Top

A Mathematician's Favorite Problem

A very famous mathematician liked the problem that is posed below. Play with it…Think about it… Ask questions…Go deeply into it.

What is the sum of the first 100 cubic numbers?

Top

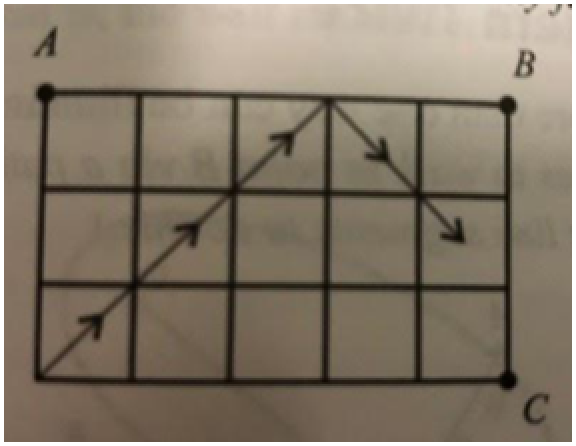
Zeros

How many digits of zero does **5555!** end with?

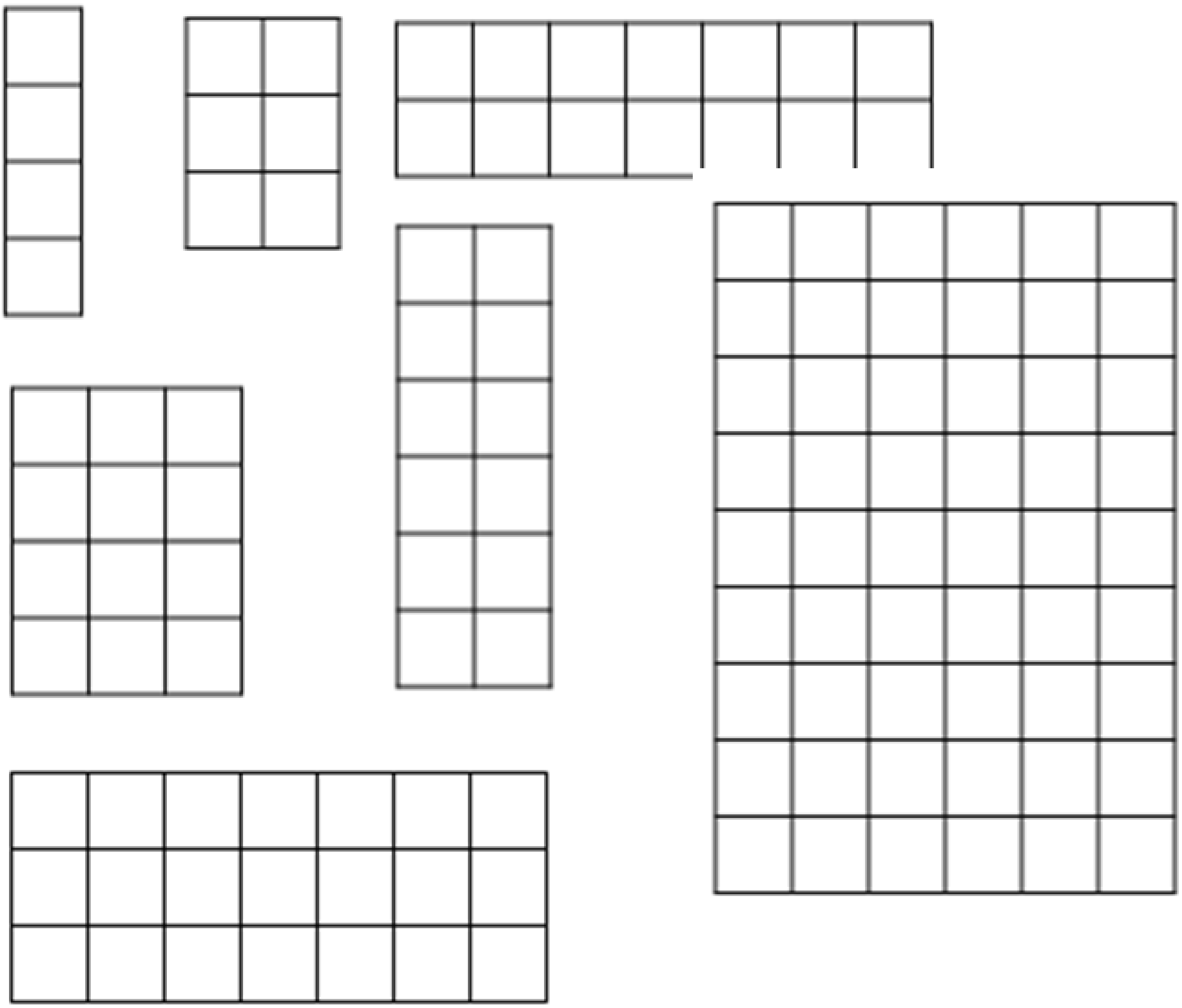
Top

On Reflection

Part 1:
A ball is shot from the bottom left corner of a 3’ x 5’ pool (billiards) table at a 45-degree angle. The ball traverses the diagonals of individual squares drawn on the table, bouncing off the sides of the table at equal angles. Into which pocket--A, B, or C--will it eventually fall?



Part 2:
Experiment with the tables below. What do you notice about those tables that have the ball fall into the top-left pocket A? Into the top-right pocket B? Into the bottom-right pocket C? Test your hypothesis with tables of different sizes.



Must every ball land in pockets A, B, or C? Is it possible for a ball to return to where it started? Is it possible for a ball to enter an infinite loop and never fall into a pocket?

Top

Unknown Dumbbells

Frank is not known for making good decisions, even though he is really smart. He works as a cleaner at the local gym. He noticed that FIVE dumbbells were looking a bit dirty, and being a really hard worker, he decided to clean them. Unfortunately he cleaned the labels right off, so no one knew how much each one weighed.

The manager yelled at Frank, “Frank, you DUMBBELL!!! How can our customers know how much each one of these things weighs???? You better fix this problem or you are FIRED!”

Frank felt bad about creating the problem, so he decided to fix it, but he also didn’t like being yelled at or being called names, so he made things a bit tricky for the manager.

He told the manager: “I decided to weigh the dumbbells in pairs. I could tell which was the lightest, 2nd lightest, etc. I labeled them so that Dumbbell A was the lightest, Dumbbell B was the 2nd lightest, C is the middle, D is the 2nd heaviest and E is the heaviest.

Then I weighed them in pairs. I weighed A and B together, A and C together, D and E together, etc. Here are all the totals:

80 kg, 82 kg, 83 kg, 84 kg, 85 kg, 86 kg, 87 kg, 88 kg, 90 kg, 91 kg.

There you go! Problem solved! If you can’t figure it out, then I guess I am not the dumbbell, because I know how much each one weighs. Oh, and I QUIT!”

- How much does each dumbbell weigh?
- Change the weight of the original pairings. When is it possible to figure out how much each dumbbell weighs?
- Extension: Using just that information, Smartypants Frank was able to figure out the answer in less than 5 minutes. How did he do it?

Top

I Take, You Take

Imagine 20 numbers, from 1-20, laid out on a table. The game is played between you and your opponent. The rules are as follows:

1. Every time you take a number, your opponent takes all its remaining factors. For example, if you take 8, your opponent takes 1, 2, and 4.

2. You are only allowed a move in which your opponent gets at least one number. For example, if you try to take 16, you cannot because all its factors have been taken already.

3. When you cannot take any more numbers, your opponent gets the rest.

4. The winner is the person with the highest sum at the end of the game.

What is the highest score you can get? Show that the score is maximal.

What is the highest score you can get if you play with numbers up to 24? 100?

Top

1001 Coins

There are 1001 pennies lined up on a table.

Starting at one end of the line, I replace every second coin with a nickel. Then, I return to the beginning and replace every third coin with a dime. Finally, I return to the beginning and replace every fourth coin with a quarter.

How much money is now on the table?

Top

Weight Problem

You have a 1kg weight, a 2kg weight, a 4kg weight, an 8kg weight and a 16kg weight to use on a 2-pan balance.

- Can you weigh something that has a mass of 15kg? How?
- Which masses can you weigh with these tools?
- Which masses can you NOT weigh with these tools?

Change the original weights, but keep a similar pattern.

What happens?

Top

Decimal Expansion

Decimal expansion can be interesting and wild.

Here are a set of questions I saw written:

1. Given 1/n is a terminating decimal, what must be true about n?

2. Given a terminating decimal 1/*n*, explain how you could determine the length of the decimal without carrying out the long division. (i.e., using only information about the number *n*.)

3. If 1/*n* is a terminating decimal, what could you say about 2/*n*? About *m*/*n*, if *m* is a counting number less than *n*?

4. What about non-terminating decimals? What patterns do you see with the repeating decimals and the non-repeating decimals?

5. And what about “eventually-repeating” decimals?

What do you want to find out about terminating decimals?

Discuss with your teacher. Set your plan. Have fun. Let us know what you have discovered.

Section 3:

For Students who struggle in math / who have had an interrupted or limited math education

Top

Jose's Rabbit

Jose is training his pet white rabbit, Blanco, to climb a flight of ten steps. Blanco can hop up 1 or 2 steps each time he hops. He never hops down, only up. How many different ways can Blanco hop up the flight of 10 steps? Can you generalize this for any number of steps?

Top

Cops and Robbers

Robin Banks robs a bank and drives off. A short time later he passes a truck stop at which police officer Willie Katchup is dining. Willie receives a call from his dispatcher and takes off in pursuit of Robin. Two minutes after he passes the truck stop, Robin is 1.5 km away.

Willie takes off and six minutes after Robin has passed, he is 2 km from the truck stop. Seven minutes after Robin passed, Willie is 4 km from the truck stop.

What question(s) would you like to think about and answer?

Top

How to Meet New Friends and Triangular Numbers

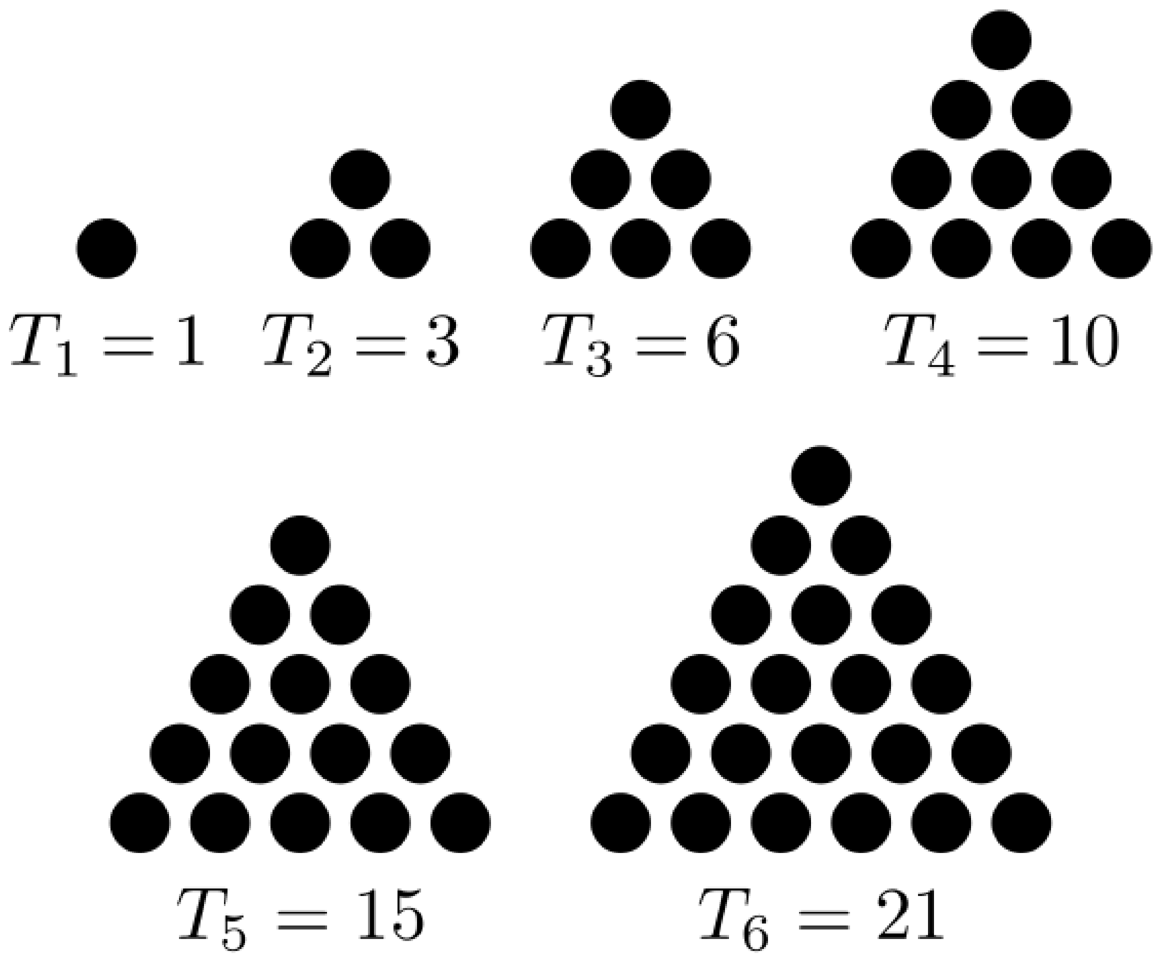
If 5 people meet and shake each other's hand exactly once how many handshakes would there be?

Share all the thoughts you had in trying to find the solution.

> What would happen if there were 10 people?
> Can you find a way to answer the question for any number of people?

Now you are going to compare the first problem to Triangular Numbers.

Look at the pattern below….



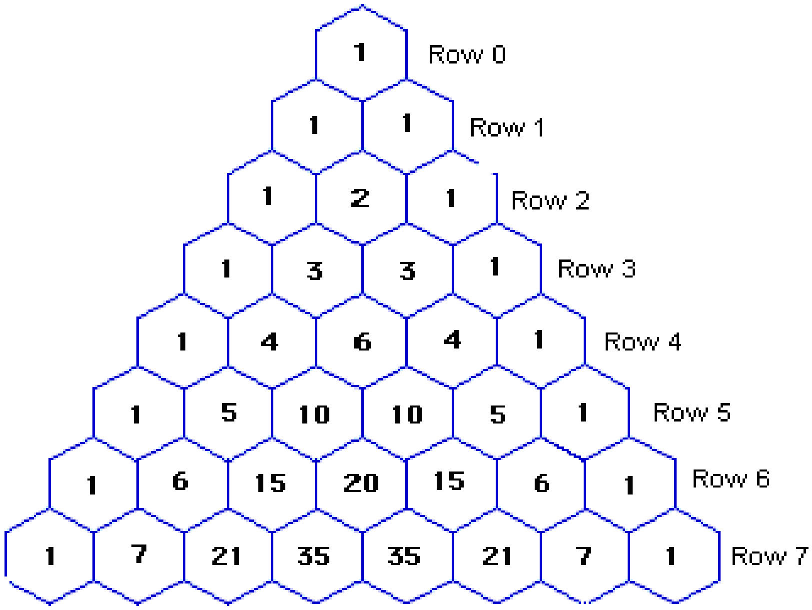
Compare and contrast the two patterns. Create two questions you would like to answer that shows the similarities and differences between these two problems.

Top

Indiana Jones and the Mystery of the Pyramid's Treasure

You and your friend Malik have been selected as game testers to try out a new video game for the Nintendo Wii. Based on the new Indiana Jones movie, the game requires you to crack codes to open secret chambers in a pyramid to find ancient treasure.

- You need to crack a code to open each chamber.
- Each code is based on different patterns.



LEVEL 1

Over time, many layers of the pyramid have worn away, and you cannot see the numbers written on them. To crack the first code, you need to find the <u>missing numbers for Row 8</u> of the pyramid.

Malik is having trouble beating Level 1. <u>Explain for him how you found EACH of these numbers using words and diagrams.</u>

LEVEL 2

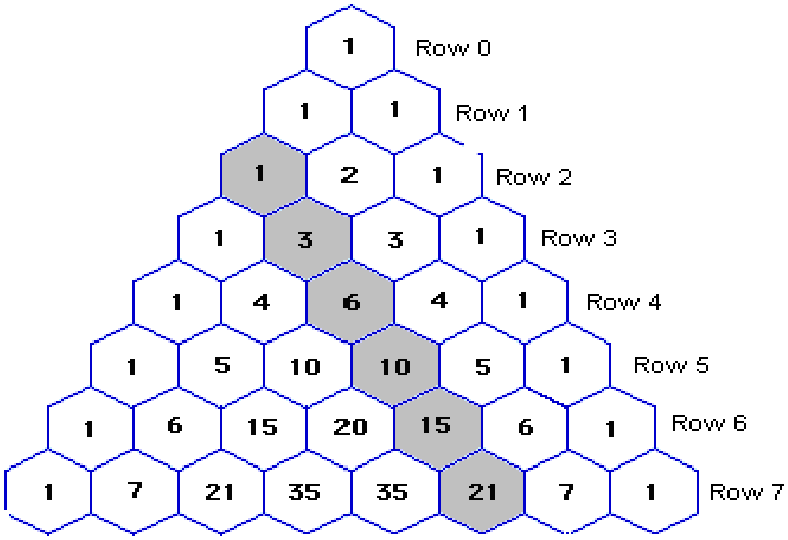
There are many levels of the pyramid buried deep underground. They have numbers, too, but there's no way you could dig down to see them.

Add up the numbers in each row, looking for patterns. To crack the second code, you need to <u>get the **sum** of the numbers written on Row 30</u>.

Malik is having trouble again, and he asks for help. <u>Explain to him how you found your answer using words and diagrams</u>.

BONUS code: Most patterns can be written using a mathematical rule or formula. To open up a secret level, <u>write a rule/formula for the sum of Row N</u>, where N can be any number.

LEVEL 3



At the start of Level 3, one diagonal of the pyramid begins to glow.
To open the third chamber, you need to <u>find the number that would be written in Row 20 of this glowing diagonal</u>.

Malik is having trouble again. Help him out! <u>Explain to him how you found your answer using words and diagrams</u>.

LEVEL 4
<u>Look at Row 0 – Row 4.</u>
Think about the digits forming a number.
    In Row 1 you should read 1 1 as eleven.
    In Row 2 you should read 1 2 1 as one hundred twenty one.
Do you notice something unique about those numbers? Write down your discovery.
What should Row 5 look like? Why does it look different?

LEVEL 5
<u>Look at Rows 5 and 7 and compare them to Rows 4 and 6</u>
How do they differ in relation to factors and multiples?
What is the next Row that will be similar to Row 5 and 7? Show your evidence by writing the full rows.
Why do you think these rows have this similar pattern?
What do you think would be the next rows that would follow that pattern?

Top

Odds and Evens

What is the sum of the first 100 odd counting numbers? How can you answer without adding up all the numbers?

How can you find the sum of the first 100 even counting numbers without adding them up? What is the sum?

How can you find the sum of the first 100 counting numbers without adding the numbers up? What is the sum?

Generalize the sum of *n* odd counting numbers? *n* even counting numbers? The first *n counting numbers?* What is the connection between the three generalizations?

Top

The Race

Sitting in his math class, Abranny is trying to figure out how he is going to get an A for the year, especially since he has missed handing in so much of the homework. Then, he has a brilliant idea.

Abranny's teacher is always bragging about what a great runner she was in college and how many trophies she won. So Abranny says to his teacher, "I have a cousin who is a pretty good runner. Would you like to race him? He's a little bit younger than you so I bet he would even give you a head start." Abranny dares his teacher and says, "If my cousin beats you, you give me an A in math for the year. If you win, I'll make up all my homework and do any extra work that you give me." Abranny's teacher thinks for a minute, "After I win, I'll be able to get Abranny to catch up on his work." Then she says, "Okay. I'll do it. Who's your cousin?"

Abranny: "Usain Bolt."
Teacher: "*Usain Bolt*! You mean the Jamaican guy who made world records for the 100 and 200 meter races in the 2008 and 2012 Olympics?"
Abranny: "That's rrriiigghhttt. But you can't back out now."
Teacher: "Okay, I'll do it. But I better get a good head start."
Abranny: "Okay. I'll lay out the race for you and I'll make sure you have a good headstart."

The Day of the Race
When the starting buzzer sounds, Abranny's teacher springs from the starting line and tears down the course. Usain takes off some time later.

NOTE: ALL TIMES ARE GIVEN SINCE THE STARTING BUZZER SOUNDED.

The Teacher:
15 seconds after the buzzer, the teacher still has 135 meters to go to the finish line. 32 seconds after he left the starting line, the teacher has only 50 meters more to run.

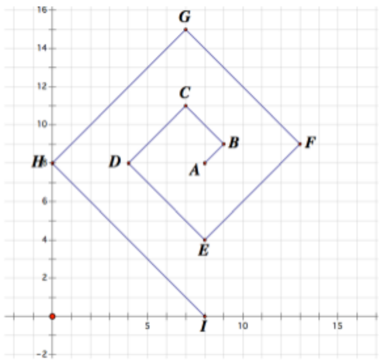
Usain Bolt (Usain starts running at some point after the teacher starts)
22 seconds after the buzzer, Usain is 189 meters from the finish line. 36 seconds after the buzzer, he is 42 meters from the finish line.

BOTH PEOPLE ARE RUNNING AT A CONSTANT SPEED.

Your task is to determine the outcome of the race. You need to explain the process (What did you do and why did you do it?) you use and show all the mathematics. Once you have completed your work discuss the following: What other questions can you now answer based on your work?

Top

The Spiral Alphabet



Based on the pattern shown above, find the coordinates of the following letters:

J: ________

N: ________

R: ________

Z: ________

Is there a shortcut to figuring out these coordinates? If so, how?

What new questions would you like to ask and try to answer?

Top

Powers of 2

Look at these numbers:

1, 2, 4, 8,16, 32, 64,......

1. What makes these numbers unique?
2. Why are they called powers of 2?
3. Think about the factors of each of the numbers. What do you notice? How does it compare to even numbers that are not powers of 2 or odd numbers? How about powers of 3? powers of 6?

Now you are going to be given two problems based on the powers of 2.
As you work on these problems, think about the uniqueness of these numbers and the meaning of patterns in mathematics.

a) What is the unit digit in the expansion of 2^{1000}?

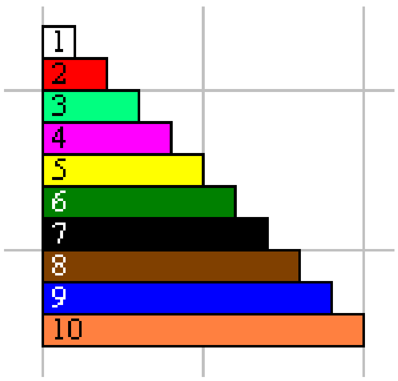
b) What is the remainder when 2^{1000} is divided by 7?

Top

Trains

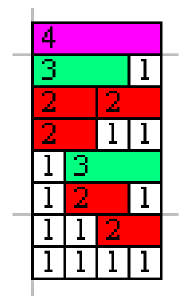
Cuisenaire rods are a math manipulative that can be used for learning many different types of math: fractions, percents, ratios, integers, etc. For this problem, you will use the various rod lengths to make Cuisenaire rod trains.

Cuisenaire rods come in whole-number lengths of 1 cm up to 10 cm.



Below is a 4 cm rod. We'll call it a 4-unit *train*.

There are 8 different ways to construct a 4-unit train. Here are the 8 ways:



How many different ways can a 10-unit train be constructed?

How many different ways can a *n*-unit train be constructed?

Top

The Three Problems

Question 1: If 5 people meet and shake hands exactly once, how many handshakes would there be?

Share all the thoughts you had in trying to find the solution.

What would happen if there were 10 people?

Can you find a way to answer the question for any number of people?

Now, you are going to compare the first problem with two other problems.

Question 2: How many three-digit numbers are "increasing numbers"?

Increasing numbers are numbers whose three digits are in increasing order. For example,

Are all increasing numbers. There are many more.

Examples of numbers that are not “increasing numbers” are 255, 444, or 351.

Question 3: What is the sum of the first 100 counting numbers?

Now that you have worked on these three problems, discuss how they are connected to each other. How does each problem help you to understand the other two problems with greater depth?