The most efficient setting for 3-5 gears

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Abstract. In this paper, for 3-5 gears, the most efficient setting is to be demonstrated and proven, including the most efficient deployment of gears, and the most efficient speed of the 'starting gear' in a machine, which is defined as the gear whose movement can cause the 'igniting gear' to move, which is defined as the gear whose movement causes the machine to operate, so that it causes the machine to operate, by applying Conservation of momentum, some geometric principles and gradient. The 'efficacy' is defined as the total momentum obtained by the gear in a machine that ignites the operation of it, contributed by the momentum of the gears engaging it and the momentum of the air around it that strikes it, according to Newton's second and third laws of motion, which can be calculated by applying the formula that describe the transfer of the momentum of a solid to another and the one of the momentum of a fluid to a solid. After calculating the total momentum of the 'igniting gear' that can obtain of every deployment, by the comparison of the total momentum of every deployment of 3-5 gears, under what circumstances which deployments are the optimal ones of 3-5 gears, if there is any, is known by applying the definite integrals and scaling the inequalities. And it is known how to judge which group of gears and machine is better in 'working efficacy' with the same deployment by a law named as 'The Zhang's Law' and found by the gradients, Lagrange Multiplier Method, functionalization and vectorization. Some examples that can preach how to construct the most efficient setting are also to be given in this paper.

Keywords: deployment, Conservation of momentum, gear, Newton's law of motion, definite integral, inequality scaling, gradient, Lagrange Multiplier Method, functionalization, vectorization.

1. Introduction

Not only do the qualities of the gears in a machine contribute to the working efficacy of it, but also the deployment of them does, eliciting the subject of the paper, aka the most efficient deployment of n gears. After having the idea of how to deploy the gears in a machine in the most efficient way, we can reduce the number of the gear of a machine needs.

As is known to all, there is the conservation of momentum. When the gear, let's say, the 'starting gear' in a machine invokes the movements of the other gears after it starts to move, the other gears engaging it will obtain the total momentum that is contributed by the momentum of the starting gear and the air striking them, according to Newton's second and third laws of motion.[1]

In collisions' physics, in particular imperfect elastic collision, the transfer of momentum between a solid body to another and between two solid bodies, can be described with the formula:[2]

$$p_{1i} + p_{2i} = p_{1f} + p_{2f}$$

with:

$$p_{1f} = m_1[(\frac{m_1-m_2}{m_1+m_2}v_{1i}) + (\frac{2m_2}{m_1+m_2}v_{2i})]$$

$$p_{2f} = m_2[(\frac{2m_1}{m_1+m_2}v_{1i}) + (\frac{m_2-m_1}{m_1+m_2}v_{2i})]$$

By dividing both of the hands of two expressions by m_1 and m_2 respectively, we get:

$$v_{1f} = (\frac{m_1-m_2}{m_1+m_2}v_{1i}) + (\frac{2m_2}{m_1+m_2}v_{2i})$$

$$v_{2f} = (\frac{2m_1}{m_1+m_2}v_{1i}) + (\frac{m_2-m_1}{m_1+m_2}v_{2i})$$

With such formula, we can calculate the velocities of two objects after the momentum of them is changed because the gear that engages one of them, viz, the engaged gear moves.

It's obvious that to make the 'efficacy' as high as possible, we need to make the total momentum transferred to the igniting gear as large as possible.

To increase it, we are supposed to make the gears engaging the igniting gear as many as possible and the volume of the air around it as small as possible since the velocity of flow and the tempature of it are constant, given space with a certain width and height, by applying some geometric principles, by the most efficient deployment that is to be demonstrated in this paper.

However, not only does the deployment of gears contribute to the working efficacy of a machine, but also the speed of the 'starting gear' does so as well.

As is known to all, in vector calculus, the gradient of a scalar-valued differentiable function f of several variables is the vector field (or vector-valued function) ∇f whose value at a point p gives the direction and the rate of fastest increase.[3] Further, a point where the gradient is the zero vector is known as a stationary point. And a stationary point of a differentiable function of one variable is a point on the graph of the function where the function's derivative is zero.[4][5][6] Informally, it is a point where the function "stops" increasing or decreasing (hence the name). Determining the position and nature of stationary points aids in curve sketching of differentiable functions. Solving the equation f'(x) = 0 returns the x-coordinates of all stationary points; the y-coordinates are trivially the function values at those x-coordinates. The specific nature of a stationary point at x can in some cases be determined by examining the second derivative:

If f″(x) < 0, the stationary point at x is concave down; a maximal extremum.

If f″(x) > 0, the stationary point at x is concave up; a minimal extremum.

If f″(x) = 0, the nature of the stationary point must be determined by way of other means, often by noting a sign change around that point.

Thus, we have to calculate the second derivative of the momentum transferrred from other gears and the one of the momentum transferred from the air to judge whether the specific natures of the stationary points are a maximal extremum or a minimal extremum or not able to be determined by the second derivatives.

Since $\{v_g = \frac{m_{air} - m_g}{m_{air} + m_g} v_{air}\ v_{gear} = \frac{2m_g}{m_g + m_{air}} v_g$, we can figure out what values are supposed to be put to m_{air}, m_g and v_{air} by calculating the gradient and then the stationary point of the gradient to maximize the working efficacy.

The paper is to start with figuring out the most efficient deployment by the definite integrals and scaling the inequalities, and figure out the values that are put so that they maximize the working efficacy by the gradients, Lagrange Multiplier Method, functionalization and vectorization.

2. Principle

According to Newton's third law of motion, if two bodies exert forces on each other, these forces have the same magnitude but opposite directions.[7] Therefore, when the teeth of a gear strike the air which exerts a force on the air, the air also strikes the teeth, which exerts a force on them and gives momentum to the gear because according to Newton's second law of motion, the net force on a body is equal to the body's acceleration multiplied by its mass or, equivalently, the rate at which the body's momentum changes with time,[8] therefore, the air gives momentum to the gear, and the momentum given by the air is equal to $m_g(\frac{m_{air}-m_g}{m_{air}+m_g}v_{air})$, which is changed from the formula $p_{1f} = m_1(\frac{m_1-m_2}{m_1+m_2}v_{1i})$, so that the speed achieved from the air is equal to $\frac{m_{air}-m_g}{m_{air}+m_g}v_{air}$.

And since the momentum of one gear achieved from another is equal to $m_g(\frac{2m_g}{m_g+m_{air}}v_g)$, which is changed from the formula $p_{2f} = m_2(\frac{2m_1}{m_1+m_2}v_{1i})$, so that the speed achieved from the air is equal to $v_{gear} = \frac{2m_g}{m_g+m_{air}}v_g$, after the 'starting' gear starts to move, the total speed of the other gears is $\frac{2m_g}{m_g+m_{air}}v_g$ - $\frac{m_{air}-m_g}{m_{air}+m_g}v_{air}$, since the direction of the force a gear gives to the air is opposite to the one of the force the air gives to a gear as per Newton's third law of motion, so that the total momentum of a gear is equal to the momentum transferred from other gears minus the momentum transferred from the air.

3. Deployment

Suppose the width of space is m and the height is n, the radii of the gears are r, and the lengths of the teeth of them are x and the widths are y, so that since the area of a gear can be approximately equated to the sum of the area of a circle and the total area of an even number of rectangles, the area of one gear is nearly equal to $\pi r^2 + kxy,\ \frac{k}{2}\epsilon Z^+$.

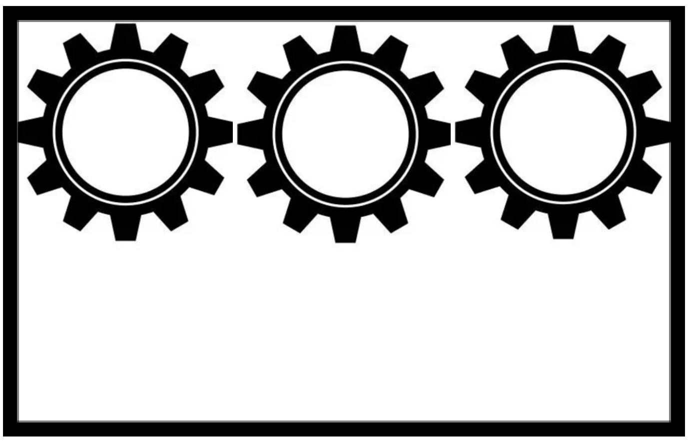
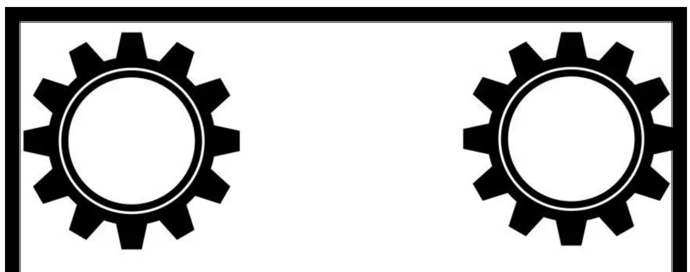


Fig 1

The most efficient deployment is non-existent



for only one gear, because no matter how you deploy it in a machine, it can't ignite the operation of the machine, and it's the same with 2 gears, so, let's start with 3 gears.

There are only two feasible deployments that can ignite the operation of a machine.

As can be seen, one of the only feasible deployments is just to deploy all of the gears in a line, as demonstrated by the figure 1

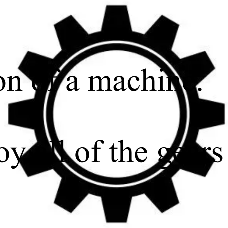


Fig 2

The other deployment that all of the gears are deployed in such a way that if we connect the centers of every two gears, there will be a triangle whose vertexes are the centers of the gears, as demonstrated by the figure 2.

For 4 gears, there are three feasible deployments.

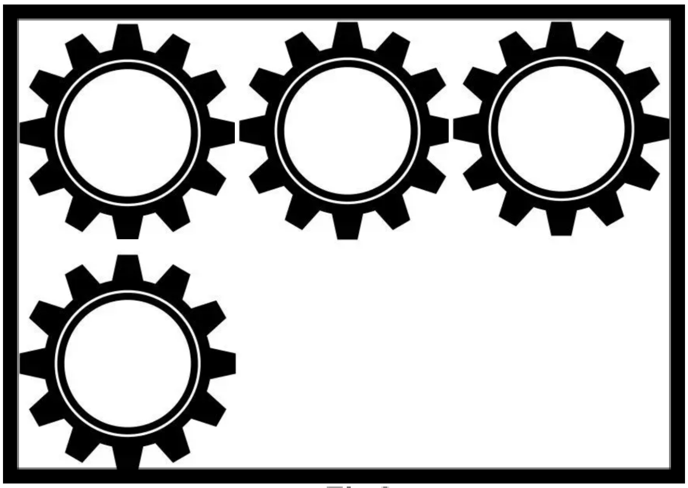


Fig 3

One of the deployments is that three of the gears are deployed in a line, and the rest one is placed on the vertical line that crosses the center of the gear in the top left corner, as demonstrated by the figure 3.

The second deployment is that three of the gears are deployed in a line, and the rest one is placed on the vertical line that crosses the center of the gear in the top middle, as demonstrated by the figure 4.

The second deployment is that three of the gears are deployed in a line, and the rest one is placed on the vertical line that crosses the center of the gear in the top middle, as demonstrated by the figure 4.

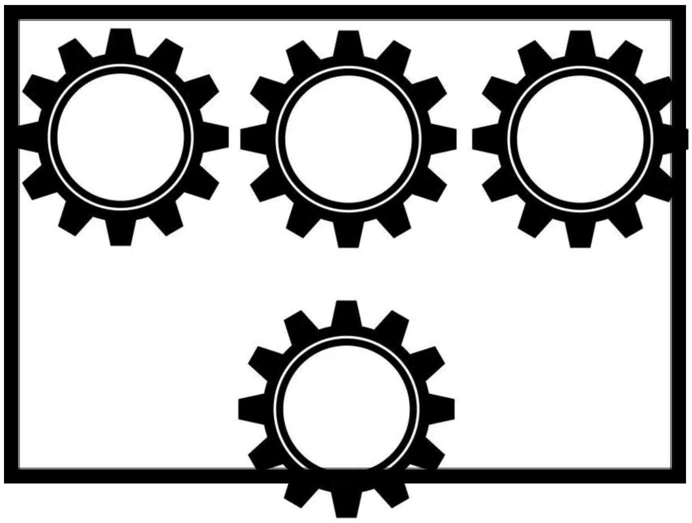
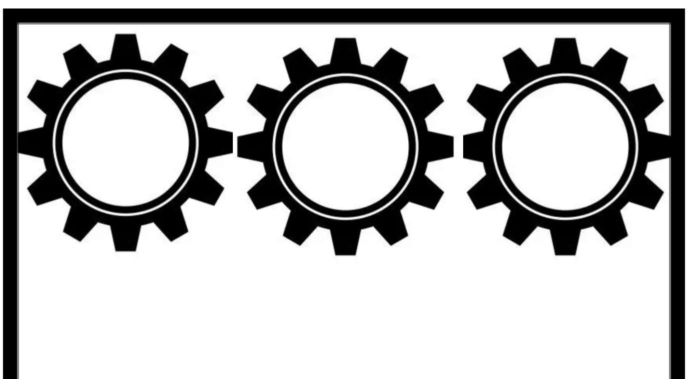


Fig 4

The rest deployment is that three of the gears are deployed in a line, and the rest one is placed on the vertical line that crosses the center of the gear in the top right corner, as demonstrated by the figure 5.



For 5 gears, there are four feasible deployments.

One of the deployments is that three of the gears are deployed on a line, and the rest two gears are deployed engaging each other on another line, one of which also engages the gear in the top left corner, as demonstrated by the figure 6.

The second deployment is that three of the gears are deployed on a line, and the rest two gears are deployed on another line not engaging each other, as demonstrated by the figure 7.

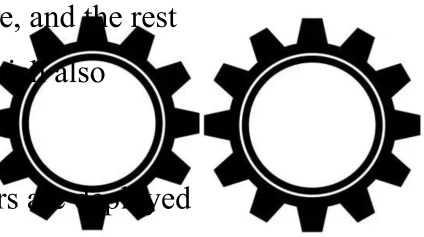


Fig 5

The third deployment is that four of the gears are deployed on two lines not juxtaposedly, and the rest one is deployed on the middle line that is between these two lines, as demonstrated by the figure 8.

The rest deployment is that three of the gears are deployed on a line, and the rest two gears are deployed engaging each other on another line, one of which also engages the gear in the top right corner, as demonstrated by the figure 9.

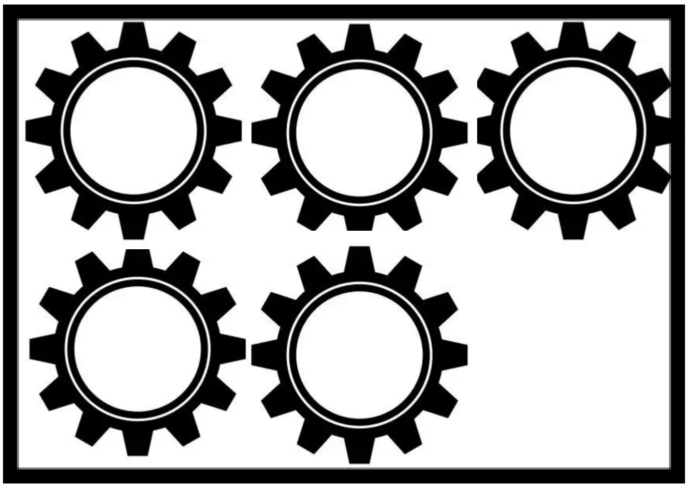


Fig 6

Fig 7

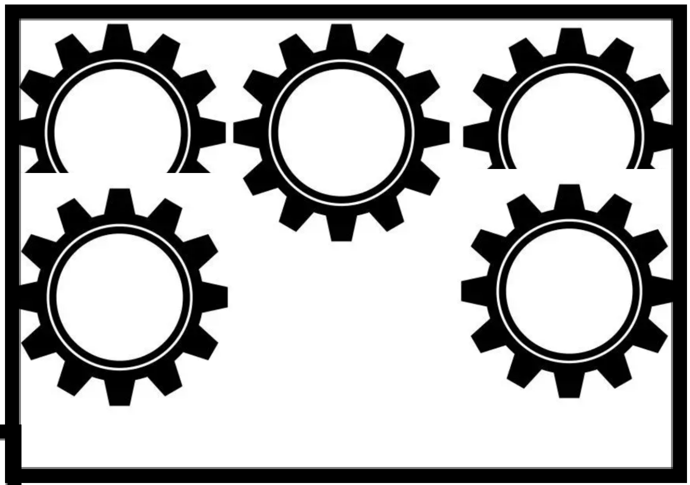


Fig 7

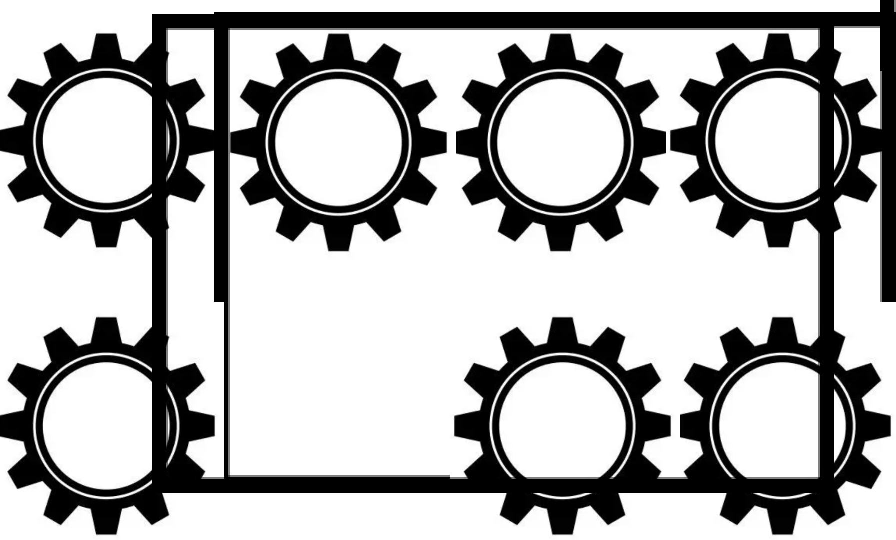


Fig 9

4. Calculation

Seeing from the formula $\{v_g = \frac{m_{air}-m_g}{m_{air}+m_g} v_{air}\ v_{gear} = \frac{2m_g}{m_g+m_{air}} v_g$, it does not take any genius to figure out that we must obtain the rotational speed and the mass of the gear and the air to be able to apply this formula, while all of them are certain, and let's just name them as v_g, m_g, and m_{air} respectively.

Furthermore, m_{air} = $\rho_{air} V_{air}$ = $1.225V_{air}$, and V_{air} is equal to the total surface area of the surface area of all of the teeth of a gear that are exposed to the air times the thickness of them, let's name, t, which varies by deployment, thus, V_{air} = kxyt, $\frac{k}{2} \epsilon Z^+$, so, m_{air} = 1.225kxyt, $\frac{k}{2} \epsilon Z^+$. Let's stand for 1.225xyt by α to make the right-hand as concise as we can, so, m_{air} = kα, $\frac{k}{2} \epsilon Z^+$.

Let's start with calculating the total momentum of the 'igniting' gear for the deployment demonstrated by the figure 1:

If we number the three gears as 1, 2, and 3 respectively, then the gear 3 is just the 'igniting gear'.

$$\because\ \{v_g = \frac{m_{air}-m_g}{m_{air}+m_g} v_{air}\ v_{gear} = \frac{2m_g}{m_g+m_{air}} v_g$$

$$\therefore\ v_{g_1}' = v_{g_1} - \frac{m_{air}-m_{g_1}}{m_{air}+m_{g_1}} v_{air}$$

$$\therefore\ v_{g_2} = \frac{2m_g}{m_g+m_{air}} v_{g_1}' = \frac{2m_g}{m_g+m_{air}} (v_{g_1} - \frac{m_{air}-m_{g_1}}{m_{air}+m_{g_1}} v_{air})$$

$$\therefore\ v_{g_2}' = v_{g_2} - \frac{m_{air}-m_{g_2}}{m_{air}+m_{g_2}} v_{air} = \frac{2m_g}{m_g+m_{air}} (v_{g_1} - \frac{m_{air}-m_{g_1}}{m_{air}+m_{g_1}} v_{air}) - \frac{m_{air}-m_{g_2}}{m_{air}+m_{g_2}} v_{air}$$

$$\therefore\ v_{g_3} = \frac{2m_g}{m_g+m_{air}} [\frac{2m_g}{m_g+m_{air}} (v_{g_1} - \frac{m_{air}-m_{g_1}}{m_{air}+m_{g_1}} v_{air}) - \frac{m_{air}-m_{g_2}}{m_{air}+m_{g_2}} v_{air}]$$

$$\therefore\ v_{g_3}' = \frac{2m_g}{m_g+m_{air}} [\frac{2m_g}{m_g+m_{air}} (v_{g_1} - \frac{m_{air}-m_{g_1}}{m_{air}+m_{g_1}} v_{air}) - \frac{m_{air}-m_{g_2}}{m_{air}+m_{g_2}} v_{air}] - \frac{m_{air}-m_{g_3}}{m_{air}+m_{g_3}} v_{air}$$

$$\therefore\ v_{g_3}' = \frac{4m_{g_1}^{\ 2}}{(m_{g_1}+m_{air})^2} (v_{g_1} - \frac{m_{air}-m_{g_1}}{m_{air}+m_{g_1}} v_{air}) - \frac{2m_{g_1}}{m_{g_1}+m_{air}} \frac{m_{air}-m_{g_1}}{m_{air}+m_{g_1}} v_{air} - \frac{m_{air}-m_{g_1}}{m_{air}+m_{g_1}} v_{air}$$

$$\therefore p_{g_3} = m_{g_1}[\frac{4m_{g_1}^{\ 2}}{(m_{g_1}+m_{air})^2}(v_{g_1} - \frac{m_{air}-m_{g_1}}{m_{air}+m_{g_1}}v_{air}) - \frac{2m_{g_1}}{m_{g_1}+m_{air}}\frac{m_{air}-m_{g_1}}{m_{air}+m_{g_1}}v_{air} - \frac{m_{air}-m_{g_1}}{m_{air}+m_{g_1}}v_{air}]$$

Because the total surface area of the teeth of all of the gears striken by the air is approximately half more than the total surface area,

Therefore, the 'efficacy' of this deployment is $m_{g_1}[\frac{4m_{g_1}^{\ 2}}{(m_{g_1}+\frac{k\alpha}{2})^2}(v_{g_1} - \frac{\frac{k\alpha}{2}-m_{g_1}}{\frac{k\alpha}{2}+m_{g_1}}v_{air}) - \frac{k\alpha-m_{g_1}^{\ 2}}{(\frac{k\alpha}{2}+m_{g_1})^2}v_{air} - \frac{\frac{k\alpha}{2}-m_{g_1}}{\frac{k\alpha}{2}+m_{g_1}}v_{air}]$, $\frac{k}{2}\epsilon Z^{+}$.

For the deployment demonstrated by the figure 2:

If we number the three gears as 1, 2, and 3 respectively, then the gear 3 is just the 'igniting gear'.

$$\because \{v_g = \frac{m_{air}-m_g}{m_{air}+m_g}v_{air}\ v_{gear} = \frac{2m_g}{m_g+m_{air}}v_g$$

$\therefore v_{g_1}' = v_{g_1} - \frac{m_{air}-m_{g_1}}{m_{air}+m_{g_1}}v_{air} = v_{g_1} - \frac{\frac{2k}{3}\alpha-m_{g_1}}{\frac{2k}{3}\alpha+m_{g_1}}v_{air}$, $\frac{k}{2}\epsilon Z^{+}$(because the total surface area of the teeth of the gear striken by the air is approximately two-thirds more than the total surface area)

$$\therefore v_{g_2} = \frac{2m_g}{m_g+m_{air}}v_{g_1}' = \frac{2m_g}{m_g+m_{air}}(v_{g_1} - \frac{\frac{2k}{3}\alpha-m_{g_1}}{\frac{2k}{3}\alpha+m_{g_1}}v_{air}),\ \frac{k}{2}\epsilon Z^{+}$$

$$\therefore v_{g_2}' = v_{g_2} - \frac{m_{air}-m_{g_2}}{m_{air}+m_{g_2}}v_{air} = \frac{2m_g}{m_g+\frac{2k}{3}\alpha}(v_{g_1} - \frac{\frac{2k}{3}\alpha-m_{g_1}}{\frac{2k}{3}\alpha+m_{g_1}}v_{air}) - \frac{\frac{4k}{5}\alpha-m_{g_2}}{\frac{4k}{5}\alpha+m_{g_2}}v_{air},\ \frac{k}{2}\epsilon Z^{+}$$

(because the total surface area of the teeth of the gear striken by the air is approximately four-fifths more than the total surface area)

$$\therefore v_{g_3} = \frac{2m_g}{m_g+m_{air}}v_{g_2}' = \frac{2m_g}{m_g+\frac{2k}{3}\alpha}[\frac{2m_g}{m_g+\frac{2k}{3}\alpha}(v_{g_1} - \frac{\frac{2k}{3}\alpha-m_{g_1}}{\frac{2k}{3}\alpha+m_{g_1}}v_{air}) - \frac{\frac{4k}{5}\alpha-m_{g_2}}{\frac{4k}{5}\alpha+m_{g_2}}v_{air}],\ \frac{k}{2}\epsilon Z^{+}$$

$$\therefore v_{g_3}' = v_{g_3} - \frac{m_{air}-m_{g_3}}{m_{air}+m_{g_3}}v_{air} = \frac{2m_g}{m_g+\frac{2k}{3}\alpha}v_{g_2}' - \frac{m_{air}-m_{g_3}}{m_{air}+m_{g_3}}v_{air} = \frac{2m_g}{m_g+\frac{2k}{3}\alpha}[\frac{2m_g}{m_g+\frac{2k}{3}\alpha}(v_{g_1} -$$

$\frac{\frac{2k}{3}\alpha - m_{g_1}}{\frac{2k}{3}\alpha + m_{g_1}} v_{air})$ - $\frac{\frac{4k}{5}\alpha - m_{g_2}}{\frac{4k}{5}\alpha + m_{g_2}} v_{air}]$ - $\frac{\frac{2k}{3}\alpha - m_{g_3}}{\frac{2k}{3}\alpha + m_{g_3}} v_{air}$, $\frac{k}{2} \epsilon Z^{+}$(because the total surface area of the teeth of the gear striken by the air is approximately two-thirds more than the total surface area)

$$\therefore\ p_{g_3} = m_{g_3} v_{g_3}' = m_{g_1} \{ \frac{2m_{g_1}}{m_{g_1} + \frac{2k}{3}\alpha} [\frac{2m_{g_1}}{m_{g_1} + \frac{2k}{3}\alpha} (v_{g_1} - \frac{\frac{2k}{3}\alpha - m_{g_1}}{\frac{2k}{3}\alpha + m_{g_1}} v_{air}) - \frac{\frac{4k}{5}\alpha - m_{g_2}}{\frac{4k}{5}\alpha + m_{g_2}} v_{air}] - \frac{\frac{2k}{3}\alpha - m_{g_3}}{\frac{2k}{3}\alpha + m_{g_3}} v_{air} \},\ \frac{k}{2} \epsilon Z^{+}$$

Therefore, the 'efficacy' of this deployment is $m_{g_1} \{ \frac{2m_{g_1}}{m_{g_1} + \frac{2k}{3}\alpha} [\frac{2m_{g_1}}{m_{g_1} + \frac{2k}{3}\alpha} (v_{g_1} - \frac{\frac{2k}{3}\alpha - m_{g_1}}{\frac{2k}{3}\alpha + m_{g_1}} v_{air})$ - $\frac{\frac{4k}{5}\alpha - m_{g_1}}{\frac{4k}{5}\alpha + m_{g_1}} v_{air}]$ - $\frac{\frac{2k}{3}\alpha - m_{g_1}}{\frac{2k}{3}\alpha + m_{g_1}} v_{air} \}$, $\frac{k}{2} \epsilon Z^{+}$.

For the deployment demonstrated by the figure 3:

Because there is also only one gear engaging the 'igniting gear' and the total surface area of it striken by the air is also half more than the total surface area of it, being the same as the 'efficacy' of the deployment demonstrated by the figure 1, the one of this deployment is also $m_{g_1} [\frac{4m_{g_1}^{2}}{(m_{g_1} + \frac{k\alpha}{2})^2} (v_{g_1} - \frac{\frac{k\alpha}{2} - m_{g_1}}{\frac{k\alpha}{2} + m_{g_1}} v_{air})$ - $\frac{k\alpha - m_{g_1}^{2}}{(\frac{k\alpha}{2} + m_{g_1})^{2}} v_{air}$ - $\frac{\frac{k\alpha}{2} - m_{g_1}}{\frac{k\alpha}{2} + m_{g_1}} v_{air}$], $\frac{k}{2} \epsilon Z^{+}$.

For the deployment demonstrated by the figure 4:

Because there is also only one gear engaging the 'igniting gear' and the total surface area of it striken by the air is also half more than the total surface area of it, being the same as the 'efficacy' of the deployment demonstrated by the figure 1, the one of this deployment is also $m_{g_1} [\frac{4m_{g_1}^{2}}{(m_{g_1} + \frac{k\alpha}{2})^2} (v_{g_1} - \frac{\frac{k\alpha}{2} - m_{g_1}}{\frac{k\alpha}{2} + m_{g_1}} v_{air})$ - $\frac{k\alpha - m_{g_1}^{2}}{(\frac{k\alpha}{2} + m_{g_1})^{2}} v_{air}$ - $\frac{\frac{k\alpha}{2} - m_{g_1}}{\frac{k\alpha}{2} + m_{g_1}} v_{air}$], $\frac{k}{2} \epsilon Z^{+}$.

For the deployment demonstrated by the figure 5:

Let’s number the three gears in the upper line as 1, 2, and 3 respectively.

$$\because \{v_g = \frac{m_{air} - m_g}{m_{air} + m_g} v_{air} \quad v_{gear} = \frac{2m_g}{m_g + m_{air}} v_g$$

$$\therefore v_{g_1}' = v_{g_1} - \frac{m_{air} - m_{g_1}}{m_{air} + m_{g_1}} v_{air}$$

$$\therefore v_{g_2} = \frac{2m_g}{m_g + m_{air}} v_{g_1}' = \frac{2m_g}{m_g + \frac{k}{2\alpha}} \left(v_{g_1} - \frac{\frac{k\alpha}{2} - m_{g_1}}{\frac{k\alpha}{2} + m_{g_1}} v_{air}\right),\ \frac{k}{2} \in Z^+$$

(because the total surface area of the teeth of the gear striken by the air is approximately half more than the total surface area)

$$\therefore v_{g_2}' = v_{g_2} - \frac{m_{air} - m_{g_2}}{m_{air} + m_{g_2}} v_{air} = \frac{2m_g}{m_g + \frac{k\alpha}{2}} \left(v_{g_1} - \frac{\frac{k\alpha}{2} - m_{g_1}}{\frac{k\alpha}{2} + m_{g_1}} v_{air}\right) - \frac{\frac{k\alpha}{2} - m_{g_2}}{\frac{k\alpha}{2} + m_{g_2}} v_{air},\ \frac{k}{2} \in Z^+$$

(because the total surface area of the teeth of the gear striken by the air is approximately half more than the total surface area)

$$\therefore v_{g_3} = \frac{2m_g}{m_g + m_{air}} v_{g_2}' = \frac{4m_{g_1}}{m_{g_1} + \frac{k}{2\alpha}} \left(v_{g_1} - \frac{\frac{k\alpha}{2} - m_{g_1}}{\frac{k\alpha}{2} + m_{g_1}} v_{air}\right) - \frac{\frac{k\alpha}{2} - 2m_{g_2}}{\frac{k\alpha}{2} + m_{g_2}} v_{air},\ \frac{k}{2} \in Z^+$$

(because the total surface area of the teeth of the gear striken by the air is approximately 0)

$$\therefore p_{g_3} = m_{g_3} v_{g_3} = m_{g_1} \left[\frac{4m_{g_1}}{m_{g_1} + \frac{k}{2\alpha}} \left(v_{g_1} - \frac{\frac{k\alpha}{2} - m_{g_1}}{\frac{k\alpha}{2} + m_{g_1}} v_{air}\right) - \frac{\frac{k\alpha}{2} - 2m_{g_2}}{\frac{k\alpha}{2} + m_{g_2}} v_{air}\right],\ \frac{k}{2} \in Z^+$$

Therefore, the ‘efficacy’ of this deployment is $m_{g_1} \left[\frac{4m_{g_1}}{m_{g_1} + \frac{k\alpha}{2}} \left(v_{g_1} - \frac{\frac{k\alpha}{2} - m_{g_1}}{\frac{k\alpha}{2} + m_{g_1}} v_{air}\right) - \frac{\frac{k\alpha}{2} - 2m_{g_1}}{\frac{k\alpha}{2} + m_{g_1}} v_{air}\right],\ \frac{k}{2} \in Z^+$.

For the deployment demonstrated by the figure 6:

Let’s number all of the gears in the upper line as 1, 2, and 3 respectively.

$$\because \{v_g = \frac{m_{air} - m_g}{m_{air} + m_g} v_{air} \quad v_{gear} = \frac{2m_g}{m_g + m_{air}} v_g$$

$\therefore\ v_{g_2} = \frac{2m_g}{m_g+m_{air}} v_{g_1} = 2v_{g_1}$ (because the total surface area of the teeth of the gear striken by the air is approximately 0)

$\therefore\ v_{g_3} = \frac{2m_g}{m_g+m_{air}} v_{g_2} = \frac{2m_{g_3}}{m_{g_3}+\frac{k\alpha}{2}} v_{g_2},\ \frac{k}{2}\epsilon Z^+$ (because the total surface area of the teeth of the gear striken by the air is approximately half more than the total surface area of it)

$\therefore\ v_{g_3}' = v_{g_3} - \frac{m_{air}-m_{g_3}}{m_{air}+m_{g_3}} v_{air} = \frac{2m_{g_3}}{m_{g_3}+\frac{k\alpha}{2}} v_{g_2} - \frac{\frac{k\alpha}{2}-m_{g_3}}{\frac{k\alpha}{2}+m_{g_3}} v_{air},\ \frac{k}{2}\epsilon Z^+$

$\therefore\ p_{g_3} = m_{g_3} v_{g_3}' = m_{g_1}(\frac{2m_{g_1}}{m_{g_1}+\frac{k\alpha}{2}} v_{g_2} - \frac{\frac{k\alpha}{2}-m_{g_1}}{\frac{k\alpha}{2}+m_{g_1}} v_{air}),\ \frac{k}{2}\epsilon Z^+$

Therefore, the 'efficacy' of the deployment is $m_{g_1}(\frac{2m_{g_1}}{m_{g_1}+\frac{k\alpha}{2}} v_{g_1} - \frac{\frac{k\alpha}{2}-m_{g_1}}{\frac{k\alpha}{2}+m_{g_1}} v_{air}),\ \frac{k}{2}\epsilon Z^+$.

For the deployment demonstrated by the figure 7:

Let's number all of the gears in the upper line as 1, 2, and 3 respectively.

$\because \{v_g = \frac{m_{air}-m_g}{m_{air}+m_g} v_{air}\ v_{gear} = \frac{2m_g}{m_g+m_{air}} v_g$

$\therefore\ v_{g_2} = \frac{2m_g}{m_g+m_{air}} v_{g_1} = \frac{2m_{g_2}}{m_{g_2}+\frac{k\alpha}{2}} v_{g_1}$ (because the total surface area of the teeth of the gear striken by the air is approximately half more than the total surface area)

$\therefore\ v_{g_2}' = \frac{2m_{g_2}}{m_{g_2}+\frac{k\alpha}{2}} v_{g_1} - \frac{m_{air}-m_g}{m_{air}+m_g} v_{air} = \frac{2m_{g_2}}{m_{g_2}+\frac{k\alpha}{2}} v_{g_1} - \frac{\frac{k\alpha}{2}-m_{g_2}}{\frac{k\alpha}{2}+m_{g_2}} v_{air},\ \frac{k}{2}\epsilon Z^+$

$\therefore\ v_{g_3} = \frac{2m_g}{m_g+m_{air}} v_{g_2}' = 2(\frac{2m_{g_2}}{m_{g_2}+\frac{k\alpha}{2}} v_{g_1} - \frac{\frac{k\alpha}{2}-m_{g_2}}{\frac{k\alpha}{2}+m_{g_2}} v_{air}) = \frac{4m_{g_1}}{m_{g_1}+\frac{k\alpha}{2}} v_{g_1} - \frac{k\alpha-m_{g_1}}{\frac{k\alpha}{2}+m_{g_1}} v_{air},\ \frac{k}{2}\epsilon Z^+$

(because the total surface area of the teeth of the gear striken by the air is approximately 0)

$\therefore p_{g_3} = m_{g_1}(\frac{4m_{g_1}}{m_{g_1}+\frac{k\alpha}{2}} v_{g_1} - \frac{k\alpha-m_{g_1}}{\frac{k\alpha}{2}+m_{g_1}} v_{air}),\ \frac{k}{2}\epsilon Z^+$

Therefore, the 'efficacy' of this deployment is $m_{g_1}(\frac{4m_{g_1}}{m_{g_1}+\frac{k\alpha}{2}}v_{g_1} - \frac{k\alpha-m_{g_1}}{\frac{k\alpha}{2}+m_{g_1}}v_{air})$, $\frac{k}{2}\epsilon Z^{+}$.

For the deployment demonstrated by the figure 8:

Let's number all of the gears in the upper line as 1 and 2 respectively, the gear in the middle line between the upper line and the lower line as 3, and all of the gears in the lower line as 4 and 5 respectively.

$\because \{v_g = \frac{m_{air}-m_g}{m_{air}+m_g}v_{air}\ v_{gear} = \frac{2m_g}{m_g+m_{air}}v_g$

$\therefore v_{g_1}' = v_{g_1} - \frac{m_{air}-m_{g_1}}{m_{air}+m_{g_1}}v_{air} = v_{g_1} - \frac{\frac{k\alpha}{3}-m_{g_1}}{\frac{k\alpha}{3}+m_{g_1}}v_{air}$, $\frac{k}{2}\epsilon Z^{+}$(because the total surface area of the teeth of the gear striken by the air is approximately one-third more than the total surface area)

$\therefore v_{g_3} = \frac{2m_g}{m_g+m_{air}}v_{g_1}' = \frac{2m_{g_3}}{m_{g_3}+\frac{5k\alpha}{6}}(v_{g_1} - \frac{\frac{k\alpha}{3}-m_{g_1}}{\frac{k\alpha}{3}+m_{g_1}}v_{air})$, $\frac{k}{2}\epsilon Z^{+}$(because the total surface area of the teeth of the gear striken by the air is approximately five-sixths more than the total surface area)

$\therefore v_{g_3}' = v_{g_3} - \frac{m_{air}-m_{g_3}}{m_{air}+m_{g_3}}v_{air} = \frac{2m_{g_3}}{m_{g_3}+\frac{5k\alpha}{6}}(v_{g_1} - \frac{\frac{k\alpha}{3}-m_{g_1}}{\frac{k\alpha}{3}+m_{g_1}}v_{air}) - \frac{\frac{5k\alpha}{6}-m_{g_3}}{\frac{5k\alpha}{6}+m_{g_3}}v_{air}$, $\frac{k}{2}\epsilon Z^{+}$

$\therefore v_{g_2} = \frac{2m_g}{m_g+m_{air}}v_{g_3}' = \frac{2m_{g_2}}{m_{g_2}+\frac{k}{3\alpha}}[\frac{2m_{g_3}}{m_{g_3}+\frac{5k\alpha}{6}}(v_{g_1} - \frac{\frac{k\alpha}{3}-m_{g_1}}{\frac{k\alpha}{3}+m_{g_1}}v_{air}) - \frac{\frac{5k\alpha}{6}-m_{g_3}}{\frac{5k\alpha}{6}+m_{g_3}}v_{air}]$, $\frac{k}{2}\epsilon Z^{+}$

(because the total surface area of the teeth of the gear striken by the air is approximately one-third more than the total surface area)

$\therefore v_{g_2}' = v_{g_2} - \frac{m_{air}-m_{g_2}}{m_{air}+m_{g_2}}v_{air} = \frac{2m_{g_2}}{m_{g_2}+\frac{k\alpha}{3}}[\frac{2m_{g_3}}{m_{g_3}+\frac{5k\alpha}{6}}(v_{g_1} - \frac{\frac{k\alpha}{3}-m_{g_1}}{\frac{k\alpha}{3}+m_{g_1}}v_{air}) - \frac{\frac{5k\alpha}{6}-m_{g_3}}{\frac{5k\alpha}{6}+m_{g_3}}v_{air}] - \frac{\frac{k\alpha}{3}-m_{g_2}}{\frac{k\alpha}{3}+m_{g_2}}v_{air}$, $\frac{k}{2}\epsilon Z^{+}$

$\therefore p_{g_2} = m_{g_1}\{\frac{2m_{g_2}}{m_{g_2}+\frac{k\alpha}{3}}[\frac{2m_{g_3}}{m_{g_3}+\frac{5k\alpha}{6}}(v_{g_1} - \frac{\frac{k\alpha}{3}-m_{g_1}}{\frac{k\alpha}{3}+m_{g_1}}v_{air}) - \frac{\frac{5k\alpha}{6}-m_{g_3}}{\frac{5k\alpha}{6}+m_{g_3}}v_{air}] - \frac{\frac{k\alpha}{3}-m_{g_2}}{\frac{k\alpha}{3}+m_{g_2}}v_{air}\}$, $\frac{k}{2}\epsilon Z^{+}$

Therefore, the 'efficacy' of this deployment is $m_{g_1}\{\frac{2m_{g_1}}{m_{g_1}+\frac{k\alpha}{3}}[\frac{2m_{g_1}}{m_{g_1}+\frac{5k\alpha}{6}}(v_{g_1}$ - $\frac{\frac{k\alpha}{3}-m_{g_1}}{\frac{k\alpha}{3}+m_{g_1}}v_{air}$) - $\frac{\frac{5k\alpha}{6}-m_{g_1}}{\frac{5k\alpha}{6}+m_{g_1}}v_{air}]$ - $\frac{\frac{k\alpha}{3}-m_{g_1}}{\frac{k\alpha}{3}+m_{g_1}}v_{air}\}$, $\frac{k}{2}\epsilon Z^{+}$.

For the deployment demonstrated by the figure 9:

Let's number all of the gears in the upper line as 1,2 and 3 respectively, and all of the gears in the lower line as 4 and 5 respectively.

$\because \{v_g = \frac{m_{air}-m_g}{m_{air}+m_g}v_{air}\ v_{gear} = \frac{2m_g}{m_g+m_{air}}v_g$

$\therefore v_{g_1}' = v_{g_1} - \frac{m_{air}-m_g}{m_{air}+m_g}v_{air} = v_{g_1} - \frac{\frac{k\alpha}{2}-m_{g_1}}{\frac{k\alpha}{2}+m_{g_1}}v_{air}$, $\frac{k}{2}\epsilon Z^{+}$(because the total surface area of the teeth of the gear striken by the air is approximately half more than the total surface area)

$\therefore v_{g_2} = \frac{2m_g}{m_g+m_{air}}v_{ig} = 2(v_{g_1} - \frac{\frac{k\alpha}{2}-m_{g_1}}{\frac{k\alpha}{2}+m_{g_1}}v_{air})$, $\frac{k}{2}\epsilon Z^{+}$(because the total surface area of the teeth of the gear striken by the air is approximately 0)

$\therefore v_{g_4} = \frac{2m_g}{m_g+m_{air}}v_{ig} = 4(v_{g_1} - \frac{\frac{k\alpha}{2}-m_{g_1}}{\frac{k\alpha}{2}+m_{g_1}}v_{air})$, $\frac{k}{2}\epsilon Z^{+}$(because the total surface area of the teeth of the gear striken by the air is approximately 0)

$\therefore v_{g_5} = \frac{2m_g}{m_g+m_{air}}v_{ig} = 8(v_{g_1} - \frac{\frac{k\alpha}{2}-m_{g_1}}{\frac{k\alpha}{2}+m_{g_1}}v_{air})$, $\frac{k}{2}\epsilon Z^{+}$(because the total surface area of the teeth of the gear striken by the air is approximately 0)

$\therefore v_{g_3} = \frac{2m_g}{m_g+m_{air}}v_{g_2}$ - $\frac{2m_g}{m_g+m_{air}}v_{g_5} = 20(v_{g_1} - \frac{\frac{k\alpha}{2}-m_{g_1}}{\frac{k\alpha}{2}+m_{g_1}}v_{air})$, $\frac{k}{2}\epsilon Z^{+}$(because the total surface area of the teeth of the gear striken by the air is approximately 0)

Therefore, the 'efficacy' of the deployment is $20(v_{g_1} - \frac{\frac{k\alpha}{2}-m_{g_1}}{\frac{k\alpha}{2}+m_{g_1}}v_{air})$, $\frac{k}{2}\epsilon Z^{+}$.

5. Discussion

To elicit the vital conclusion, we have to showcase the ‘efficacies’ of all of the deployments:

Deployment	Efficacy
1	$m_{g_1}[\frac{4m_{g_1}^2}{(m_{g_1}+\frac{k\alpha}{2})^2}(v_{g_1} - \frac{\frac{k\alpha}{2}-m_{g_1}}{\frac{k\alpha}{2}+m_{g_1}}v_{air}) - \frac{k\alpha-m_{g_1}^2}{(\frac{k\alpha}{2}+m_{g_1})^2}v_{air} - \frac{\frac{k\alpha}{2}-m_{g_1}}{\frac{k\alpha}{2}+m_{g_1}}v_{air}], \frac{k}{2}\epsilon Z^+$
2	$m_{g_1}\{\frac{2m_{g_1}}{m_{g_1}+\frac{2k}{3}\alpha}[\frac{2m_{g_1}}{m_{g_1}+\frac{2k}{3}\alpha}(v_{g_1} - \frac{\frac{2k}{3}\alpha-m_{g_1}}{\frac{2k}{3}\alpha+m_{g_1}}v_{air}) - \frac{\frac{4k}{5}\alpha-m_{g_1}}{\frac{4k}{5}\alpha+m_{g_1}}v_{air}] - \frac{\frac{2k}{3}\alpha-m_{g_1}}{\frac{2k}{3}\alpha+m_{g_1}}v_{air}\}, \frac{k}{2}\epsilon Z^+$
3	$m_{g_1}[\frac{4m_{g_1}^2}{(m_{g_1}+\frac{k\alpha}{2})^2}(v_{g_1} - \frac{\frac{k\alpha}{2}-m_{g_1}}{\frac{k\alpha}{2}+m_{g_1}}v_{air}) - \frac{k\alpha-m_{g_1}^2}{(\frac{k\alpha}{2}+m_{g_1})^2}v_{air} - \frac{\frac{k\alpha}{2}-m_{g_1}}{\frac{k\alpha}{2}+m_{g_1}}v_{air}], \frac{k}{2}\epsilon Z^+$
4	$m_{g_1}[\frac{4m_{g_1}^2}{(m_{g_1}+\frac{k\alpha}{2})^2}(v_{g_1} - \frac{\frac{k\alpha}{2}-m_{g_1}}{\frac{k\alpha}{2}+m_{g_1}}v_{air}) - \frac{k\alpha-m_{g_1}^2}{(\frac{k\alpha}{2}+m_{g_1})^2}v_{air} - \frac{\frac{k\alpha}{2}-m_{g_1}}{\frac{k\alpha}{2}+m_{g_1}}v_{air}], \frac{k}{2}\epsilon Z^+$
5	$m_{g_1}[\frac{4m_{g_1}}{m_{g_1}+\frac{k\alpha}{2}}(v_{g_1} - \frac{\frac{k\alpha}{2}-m_{g_1}}{\frac{k\alpha}{2}+m_{g_1}}v_{air}) - \frac{\frac{k\alpha}{2}-2m_{g_1}}{\frac{k\alpha}{2}+m_{g_1}}v_{air}], \frac{k}{2}\epsilon Z^+$
6	$m_{g_1}(\frac{2m_{g_1}}{m_{g_1}+\frac{k\alpha}{2}}v_{g_1} - \frac{\frac{k\alpha}{2}-m_{g_1}}{\frac{k\alpha}{2}+m_{g_1}}v_{air}), \frac{k}{2}\epsilon Z^+$
7	$m_{g_1}(\frac{4m_{g_1}}{m_{g_1}+\frac{k\alpha}{2}}v_{g_1} - \frac{k\alpha-m_{g_1}}{\frac{k\alpha}{2}+m_{g_1}}v_{air}), \frac{k}{2}\epsilon Z^+$

8	$m_{g_1}\{\frac{2m_{g_1}}{m_{g_1}+\frac{k\alpha}{3}}[\frac{2m_{g_1}}{m_{g_1}+\frac{5k\alpha}{6}}(v_{g_1} - \frac{\frac{k\alpha}{3}-m_{g_1}}{\frac{k\alpha}{3}+m_{g_1}}v_{air}) - \frac{\frac{5k\alpha}{6}-m_{g_1}}{\frac{5k\alpha}{6}+m_{g_1}}v_{air}] - \frac{\frac{k\alpha}{3}-m_{g_1}}{\frac{k\alpha}{3}+m_{g_1}}v_{air}\}, \frac{k}{2}\epsilon Z^+$
9	$20(v_{g_1} - \frac{\frac{k\alpha}{2}-m_{g_1}}{\frac{k\alpha}{2}+m_{g_1}}v_{air}), \frac{k}{2}\epsilon Z^+$

The deployments 1 and 2 are for 3 gears; The deployments 3 and 4 are for 4 gears; The deployments 5, 6, 7, 8 and 9 are for 5 gears.

Now we are to find out the most efficient deployments for 3-5 gears:

For 3 gears:

$$\because m_{g_1}[\frac{4m_{g_1}^2}{(m_{g_1}+\frac{k}{2\alpha})^2}(v_{g_1} - \frac{\frac{k}{2\alpha}-m_{g_1}}{\frac{k}{2\alpha}+m_{g_1}}v_{air}) - \frac{\frac{k}{\alpha}-m_{g_1}^2}{(\frac{k}{2\alpha}+m_{g_1})^2}v_{air} - \frac{\frac{k}{2\alpha}-m_{g_1}}{\frac{k}{2\alpha}+m_{g_1}}v_{air}] - m_{g_1}\{\frac{2m_{g_1}}{m_{g_1}+\frac{2k}{3}\alpha}[\frac{2m_{g_1}}{m_{g_1}+\frac{2k}{3}\alpha}(v_{g_1} - \frac{\frac{2k}{3}\alpha-m_{g_1}}{\frac{2k}{3}\alpha+m_{g_1}}v_{air}) - \frac{\frac{4k}{5}\alpha-m_{g_1}}{\frac{4k}{5}\alpha+m_{g_1}}v_{air}] - \frac{\frac{2k}{3}\alpha-m_{g_1}}{\frac{2k}{3}\alpha+m_{g_1}}v_{air}\} =$$

$$m_{g_1}\{\frac{4m_{g_1}^2}{(m_{g_1}+\frac{k}{2\alpha})^2}(v_{g_1} - \frac{\frac{k}{2\alpha}-m_{g_1}}{\frac{k}{2\alpha}+m_{g_1}}v_{air}) - \frac{\frac{k}{\alpha}-m_{g_1}^2}{(\frac{k}{2\alpha}+m_{g_1})^2}v_{air} - \frac{\frac{k}{2\alpha}-m_{g_1}}{\frac{k}{2\alpha}+m_{g_1}}v_{air} - \frac{2m_{g_1}}{m_{g_1}+\frac{2k}{3}\alpha}[\frac{2m_{g_1}}{m_{g_1}+\frac{2k}{3}\alpha}(v_{g_1} - \frac{\frac{2k}{3}\alpha-m_{g_1}}{\frac{2k}{3}\alpha+m_{g_1}}v_{air}) - \frac{\frac{4k}{5}\alpha-m_{g_1}}{\frac{4k}{5}\alpha+m_{g_1}}v_{air}] - \frac{\frac{2k}{3}\alpha-m_{g_1}}{\frac{2k}{3}\alpha+m_{g_1}}v_{air}\} < 0$$

$\therefore$ For 3 gears, the deployment 2 is the more efficient deployment.

For 4 gears:

$$\because m_{g_1}[\frac{4m_{g_1}^2}{(m_{g_1}+\frac{k}{2\alpha})^2}(v_{g_1} - \frac{\frac{k}{2\alpha}-m_{g_1}}{\frac{k}{2\alpha}+m_{g_1}}v_{air}) - \frac{\frac{k}{\alpha}-m_{g_1}^2}{(\frac{k}{2\alpha}+m_{g_1})^2}v_{air} - \frac{\frac{k}{2\alpha}-m_{g_1}}{\frac{k}{2\alpha}+m_{g_1}}v_{air}] - m_{g_1}[\frac{4m_{g_1}^2}{(m_{g_1}+\frac{k}{2\alpha})^2}(v_{g_1} - \frac{\frac{k}{2\alpha}-m_{g_1}}{\frac{k}{2\alpha}+m_{g_1}}v_{air}) - \frac{\frac{k}{\alpha}-m_{g_1}^2}{(\frac{k}{2\alpha}+m_{g_1})^2}v_{air} - \frac{\frac{k}{2\alpha}-m_{g_1}}{\frac{k}{2\alpha}+m_{g_1}}v_{air}] = 0$$

$\therefore$ For 4 gears, the working efficacies of the deployments 3 and 4 are equal.

For 5 gears, obviously, we can't know the most efficient deployment, since whether m_{g_1} is larger or smaller than 1 is unknown, as well as whether $k\alpha$, as well as $\frac{k\alpha}{2}$ and $\frac{k\alpha}{3}$, is larger or smaller than m_{g_1} by subtraction or ratio, but it will be by definite integral, since if $\int_a^b f(x)dx > \int_a^b g(x)dx$, then f(x) > g(x)[9].

But don't straightforward get down to calculating the definite integrals without thinking of what values are put to a and b beforehand so that the calculations will be way simpler than the ones if we put some values to them that will make mischief, and by taking a glance at the expressions, it does not take any genius to figure out we should put 0 and 1 to a and b respectively to simplify the calculations.

$\iiint_0^1 [\frac{4m_{g_1}}{m_{g_1}+\frac{k\alpha}{2}}(v_{g_1} - \frac{\frac{k\alpha}{2}-m_{g_1}}{\frac{k\alpha}{2}+m_{g_1}}v_{air}) - \frac{\frac{k\alpha}{2}-2m_{g_1}}{\frac{k\alpha}{2}+m_{g_1}}v_{air}]dm_{g_1}dv_{g_1}dv_{air}$(we don't need to integrate m_{g_1} into the integrands for any of the expressions of the working efficacies of the deployments 5, 6, 7 and 8 because all of them contain the factor 'm_{g_1}')

$=\iiint_0^1 [\frac{4m_{g_1}}{m_{g_1}+\frac{k\alpha}{2}}(v_{g_1} - \frac{\frac{k\alpha}{2}-m_{g_1}}{\frac{k\alpha}{2}+m_{g_1}}v_{air})]dm_{g_1}dv_{g_1}dv_{air} - \iiint_0^1 (\frac{\frac{k\alpha}{2}-2m_{g_1}}{\frac{k\alpha}{2}+m_{g_1}}v_{air})dm_{g_1}dv_{g_1}dv_{air}$

$= \iiint_0^1 (\frac{4m_{g_1}}{m_{g_1}+\frac{k\alpha}{2}}v_{g_1})dm_{g_1}dv_{g_1}dv_{air} - \iiint_0^1 (\frac{4m_{g_1}}{m_{g_1}+\frac{k\alpha}{2}})(\frac{\frac{k\alpha}{2}-m_{g_1}}{\frac{k\alpha}{2}+m_{g_1}}v_{air})dm_{g_1}dv_{g_1}dv_{air}$ - $\iiint_0^1 (\frac{\frac{k\alpha}{2}-2m_{g_1}}{\frac{k\alpha}{2}+m_{g_1}}v_{air})dm_{g_1}dv_{g_1}dv_{air}$

$= 2 - k\alpha\ln(\frac{k\alpha+2}{k\alpha}) - \frac{2k^2\alpha^2}{1+k\alpha} + 2k\alpha - k\alpha\ln(\frac{k\alpha+2}{k\alpha}) - 2 - \frac{3k\alpha}{4}\ln(\frac{k\alpha+2}{k\alpha})+1$

$= (-\frac{11}{4}k\alpha)\ln(\frac{k\alpha+2}{k\alpha}) - \frac{2k^2\alpha^2}{1+k\alpha} + 2k\alpha + 1$

$= (-\frac{11}{4}k\alpha)\ln(\frac{k\alpha+2}{k\alpha}) + \frac{2k\alpha}{1+k\alpha} + 1, \frac{k}{2}\epsilon Z^+$

$$\iiint_0^1\left(\frac{2m_{g_1}}{m_{g_1}+\frac{k\alpha}{2}}v_{g_1} - \frac{\frac{k\alpha}{2}-m_{g_1}}{\frac{k\alpha}{2}+m_{g_1}}v_{air}\right)dm_{g_1}\mathrm{d}v_{g_1}\mathrm{d}v_{air}$$

$$= \iiint_0^1\frac{2m_{g_1}}{m_{g_1}+\frac{k\alpha}{2}}v_{g_1}dm_{g_1}\mathrm{d}v_{g_1}\mathrm{d}v_{air} - \iiint_0^1\frac{\frac{k\alpha}{2}-m_{g_1}}{\frac{k\alpha}{2}+m_{g_1}}v_{air}dm_{g_1}\mathrm{d}v_{g_1}\mathrm{d}v_{air}$$

$$= 1 - \frac{k\alpha}{2}\ln\left(\frac{2+k\alpha}{k\alpha}\right) - \frac{k\alpha}{2}\ln\left(\frac{2+k\alpha}{k\alpha}\right) + \frac{1}{2}$$

$$= \frac{3}{2} - k\alpha\ln\left(\frac{2+k\alpha}{k\alpha}\right),\ \frac{k}{2}\epsilon Z^{+}$$

$$\iiint_0^1\left(\frac{4m_{g_1}}{m_{g_1}+\frac{k\alpha}{2}}v_{g_1} - \frac{k\alpha-m_{g_1}}{\frac{k\alpha}{2}+m_{g_1}}v_{air}\right)dm_{g_1}\mathrm{d}v_{g_1}\mathrm{d}v_{air}$$

$$= 2\iiint_0^1\left(\frac{2m_{g_1}}{m_{g_1}+\frac{k\alpha}{2}}v_{g_1} - \frac{\frac{k\alpha}{2}-m_{g_1}}{\frac{k\alpha}{2}+m_{g_1}}v_{air}\right)dm_{g_1}\mathrm{d}v_{g_1}\mathrm{d}v_{air}$$

$$= 3 - 2k\alpha\ln\left(\frac{2+k\alpha}{k\alpha}\right),\ \frac{k}{2}\epsilon Z^{+}$$

$$\iiint_0^1\left\{\frac{2m_{g_1}}{m_{g_1}+\frac{k\alpha}{3}}\left[\frac{2m_{g_1}}{m_{g_1}+\frac{5k\alpha}{6}}\left(v_{g_1} - \frac{\frac{k\alpha}{3}-m_{g_1}}{\frac{k\alpha}{3}+m_{g_1}}v_{air}\right) - \frac{\frac{5k\alpha}{6}-m_{g_1}}{\frac{5k\alpha}{6}+m_{g_1}}v_{air}\right] - \frac{\frac{k\alpha}{3}-m_{g_1}}{\frac{k\alpha}{3}+m_{g_1}}v_{air}\right\}dm_{g_1}\mathrm{d}v_{g_1}\mathrm{d}v_{air}$$

$$= \iiint_0^1\left[\frac{m_{g_1}}{(m_{g_1}+\frac{k\alpha}{3})(m_{g_1}+\frac{5k\alpha}{6})}\left(4m_{g_1}v_{g_1} - 4\frac{\frac{k\alpha}{3}-m_{g_1}}{\frac{k\alpha}{3}+m_{g_1}}v_{air} - v_{air}\right)\right]dm_{g_1}\mathrm{d}v_{g_1}\mathrm{d}v_{air}$$

$$= \iiint_0^1\left[\frac{m_{g_1}}{(m_{g_1}+\frac{k\alpha}{3})(m_{g_1}+\frac{5k\alpha}{6})}\left(4m_{g_1}v_{g_1} - 4\frac{\frac{k\alpha}{3}-m_{g_1}}{\frac{k\alpha}{3}+m_{g_1}}v_{air} - v_{air}\right)\right]dv_{g_1}\mathrm{d}v_{air}\mathrm{d}m_{g_1}$$ (we change the order of the variables that are to be integrated to simplify the calculations, as can be seen afterwards)

$$= \iint_0^1\left[\frac{m_{g_1}}{(m_{g_1}+\frac{k\alpha}{3})(m_{g_1}+\frac{5k\alpha}{6})}\left(2m_{g_1} - 4\frac{\frac{k\alpha}{3}-m_{g_1}}{\frac{k\alpha}{3}+m_{g_1}}v_{air} - v_{air}\right)\right]\mathrm{d}v_{air}\mathrm{d}m_{g_1}$$

$$= \int_0^1\left[\frac{m_{g_1}}{(m_{g_1}+\frac{k\alpha}{3})(m_{g_1}+\frac{5k\alpha}{6})}\left(2m_{g_1} - 2\frac{\frac{k\alpha}{3}-m_{g_1}}{\frac{k\alpha}{3}+m_{g_1}} - \frac{1}{2}\right)\right]\mathrm{d}m_{g_1}$$

$$= 4(3 - 2k\alpha)\int_0^1[(\frac{1}{3m_{g_1}+\frac{5k\alpha}{2}} - \frac{1}{3m_{g_1}+k\alpha})(m_{g_1} - \frac{k\alpha-3m_{g_1}}{k\alpha+3m_{g_1}} - \frac{1}{4})]\mathrm{d}m_{g_1}$$

$$= 4(3 - 2k\alpha)\int_0^1[\frac{m_{g_1}}{3m_{g_1}+\frac{5k\alpha}{2}} - \frac{k\alpha-3m_{g_1}}{(3m_{g_1}+\frac{5k\alpha}{2})(k\alpha+3m_{g_1})} - \frac{1}{12m_{g_1}+10k\alpha} - \frac{m_{g_1}}{3m_{g_1}+k\alpha} + \frac{k\alpha-3m_{g_1}}{(k\alpha+3m_{g_1})^2} + \frac{1}{12m_{g_1}+4k\alpha}]\mathrm{d}m_{g_1}$$

$$= 4(3 - 2k\alpha)(\frac{1}{3} - \frac{5k\alpha}{6}ln\frac{6+5k\alpha}{5k\alpha} + \frac{1}{3}ln\frac{6+5k\alpha}{5k\alpha} - \frac{1}{12}ln\frac{6+5k\alpha}{5k\alpha} - \frac{1}{3} + \frac{k\alpha}{9}ln\frac{3+k\alpha}{k\alpha} + \frac{2k\alpha-1}{3}\ln\frac{3+k\alpha}{k\alpha} + \frac{1}{12}\ln\frac{3+k\alpha}{k\alpha})$$

$$= (12\text{-}8k\alpha)(\frac{3-10k\alpha}{12}ln\frac{6+5k\alpha}{5k\alpha} + \frac{28k\alpha-9}{36}ln\frac{3+k\alpha}{k\alpha})$$

$$= (3\text{-}2k\alpha)(\frac{3-10k\alpha}{3}ln\frac{6+5k\alpha}{5k\alpha} + \frac{28k\alpha-9}{9}ln\frac{3+k\alpha}{k\alpha}),\ \frac{k}{2}\epsilon Z^+$$

$$\iiint_0^1 20(v_{g_1} - \frac{\frac{k\alpha}{2}-m_{g_1}}{\frac{k\alpha}{2}+m_{g_1}}v_{air})dm_{g_1}\mathrm{d}v_{g_1}\mathrm{d}v_{air}$$

$$= 10 - 20\iiint_0^1(\frac{\frac{k\alpha}{2}-m_{g_1}}{\frac{k\alpha}{2}+m_{g_1}}v_{air})dm_{g_1}\mathrm{d}v_{g_1}\mathrm{d}v_{air}$$

$$= 10 - 10\int_0^1(\frac{k\alpha-2m_{g_1}}{k\alpha+2m_{g_1}})dm_{g_1}$$

$$= 10 - 10[k\alpha\int_0^1(\frac{1}{k\alpha+2m_{g_1}})dm_{g_1} - \int_0^1(\frac{2m_{g_1}}{k\alpha+2m_{g_1}})dm_{g_1}]$$

$$= 10 - 10[k\alpha\int_0^1(\frac{1}{k\alpha+2m_{g_1}})dm_{g_1} - \int_0^1 1dm_{g_1} + k\alpha\int_0^1(\frac{1}{k\alpha+2m_{g_1}})dm_{g_1}]$$

$$= 10 - 10(k\alpha\ln\frac{2+k\alpha}{k\alpha}\text{-}1)$$

$$= 10(2 - k\alpha\ln\frac{2+k\alpha}{k\alpha}),\ \frac{k}{2}\epsilon Z^+$$

As can be seen, except that the working efficacy of the deployment 7 is definitely higher than the one of the deployment 6, it’s actually tentative whether the working efficacy of the deployment 5 is higher than the one of the deployment 8 or not, depending on the value of $k\alpha$, and the value of $k\alpha$ that makes the working efficacy of the deployment 5 higher than the one of the deployment 8 and the value of $k\alpha$ that makes the working efficacy of the deployment 5 not higher than the one of the deployment 8 are supposed to be attained by solving inequalities.

Since we can't attain the result of the comparison between the working efficacy of the deployment 5 and the one of the deployment 8 by whether the ratio of them, namely $\frac{(-\frac{11}{4}k\alpha)ln(\frac{k\alpha+2}{k\alpha})+\frac{2k\alpha}{1+k\alpha}}{(3-2k\alpha)(\frac{3-10k\alpha}{3}ln\frac{6+5k\alpha}{5k\alpha}+\frac{28k\alpha-9}{9}ln\frac{3+k\alpha}{k\alpha})}$ is larger than 1 or else, it necessitates it to solicit the result of the comparison from Inequality Scaling.

And because $\frac{k\alpha+2}{k\alpha}$, $\frac{6+5k\alpha}{5k\alpha}$, and $\frac{3+k\alpha}{k\alpha}$ are all bigger than 1, $\frac{(-\frac{11}{4}k\alpha)ln(\frac{k\alpha+2}{k\alpha})+\frac{2k\alpha}{1+k\alpha}}{(3-2k\alpha)(\frac{3-10k\alpha}{3}ln\frac{6+5k\alpha}{5k\alpha}+\frac{28k\alpha-9}{9}ln\frac{3+k\alpha}{k\alpha})}$ is definitely smaller than $\frac{(-\frac{11}{4}k\alpha)+\frac{2k\alpha}{1+k\alpha}}{(3-2k\alpha)(\frac{3-10k\alpha}{3}+\frac{28k\alpha-9}{9})}$, and if $\frac{(-\frac{11}{4}k\alpha)+\frac{2k\alpha}{1+k\alpha}}{(3-2k\alpha)(\frac{3-10k\alpha}{3}+\frac{28k\alpha-9}{9})}$ is smaller than 1, then of course so will be $\frac{(-\frac{11}{4}k\alpha)ln(\frac{k\alpha+2}{k\alpha})+\frac{2k\alpha}{1+k\alpha}}{(3-2k\alpha)(\frac{3-10k\alpha}{3}ln\frac{6+5k\alpha}{5k\alpha}+\frac{28k\alpha-9}{9}ln\frac{3+k\alpha}{k\alpha})}$, whereby we don't need to integrate the logarithms into the fraction.

$\frac{(-\frac{11}{4}k\alpha)+\frac{2k\alpha}{1+k\alpha}}{(3-2k\alpha)(\frac{3-10k\alpha}{3}+\frac{28k\alpha-9}{9})}=\frac{k\alpha[(-\frac{11}{4})+\frac{2}{1+k\alpha}]}{(3-2k\alpha)(\frac{3-10k\alpha}{3}+\frac{28k\alpha-9}{9})}$, which solicits Inequality Scaling again.

$\because -2k\alpha+3>-2k\alpha$

$\therefore \frac{k\alpha[(-\frac{11}{4})+\frac{2}{1+k\alpha}]}{(3-2k\alpha)(\frac{3-10k\alpha}{3}+\frac{28k\alpha-9}{9})}<\frac{k\alpha[(-\frac{11}{4})+\frac{2}{1+k\alpha}]}{k\alpha(\frac{3-10k\alpha}{3}+\frac{28k\alpha-9}{9})}=\frac{(-\frac{11}{4})+\frac{2}{1+k\alpha}}{\frac{3-10k\alpha}{3}+\frac{28k\alpha-9}{9}}$

If $\frac{(-\frac{11}{4})+\frac{2}{1+k\alpha}}{\frac{3-10k\alpha}{3}+\frac{28k\alpha-9}{9}}$ is smaller than 1, then of course so will be $\frac{(-\frac{11}{4}k\alpha)ln(\frac{k\alpha+2}{k\alpha})+\frac{2k\alpha}{1+k\alpha}}{(3-2k\alpha)(\frac{3-10k\alpha}{3}ln\frac{6+5k\alpha}{5k\alpha}+\frac{28k\alpha-9}{9}ln\frac{3+k\alpha}{k\alpha})}$.

$\because \frac{(-\frac{11}{4})+\frac{2}{1+k\alpha}}{\frac{3-10k\alpha}{3}+\frac{28k\alpha-9}{9}}=\frac{(-\frac{99}{4})+\frac{18}{1+k\alpha}}{9-30k\alpha+28k\alpha-9}=\frac{(-\frac{99}{4})+\frac{18}{1+k\alpha}}{-2k\alpha}=\frac{-99+\frac{72}{1+k\alpha}}{-8k\alpha}=\frac{-99(1+k\alpha)+72}{-8k\alpha}=$
$\frac{-99-99k\alpha+72}{-8k\alpha}=\frac{-27-99k\alpha}{-8k\alpha}=\frac{27+99k\alpha}{8k\alpha}>1$

Unfortunately we find out it is larger than 1, so it turns out that we can't prove

$\frac{(-\frac{11}{4}k\alpha)ln(\frac{k\alpha+2}{k\alpha})+\frac{2k\alpha}{1+k\alpha}}{(3-2k\alpha)(\frac{3-10k\alpha}{3}ln\frac{6+5k\alpha}{5k\alpha}+\frac{28k\alpha-9}{9}ln\frac{3+k\alpha}{k\alpha})}$ as smaller than 1 by scaling down $\frac{k\alpha[(-\frac{11}{4})+\frac{2}{1+k\alpha}]}{(3-2k\alpha)(\frac{3-10k\alpha}{3}+\frac{28k\alpha-9}{9})}$ to $\frac{(-\frac{11}{4})+\frac{2}{1+k\alpha}}{\frac{3-10k\alpha}{3}+\frac{28k\alpha-9}{9}}$, however the idea is elicitted that $\frac{k\alpha}{3-2k\alpha}$ *and* $\frac{(-\frac{11}{4})+\frac{2}{1+k\alpha}}{\frac{3-10k\alpha}{3}+\frac{28k\alpha-9}{9}}$ are separated from each other, so that we can know the values of $k\alpha$ that make $\frac{k\alpha[(-\frac{11}{4})+\frac{2}{1+k\alpha}]}{(3-2k\alpha)(\frac{3-10k\alpha}{3}+\frac{28k\alpha-9}{9})}$ smaller than 1.

When $\frac{k\alpha}{3-2k\alpha}\cdot\frac{(-\frac{11}{4})+\frac{2}{1+k\alpha}}{\frac{3-10k\alpha}{3}+\frac{28k\alpha-9}{9}}<1$

$\frac{k\alpha}{3-2k\alpha}<\frac{8k\alpha}{27+99k\alpha}$

$\frac{1}{3-2k\alpha}<\frac{8}{27+99k\alpha}$

$\frac{27+99k\alpha}{3-2k\alpha}<8$

$\frac{27+99k\alpha-8(3-2k\alpha)}{3-2k\alpha}<0$

$\frac{115k\alpha+3}{3-2k\alpha}<0$

$(115k\alpha+3)(2k\alpha-3)>0$

$\because\ k\alpha>0$

$\therefore$ When $k\alpha\ \geq\ \frac{3}{2}$, $\frac{(-\frac{11}{4}k\alpha)ln(\frac{k\alpha+2}{k\alpha})+\frac{2k\alpha}{1+k\alpha}}{(3-2k\alpha)(\frac{3-10k\alpha}{3}ln\frac{6+5k\alpha}{5k\alpha}+\frac{28k\alpha-9}{9}ln\frac{3+k\alpha}{k\alpha})}<1$

When $0<k\alpha<\frac{3}{2}$, $\frac{(-\frac{11}{4}k\alpha)ln(\frac{k\alpha+2}{k\alpha})+\frac{2k\alpha}{1+k\alpha}}{(3-2k\alpha)(\frac{3-10k\alpha}{3}ln\frac{6+5k\alpha}{5k\alpha}+\frac{28k\alpha-9}{9}ln\frac{3+k\alpha}{k\alpha})}>1$

And $\because$ When $k\alpha\ \geq\ \frac{3}{2}$, $\iiint_0^1[\frac{4m_{g_1}}{m_{g_1}+\frac{k\alpha}{2}}(v_{g_1}-\frac{\frac{k\alpha}{2}-m_{g_1}}{\frac{k\alpha}{2}+m_{g_1}}v_{air})-\frac{\frac{k\alpha}{2}-2m_{g_1}}{\frac{k\alpha}{2}+m_{g_1}}v_{air}]dm_{g_1}\,dv_{g_1}dv_{air}$ $<$ $\iiint_0^1\{\frac{2m_{g_1}}{m_{g_1}+\frac{k\alpha}{3}}[\frac{2m_{g_1}}{m_{g_1}+\frac{5k\alpha}{6}}(v_{g_1}-\frac{\frac{k\alpha}{3}-m_{g_1}}{\frac{k\alpha}{3}+m_{g_1}}v_{air})-\frac{\frac{5k\alpha}{6}-m_{g_1}}{\frac{5k\alpha}{6}+m_{g_1}}v_{air}]-\frac{\frac{k\alpha}{3}-m_{g_1}}{\frac{k\alpha}{3}+m_{g_1}}v_{air}\}dm_{g_1}\,dv_{g_1}dv_{air}$

$\therefore$ When $k\alpha \geq \frac{3}{2}$, $\frac{4m_{g_1}}{m_{g_1}+\frac{k\alpha}{2}}(v_{g_1} - \frac{\frac{k\alpha}{2}-m_{g_1}}{\frac{k\alpha}{2}+m_{g_1}}v_{air}) - \frac{\frac{k\alpha}{2}-2m_{g_1}}{\frac{k\alpha}{2}+m_{g_1}}v_{air} < \frac{2m_{g_1}}{m_{g_1}+\frac{k\alpha}{3}}[\frac{2m_{g_1}}{m_{g_1}+\frac{5k\alpha}{6}}(v_{g_1} - \frac{\frac{k\alpha}{3}-m_{g_1}}{\frac{k\alpha}{3}+m_{g_1}}v_{air}) - \frac{\frac{5k\alpha}{6}-m_{g_1}}{\frac{5k\alpha}{6}+m_{g_1}}v_{air}] - \frac{\frac{k\alpha}{3}-m_{g_1}}{\frac{k\alpha}{3}+m_{g_1}}v_{air}$

$\because$ When $k\alpha < \frac{3}{2}$, $\iiint_0^1[\frac{4m_{g_1}}{m_{g_1}+\frac{k\alpha}{2}}(v_{g_1} - \frac{\frac{k\alpha}{2}-m_{g_1}}{\frac{k\alpha}{2}+m_{g_1}}v_{air}) - \frac{\frac{k\alpha}{2}-2m_{g_1}}{\frac{k\alpha}{2}+m_{g_1}}v_{air}]dm_{g_1}\mathrm{d}v_{g_1}\mathrm{d}v_{air} > \iiint_0^1\{\frac{2m_{g_1}}{m_{g_1}+\frac{k\alpha}{3}}[\frac{2m_{g_1}}{m_{g_1}+\frac{5k\alpha}{6}}(v_{g_1} - \frac{\frac{k\alpha}{3}-m_{g_1}}{\frac{k\alpha}{3}+m_{g_1}}v_{air}) - \frac{\frac{5k\alpha}{6}-m_{g_1}}{\frac{5k\alpha}{6}+m_{g_1}}v_{air}] - \frac{\frac{k\alpha}{3}-m_{g_1}}{\frac{k\alpha}{3}+m_{g_1}}v_{air}\}dm_{g_1}\mathrm{d}v_{g_1}\mathrm{d}v_{air}$

$\therefore$ When $k\alpha < \frac{3}{2}$, $\frac{4m_{g_1}}{m_{g_1}+\frac{k\alpha}{2}}(v_{g_1} - \frac{\frac{k\alpha}{2}-m_{g_1}}{\frac{k\alpha}{2}+m_{g_1}}v_{air}) - \frac{\frac{k\alpha}{2}-2m_{g_1}}{\frac{k\alpha}{2}+m_{g_1}}v_{air} > \frac{2m_{g_1}}{m_{g_1}+\frac{k\alpha}{3}}[\frac{2m_{g_1}}{m_{g_1}+\frac{5k\alpha}{6}}(v_{g_1} - \frac{\frac{k\alpha}{3}-m_{g_1}}{\frac{k\alpha}{3}+m_{g_1}}v_{air}) - \frac{\frac{5k\alpha}{6}-m_{g_1}}{\frac{5k\alpha}{6}+m_{g_1}}v_{air}] - \frac{\frac{k\alpha}{3}-m_{g_1}}{\frac{k\alpha}{3}+m_{g_1}}v_{air}$

Therefore, when $k\alpha \geq \frac{3}{2}$, the working efficacy of the deployment 5 is lower than the one of the deployment 8; when $0 < k\alpha < \frac{3}{2}$, the working efficacy of the deployment 5 is higher than the one of the deployment 8.

And it's better to straightforward compare the working efficacy of the deployment 8 with the one of the deployment 7 by subtraction than doing so however with having beforehand and cumbersomely compared the working efficacy of one deployment except the deployment 8 with the one of another except the deployment 7.

When $k\alpha \geq \frac{3}{2}$,

$\because (3 - 2k\alpha)(\frac{3-10k\alpha}{3}ln\frac{6+5k\alpha}{5k\alpha} + \frac{28k\alpha-9}{9}ln\frac{3+k\alpha}{k\alpha}) - [3 - 2k\alpha ln(\frac{2+k\alpha}{k\alpha})]$

$< (3 - 2k\alpha)(\frac{3-10k\alpha}{3}ln\frac{6+5k\alpha}{5k\alpha} + \frac{28k\alpha-9}{9}ln\frac{3+k\alpha}{k\alpha}) - (3 - 2k\alpha)$

$= (3 - 2k\alpha)(\frac{3-10k\alpha}{3}ln\frac{6+5k\alpha}{5k\alpha} + \frac{28k\alpha-9}{9}ln\frac{3+k\alpha}{k\alpha}$-1)

$< (3 - 2k\alpha)(\frac{3-10k\alpha}{3} + \frac{28k\alpha-9}{9}$-1)

$= (3 - 2k\alpha)(\frac{3-10k\alpha}{3} + \frac{28k\alpha-9}{9}$-1)

$= (3 - 2k\alpha)(-9 - 2k\alpha), \frac{k}{2}\epsilon Z^{+}$

And $\because k\alpha \geq \frac{3}{2}$

$\therefore \{3 - 2k\alpha \leq 0 \quad -9 - 2k\alpha \leq -12$

$\therefore (3 - 2k\alpha)(-9 - 2k\alpha) \leq 0$

$\therefore (3 - 2k\alpha)(\frac{3-10k\alpha}{3} ln\frac{6+5k\alpha}{5k\alpha} + \frac{28k\alpha-9}{9} ln\frac{3+k\alpha}{k\alpha}) < 3 - 2k\alpha ln(\frac{2+k\alpha}{k\alpha}), \frac{k}{2}\epsilon Z^{+}$

Therefore the working efficacy of the deployment 8 is higher than the one of the deployment 7, so we are supposed to compare the one of the deployment 8 with the one of the deployment 9.

But if we know which one is higher still by definite integral of three variables, the definite integral of three variables of the working efficacy of the deployment 9 will be easy to attain, but the one of the working efficacy of the deployment 8 will be extremely hard to attain, so, so as to simplify the calculations, we can firstly attain the definite integrals of only v_{g_1} and v_{air} of the working efficacies so that the difficulty of the integration of the working efficacy of the deployment 9 is lowered extremely a lot since m_{g_1} is constantized when integrating the working efficacy, then see what ranges of value m_{g_1} lies within so that one of the working efficacies is higher than or lower than or equal to the other, and finally see whether m_{g_1} can lie within the ranges of value. If m_{g_1} can't lie within certain ranges of value, then it means one of the working efficacies is always higher than or lower than or equal to the other.

$$\iiint_0^1 m_{g_1}\{\frac{2m_{g_1}}{m_{g_1}+\frac{k\alpha}{3}}[\frac{2m_{g_1}}{m_{g_1}+\frac{5k\alpha}{6}}(v_{g_1} - \frac{\frac{k\alpha}{3}-m_{g_1}}{\frac{k\alpha}{3}+m_{g_1}}v_{air}) - \frac{\frac{5k\alpha}{6}-m_{g_1}}{\frac{5k\alpha}{6}+m_{g_1}}v_{air}] - \frac{\frac{k\alpha}{3}-m_{g_1}}{\frac{k\alpha}{3}+m_{g_1}}v_{air}\}dv_{g_1}dv_{air}$$

$=$

$$m_{g_1}\frac{2m_{g_1}}{m_{g_1}+\frac{k\alpha}{3}}\iiint_0^1[\frac{2m_{g_1}}{m_{g_1}+\frac{5k\alpha}{6}}(v_{g_1} - \frac{\frac{k\alpha}{3}-m_{g_1}}{\frac{k\alpha}{3}+m_{g_1}}v_{air}) - \frac{\frac{5k\alpha}{6}-m_{g_1}}{\frac{5k\alpha}{6}+m_{g_1}}v_{air}]dv_{g_1}dv_{air} - m_{g_1}\iiint_0^1\frac{\frac{k\alpha}{3}-m_{g_1}}{\frac{k\alpha}{3}+m_{g_1}}v_{air}dv_{g_1}d$$

$$= m_{g_1}\frac{2m_{g_1}}{m_{g_1}+\frac{k\alpha}{3}}\frac{2m_{g_1}}{m_{g_1}+\frac{5k\alpha}{6}}\iiint_0^1\left(v_{g_1} - \frac{\frac{k\alpha}{3}-m_{g_1}}{\frac{k\alpha}{3}+m_{g_1}}v_{air}\right)dv_{g_1}dv_{air} -$$

$$m_{g_1}\frac{2m_{g_1}}{m_{g_1}+\frac{k\alpha}{3}}\iiint_0^1\frac{\frac{5k\alpha}{6}-m_{g_1}}{\frac{5k\alpha}{6}+m_{g_1}}v_{air}dv_{g_1}dv_{air} - m_{g_1}\iiint_0^1\frac{\frac{k\alpha}{3}-m_{g_1}}{\frac{k\alpha}{3}+m_{g_1}}v_{air}dv_{g_1}dv_{air}$$

$$= m_{g_1}\frac{2m_{g_1}}{m_{g_1}+\frac{k\alpha}{3}}\frac{2m_{g_1}}{m_{g_1}+\frac{5k\alpha}{6}}\iiint_0^1 v_{g_1}\,dv_{g_1}dv_{air} -$$

$$m_{g_1}\frac{2m_{g_1}}{m_{g_1}+\frac{k\alpha}{3}}\frac{2m_{g_1}}{m_{g_1}+\frac{5k\alpha}{6}}\iiint_0^1\frac{\frac{k\alpha}{3}-m_{g_1}}{\frac{k\alpha}{3}+m_{g_1}}v_{air}dv_{g_1}dv_{air} - m_{g_1}\frac{2m_{g_1}}{m_{g_1}+\frac{k\alpha}{3}}\iiint_0^1\frac{\frac{5k\alpha}{6}-m_{g_1}}{\frac{5k\alpha}{6}+m_{g_1}}v_{air}dv_{g_1}dv_{air}$$

$$- m_{g_1}\iiint_0^1\frac{\frac{k\alpha}{3}-m_{g_1}}{\frac{k\alpha}{3}+m_{g_1}}v_{air}dv_{g_1}dv_{air}$$

$$= \frac{2m_{g_1}^{\ 3}}{(m_{g_1}+\frac{k\alpha}{3})(m_{g_1}+\frac{5k\alpha}{6})} - \frac{2m_{g_1}^{\ 3}}{(m_{g_1}+\frac{k\alpha}{3})(m_{g_1}+\frac{5k\alpha}{6})}\frac{\frac{k\alpha}{3}-m_{g_1}}{\frac{k\alpha}{3}+m_{g_1}} - \frac{m_{g_1}^{\ 2}(\frac{5k\alpha}{6}-m_{g_1})}{(m_{g_1}+\frac{k\alpha}{3})(m_{g_1}+\frac{5k\alpha}{6})} - \frac{\frac{k\alpha}{3}m_{g_1}-m_{g_1}^{\ 2}}{2(\frac{k\alpha}{3}+m_{g_1})}$$

$$= \frac{2m_{g_1}^{\ 3}}{(m_{g_1}+\frac{k\alpha}{3})(m_{g_1}+\frac{5k\alpha}{6})}\left(1-\frac{\frac{k\alpha}{3}-m_{g_1}}{\frac{k\alpha}{3}+m_{g_1}}\right) - \frac{m_{g_1}}{m_{g_1}+\frac{k\alpha}{3}}\left(\frac{m_{g_1}(\frac{5k\alpha}{6}-m_{g_1})}{m_{g_1}+\frac{5k\alpha}{6}} - \frac{\frac{k\alpha}{3}-m_{g_1}}{2}\right)$$

$$= \frac{m_{g_1}}{m_{g_1}+\frac{k\alpha}{3}}\left[\frac{2m_{g_1}^{\ 2}}{m_{g_1}+\frac{5k\alpha}{6}}\left(1-\frac{\frac{k\alpha}{3}-m_{g_1}}{\frac{k\alpha}{3}+m_{g_1}}\right) - \frac{m_{g_1}(\frac{5k\alpha}{6}-m_{g_1})}{m_{g_1}+\frac{5k\alpha}{6}} + \frac{\frac{k\alpha}{3}-m_{g_1}}{2}\right]$$

$$= \frac{m_{g_1}}{m_{g_1}+\frac{k\alpha}{3}}\left\{\frac{1}{m_{g_1}+\frac{5k\alpha}{6}}\left[2m_{g_1}^{\ 2}\left(1-\frac{\frac{k\alpha}{3}-m_{g_1}}{\frac{k\alpha}{3}+m_{g_1}}\right)-m_{g_1}\left(\frac{5k\alpha}{6}-m_{g_1}\right)\right]+\frac{\frac{k\alpha}{3}-m_{g_1}}{2}\right\}$$

$$= \frac{m_{g_1}}{m_{g_1}+\frac{k\alpha}{3}}\left[\frac{1}{m_{g_1}+\frac{5k\alpha}{6}}\left(3m_{g_1}^{\ 2} - 2m_{g_1}^{\ 2}\frac{\frac{k\alpha}{3}-m_{g_1}}{\frac{k\alpha}{3}+m_{g_1}} - m_{g_1}\frac{5k\alpha}{6}\right) + \frac{\frac{k\alpha}{3}-m_{g_1}}{2}\right]$$

$$= \frac{m_{g_1}}{m_{g_1}+\frac{k\alpha}{3}}\left(\frac{3m_{g_1}^{\ 2}}{m_{g_1}+\frac{5k\alpha}{6}} - \frac{2m_{g_1}^{\ 2}}{m_{g_1}+\frac{5k\alpha}{6}}\frac{\frac{k\alpha}{3}-m_{g_1}}{\frac{k\alpha}{3}+m_{g_1}} - \frac{m_{g_1}}{\frac{6}{5k\alpha}m_{g_1}+1} + \frac{\frac{k\alpha}{3}-m_{g_1}}{2}\right)$$

$$= \frac{m_{g_1}}{m_{g_1}+\frac{k\alpha}{3}}\left(\frac{3m_{g_1}^{\ 2}}{m_{g_1}+\frac{5k\alpha}{6}} - \frac{\frac{2k\alpha}{3}m_{g_1}^{\ 2}}{(m_{g_1}+\frac{5k\alpha}{6})(\frac{k\alpha}{3}+m_{g_1})} + \frac{2m_{g_1}^{\ 3}}{(m_{g_1}+\frac{5k\alpha}{6})(\frac{k\alpha}{3}+m_{g_1})} - \frac{m_{g_1}}{\frac{6}{5k\alpha}m_{g_1}+1} + \frac{\frac{k\alpha}{3}-m_{g_1}}{2}\right)$$

$$= \frac{3m_{g_1}}{3m_{g_1}+k\alpha}\left(\frac{18m_{g_1}^{\ 2}}{6m_{g_1}+5k\alpha} + \frac{8m_{g_1}^{\ 2}}{5k\alpha+6m_{g_1}} - \frac{4m_{g_1}^{\ 2}}{k\alpha+3m_{g_1}} + \frac{24m_{g_1}^{\ 3}}{5k^2\alpha^2+6k\alpha m_{g_1}} - \frac{12m_{g_1}^{\ 3}}{k^2\alpha^2+3k\alpha m_{g_1}} - \frac{m_{g_1}}{\frac{6}{5k\alpha}m_{g_1}+1}\right.$$

$$\left. + \frac{k\alpha-3m_{g_1}}{6}\right), \frac{k}{2}\epsilon Z^+$$

$\because$

$$\{0 < \frac{3m_{g_1}}{3m_{g_1}+k\alpha} \leq \frac{2m_{g_1}}{2m_{g_1}+1} \quad 0 < \frac{18m_{g_1}^{\ 2}}{6m_{g_1}+5k\alpha} \leq \frac{12m_{g_1}^{\ 2}}{4m_{g_1}+5} \quad 0 < \frac{8m_{g_1}^{\ 2}}{5k\alpha+6m_{g_1}} \leq \frac{16m_{g_1}^{\ 2}}{12m_{g_1}+15} \quad -\frac{8m_{g_1}^{\ 2}}{6m_{g_1}+3} \leq$$

$$\therefore \quad \frac{2m_{g_1}}{2m_{g_1}+1}\left(-\frac{8m_{g_1}^{\ 2}-16m_{g_1}^{\ 3}}{6m_{g_1}+3} - \frac{5m_{g_1}}{4m_{g_1}+5} - \frac{1}{2}m_{g_1} + \frac{1}{4}\right) \quad \leq \quad \frac{3m_{g_1}}{3m_{g_1}+k\alpha}\left(\frac{18m_{g_1}^{\ 2}}{6m_{g_1}+5k\alpha}\right.$$

$$\left. + \frac{8m_{g_1}^{\ 2}}{5k\alpha+6m_{g_1}} - \frac{4m_{g_1}^{\ 2}}{k\alpha+3m_{g_1}} + \frac{24m_{g_1}^{\ 3}}{5k^2\alpha^2+6k\alpha m_{g_1}} - \frac{12m_{g_1}^{\ 3}}{k^2\alpha^2+3k\alpha m_{g_1}} - \frac{m_{g_1}}{\frac{6}{5k\alpha}m_{g_1}+1} + \frac{k\alpha-3m_{g_1}}{6}\right) \leq \frac{2m_{g_1}}{2m_{g_1}+1}$$

$$\left(\frac{2m_{g_1}}{2m_{g_1}+1} + \frac{12m_{g_1}^{\ 2}}{4m_{g_1}+5} + \frac{16m_{g_1}^{\ 2}+32m_{g_1}^{\ 3}}{12m_{g_1}+15}\right), \frac{k}{2}\epsilon Z^{+}$$

And $\because \iiint_0^1 20\left(v_{g_1} - \frac{\frac{k\alpha}{2}-m_{g_1}}{\frac{k\alpha}{2}+m_{g_1}}v_{air}\right)dv_{g_1}dv_{air}$

$$= 10 - 20\iiint_0^1\left(\frac{\frac{k\alpha}{2}-m_{g_1}}{\frac{k\alpha}{2}+m_{g_1}}v_{air}\right)dv_{g_1}dv_{air}$$

$$= 10 - 10\left(\frac{k\alpha-2m_{g_1}}{k\alpha+2m_{g_1}}\right)$$

$$= 10 - \frac{10}{1+\frac{2m_{g_1}}{k\alpha}} + \frac{20m_{g_1}}{k\alpha+2m_{g_1}}, \frac{k}{2}\epsilon Z^{+}$$

And $\because$ $0 < 10 - \frac{10}{1+\frac{2m_{g_1}}{k\alpha}} + \frac{20m_{g_1}}{k\alpha+2m_{g_1}} \leq 10 + \frac{10}{3+4m_{g_1}}, \frac{k}{2}\epsilon Z^{+}$

$\therefore$ It’s tentative whether the working efficacy of the deployment 8 is higher than the one of the deployment 9 or not. We need to discuss the results of the comparison betwen the working efficacy of the deployment 8 and the one of the deployment 9 by the category of what ranges of value m_{g_1} lies within. However it’s impossible to attain the solution sets of the ineqaulity $\frac{2m_{g_1}}{2m_{g_1}+1}\left(-\frac{8m_{g_1}^{\ 2}-16m_{g_1}^{\ 3}}{6m_{g_1}+3} - \frac{5m_{g_1}}{4m_{g_1}+5} - \frac{1}{2}m_{g_1} + \frac{1}{4}\right)$ - $10 + \frac{10}{3+4m_{g_1}} \geq 0$ and the inequality $\frac{2m_{g_1}}{2m_{g_1}+1}\left(\frac{2m_{g_1}}{2m_{g_1}+1} + \frac{12m_{g_1}^{\ 2}}{4m_{g_1}+5} + \frac{16m_{g_1}^{\ 2}+32m_{g_1}^{\ 3}}{12m_{g_1}+15}\right) - 0 \leq 0$ by attaining the roots of the equation $\frac{2m_{g_1}}{2m_{g_1}+1}\left(-\frac{8m_{g_1}^{\ 2}-16m_{g_1}^{\ 3}}{6m_{g_1}+3} - \frac{5m_{g_1}}{4m_{g_1}+5} - \frac{1}{2}m_{g_1} + \frac{1}{4}\right.$

) - 10 + $\frac{10}{3+4m_{g_1}}$ = 0 and the ones of the equation $\frac{2m_{g_1}}{2m_{g_1}+1}(\frac{2m_{g_1}}{2m_{g_1}+1} + \frac{12m_{g_1}^{2}}{4m_{g_1}+5} + \frac{16m_{g_1}^{2}+32m_{g_1}^{3}}{12m_{g_1}+15})$ = 0, which are impossible to attain after all. We have to let these two equations be functions and by the properties of these two fuctions, we can attain the solution sets.

$$\because \frac{d[\frac{2m_{g_1}}{2m_{g_1}+1}(-\frac{8m_{g_1}^{2}-16m_{g_1}^{3}}{6m_{g_1}+3}-\frac{5m_{g_1}}{4m_{g_1}+5}-\frac{1}{2}m_{g_1}+\frac{1}{4})-10+\frac{10}{3+4m_{g_1}}]}{dm_{g_1}}$$

$$= -\frac{8m_{g_1}^{2}-16m_{g_1}^{3}}{6m_{g_1}+3} - \frac{5m_{g_1}}{4m_{g_1}+5} + \frac{m_{g_1}}{2} + \frac{10}{4m_{g_1}+3} - \frac{39}{4}$$

If we let $\frac{2m_{g_1}}{2m_{g_1}+1}(-\frac{8m_{g_1}^{2}-16m_{g_1}^{3}}{6m_{g_1}+3} - \frac{5m_{g_1}}{4m_{g_1}+5} - \frac{1}{2}m_{g_1} + \frac{1}{4})$ - 10 + $\frac{10}{3+4m_{g_1}}$ be f(m_{g_1}) as a function, then we can figure out that f(m_{g_1}) increases after $m_{g_1} \approx 2.25045$ and f(2.25045) = 0, so when m_{g_1} > about 2.25045, the working efficacy of the deployment 8 is higher than the one of the deployment 9.

By the same method, we can also figure out when $about\ 2.25045 > m_{g_1} > 0$, the working efficacy of the deployment 9 is higher than the one of the deployment 8, and when m_{g_1} is equal to about 2.25045, they are the same.

When $0 < k\alpha < \frac{3}{2}$,

$\because (3 - 2k\alpha)(\frac{3-10k\alpha}{3}ln\frac{6+5k\alpha}{5k\alpha} + \frac{28k\alpha-9}{9}ln\frac{3+k\alpha}{k\alpha}) - [3 - 2k\alpha ln(\frac{2+k\alpha}{k\alpha})]$

$> (-k\alpha)(3 - 10k\alpha+28k\alpha - 9)-3$

$= (-k\alpha)(18k\alpha-6)-3$

$= -18k^2\alpha^2 + 6k\alpha - 3$

$= 3(-6k^2\alpha^2+2k\alpha-1)$

$= 3(k\alpha-\frac{1+\sqrt{5}i}{6})(k\alpha - \frac{1-\sqrt{5}i}{6}), \frac{k}{2}\epsilon Z^{+}$

And $\because$ When $k\alpha = 0$, $3(k\alpha\text{-}\frac{1+\sqrt{5}i}{6})(k\alpha - \frac{1-\sqrt{5}i}{6}) > 0$

$\therefore$ $(3 - 2k\alpha)(\frac{3-10k\alpha}{3}ln\frac{6+5k\alpha}{5k\alpha} + \frac{28k\alpha-9}{9}ln\frac{3+k\alpha}{k\alpha}) > [3 - 2k\alpha ln(\frac{2+k\alpha}{k\alpha})]$, $\frac{k}{2}\epsilon Z^{+}$

Therefore, when $0 < k\alpha < \frac{3}{2}$, the working efficacy of the deployment 7 is higher than the one of the deployment 8, thus we should compare the working efficacy of the deployment 7 with the one of the deployment 9.

$\because \iint_{0}^{1} m_{g_1} (\frac{4m_{g_1}}{m_{g_1}+\frac{k\alpha}{2}} v_{g_1} - \frac{k\alpha - m_{g_1}}{\frac{k\alpha}{2}+m_{g_1}} v_{air}) dv_{g_1} dv_{air}$

$= \frac{4m_{g_1}^{2}}{2m_{g_1}+k\alpha} - \frac{m_{g_1}(k\alpha - m_{g_1})}{k\alpha+2m_{g_1}}$

$= \frac{5m_{g_1}^{2} - k\alpha m_{g_1}}{2m_{g_1}+k\alpha}$, $\frac{k}{2}\epsilon Z^{+}$

$\frac{10m_{g_1}^{2} - 3m_{g_1}}{3+4m_{g_1}} < \frac{5m_{g_1}^{2} - k\alpha m_{g_1}}{2m_{g_1}+k\alpha} < \frac{5m_{g_1}}{2}$, $\frac{k}{2}\epsilon Z^{+}$

And $\because \iiint_{0}^{1} 20(v_{g_1} - \frac{\frac{k\alpha}{2}-m_{g_1}}{\frac{k\alpha}{2}+m_{g_1}} v_{air}) dv_{g_1} dv_{air}$

$= 10 - 20\iiint_{0}^{1} (\frac{\frac{k\alpha}{2}-m_{g_1}}{\frac{k\alpha}{2}+m_{g_1}} v_{air}) dv_{g_1} dv_{air}$

$= 10 - 10(\frac{k\alpha - 2m_{g_1}}{k\alpha+2m_{g_1}})$

$= 10 - \frac{10}{1+\frac{2m_{g_1}}{k\alpha}} + \frac{20m_{g_1}}{k\alpha+2m_{g_1}}$, $\frac{k}{2}\epsilon Z^{+}$

$0 < 10 - \frac{10}{1+\frac{2m_{g_1}}{k\alpha}} + \frac{20m_{g_1}}{k\alpha+2m_{g_1}} \le 10 + \frac{10}{3+4m_{g_1}}$, $\frac{k}{2}\epsilon Z^{+}$

$\therefore$ let $\frac{10m_{g_1}^{2} - 3m_{g_1} - 10}{3+4m_{g_1}}$ - 10 be $f(m_{g_1})$, when $m_{g_1} > \frac{3}{20} + \frac{\sqrt{409}}{20}$, $f(m_{g_1})$ increases, and

$\frac{10m_{g_1}^2-3m_{g_1}-10}{3+4m_{g_1}}$ - 10>0. Therefore, when $m_{g_1} > \frac{3}{20} + \frac{\sqrt{409}}{20}$, the working efficacy of the deployment 7 is higher than the one of the deployment 9; let $-\frac{5m_{g_1}}{2}$ be g(m_{g_1}), g(m_{g_1}) decreases, and when $m_{g_1} < 0$, $-\frac{5m_{g_1}}{2} > 0$, but m_{g_1} can't be negative, so there is no range of values within which m_{g_1} lies so that the working efficacy of the deployment 9 is higher than the one of the deployment 7. When $m_{g_1} = \frac{3}{20} + \frac{\sqrt{409}}{20}$, the working efficacy of the deployment 7 is equal to the one of the deployment 9.

$$\because \iint_0^1 m_{g_1}\left[\frac{4m_{g_1}}{m_{g_1}+\frac{k\alpha}{2}}\left(v_{g_1} - \frac{\frac{k\alpha}{2}-m_{g_1}}{\frac{k\alpha}{2}+m_{g_1}}v_{air}\right) - \frac{\frac{k\alpha}{2}-2m_{g_1}}{\frac{k\alpha}{2}+m_{g_1}}v_{air}\right]dv_{g_1}\,dv_{air}$$

$$= \frac{6m_{g_1}^2-\frac{k\alpha}{2}m_{g_1}}{2m_{g_1}+k\alpha} - \frac{4m_{g_1}^2(\frac{k\alpha}{2}-m_{g_1})}{(m_{g_1}+\frac{k\alpha}{2})(2m_{g_1}+k\alpha)},\ \frac{k}{2}\epsilon Z^+$$

$$\iint_0^1 20\left(v_{g_1} - \frac{\frac{k\alpha}{2}-m_{g_1}}{\frac{k\alpha}{2}+m_{g_1}}v_{air}\right)\mathrm{d}v_{g_1}\,\mathrm{d}v_{air}$$

$$= 10 - 20\iiint_0^1\left(\frac{\frac{k\alpha}{2}-m_{g_1}}{\frac{k\alpha}{2}+m_{g_1}}v_{air}\right)\mathrm{d}v_{g_1}\,\mathrm{d}v_{air}$$

$$= 10 - \frac{10k\alpha-20m_{g_1}}{k\alpha+2m_{g_1}},\ \frac{k}{2}\epsilon Z^+$$

And let $\frac{6m_{g_1}^2-\frac{k\alpha}{2}m_{g_1}-20m_{g_1}+10k\alpha}{2m_{g_1}+k\alpha} - \frac{4m_{g_1}^2(\frac{k\alpha}{2}-m_{g_1})}{(m_{g_1}+\frac{k\alpha}{2})(2m_{g_1}+k\alpha)}$ - 10 be f(m_{g_1})

$\because$ f'(m_{g_1}) = $\frac{80m_{g_1}^3+120k\alpha m_{g_1}^2-k^2\alpha^2(80+k\alpha)+2k\alpha m_{g_1}(-80+3k\alpha)}{2(2m_{g_1}+k\alpha)^3}$, f'(0) ≈ 0, and f''($m_{g_1}$) = $\frac{2k\alpha[k\alpha(80+3k\alpha)+2m_{g_1}(80+27k\alpha)]}{(2m_{g_1}+k\alpha)^4}$, f''(0) > 0

∴ x = 0 is one of the points at which f(m_{g_1}) reaches the minimal value

And ∵ f(0) = 0

$$\therefore \frac{6m_{g_1}^{\ 2} - \frac{k\alpha}{2}m_{g_1} - 20m_{g_1} + 10k\alpha}{2m_{g_1} + k\alpha} - \frac{4m_{g_1}^{\ 2}(\frac{k\alpha}{2} - m_{g_1})}{(m_{g_1} + \frac{k\alpha}{2})(2m_{g_1} + k\alpha)} - 10 > 0,\ \frac{k}{2} \epsilon Z^+$$

Therefore, the working efficacy of the deployment 5 is higher than the one of the deployment 9.

$$\because \iint_0^1 m_{g_1}\left(\frac{4m_{g_1}}{m_{g_1} + \frac{k\alpha}{2}} v_{g_1} - \frac{k\alpha - m_{g_1}}{\frac{k\alpha}{2} + m_{g_1}} v_{air}\right) dv_{g_1} dv_{air}$$

$$= \frac{4m_{g_1}^{\ 2}}{2m_{g_1} + k\alpha} - \frac{k\alpha m_{g_1} - m_{g_1}^{\ 2}}{2m_{g_1} + k\alpha}$$

$$= \frac{5m_{g_1}^{\ 2} - k\alpha m_{g_1}}{2m_{g_1} + k\alpha},\ \frac{k}{2} \epsilon Z^+$$

And let $\frac{m_{g_1}^{\ 2} + \frac{k\alpha}{2}m_{g_1}}{2m_{g_1} + k\alpha} - \frac{4m_{g_1}^{\ 2}(\frac{k\alpha}{2} - m_{g_1})}{(m_{g_1} + \frac{k\alpha}{2})(2m_{g_1} + k\alpha)}$ be g(m_{g_1})

$$\because g'(m_{g_1}) = \frac{k^3\alpha^3 - 10k^2\alpha^2 m_{g_1} + 60k\alpha m_{g_1}^{\ 2} + 40m_{g_1}^{\ 3}}{2(k\alpha + 2m_{g_1})^3},\ g''(m_{g_1}) = -\frac{8k^2\alpha^2(k\alpha - 10m_{g_1})}{(k\alpha + 2m_{g_1})^4}$$

∵ When $m_{g_1} < \frac{k\alpha}{10}$, g''($m_{g_1}$) > 0; when $m_{g_1} > \frac{k\alpha}{10}$, g''($m_{g_1}$) < 0

And ∵ $m_{g_1} > \frac{3}{20} + \frac{\sqrt{409}}{20} > \frac{k\alpha}{10}$

∴ g'(m_{g_1}) < 0

And ∵ g($\frac{3}{20} + \frac{\sqrt{409}}{20}$) > 0, and there is no root that is larger than $\frac{3}{20} + \frac{\sqrt{409}}{20}$ and can make g(m_{g_1}) 0

∴ When $m_{g_1} > \frac{3}{20} + \frac{\sqrt{409}}{20}$, g($m_{g_1}$) > 0

Therefore, the working efficacy of the deployment 7 is higher than the one of the deployment 5.

To recap, the working efficacy of the deployment 7 is definitely higher than the one of the deployment 6. When $k\alpha \geq \frac{3}{2}$, the working efficacy of the deployment 7 is higher than the one of the deployment 5, and the one of the deployment 8 is higher than the one of the deployment 7, and when m_{g_1} > about 2.25045, the working efficacy of the deployment 8 is higher than the one of the deployment 9, and when $about\ 2.25045 > m_{g_1} > 0$, the working efficacy of the deployment 9 is higher than the one of the deployment 8, and when m_{g_1} is equal to about 2.25045, they are the same; when $0 < k\alpha < \frac{3}{2}$, the working efficacy of the deployment 5 is higher than the one of the deployment 8, and when $m_{g_1} > \frac{3}{20} + \frac{\sqrt{409}}{20}$, the working efficacy of the deployment 7 is higher than the one of the deployment 9, the working efficacy of the deployment 5 is higher than the one of the deployment 9 and the working efficacy of the deployment 7 is higher than the one of the deployment 5. And when $m_{g_1} < 0$, $-\frac{5m_{g_1}}{2} > 0$, but m_{g_1} can't be negative, so there is no range of values within which m_{g_1} lies so that the working efficacy of the deployment 7 is higher than the one of the deployment 9. When $m_{g_1} = \frac{3}{20} + \frac{\sqrt{409}}{20}$, the working efficacy of the deployment 7 is equal to the one of the deployment 9, and both the working efficacy of the deployment 7 and the one of the deployment 9 are higher than the one of the deployment 5.

6. Gradient

Since the total momentum of a gear achieves is quantitatively positively related to $\frac{2m_g}{m_g+m_{air}}(\frac{m_{air}-m_g}{m_{air}+m_g}v_{air})$ and quantitatively negatively related to $\frac{m_{air}-m_g}{m_{air}+m_g}v_{air}$, suppose $f(m_g, m_{air}, v_{air}) = \frac{2m_g}{m_g+m_{air}}(\frac{m_{air}-m_g}{m_{air}+m_g}v_{air})$ and $g(m_g, m_{air}, v_{air}) = \frac{m_{air}-m_g}{m_{air}+m_g}v_{air}$.

Obviously we have to make $f(m_g, m_{air}, v_{air})$ as large as possible, and $g(m_g, m_{air}, v_{air})$ as small as possible.

$$\because \nabla f = \frac{\partial f}{\partial x}\vec{x} + \frac{\partial f}{\partial y}\vec{y} + \frac{\partial f}{\partial z}\vec{z}$$

$$\therefore \nabla f = \frac{\partial f}{\partial m_g}\vec{x} + \frac{\partial f}{\partial m_{air}}\vec{y} + \frac{\partial f}{\partial v_{air}}\vec{z}$$

When $\nabla f = \frac{\partial f}{\partial m_g}\vec{x} + \frac{\partial f}{\partial m_{air}}\vec{y} + \frac{\partial f}{\partial v_{air}}\vec{z} = \vec{0}$

$$\{\frac{\partial f}{\partial m_g} = 0 \ \frac{\partial f}{\partial m_{air}} = 0 \ \frac{\partial f}{\partial v_{air}} = 0$$

$\therefore$

$$\{2v_{air}[\frac{m_{air}(m_{air}+m_g)-2m_{air}m_g+m_g-m_{air}}{(m_g+m_{air})^3}] = 0 \ 2m_g v_{air}[\frac{(m_g+m_{air})^2-2(m_{air}-m_g)(m_g+m_{air})}{(m_g+m_{air})^4}] = 0 \ 2\frac{m_g(m_{air}-m_g)}{(m_g+m_{air})^2} =$$

$\therefore$

$$\{m_{air}(m_{air}+m_g) - 2m_{air}m_g + m_g - m_{air} = 0 \ (m_g+m_{air})^2 - 2(m_{air}-m_g)(m_g+m_{air}) = 0 \ m_a$$

$\therefore m_{air} = m_g = 0$ which is impossible.

We have known that this function does not have any stationary point. Although we can't obtain the stationary points of this function by gradient, Lagrange Multiplier Method can be derived and it is applicable for obtaining the extrema.[10]

However, without any constraint condition, it can't be applied.

To derive a constraint condition, we have to apply Conservation of momentum:

Firstly we have to isolate the 'starting gear' of a machine and the air 'touching' it into a physical system, and we can't isolate any of the other gears of it and the air 'touching' them into a physical system, because we don't know the total momentum of the other gears, not to mention that v_{air} in the expression $\frac{2m_g}{m_g+m_{air}}(\frac{m_{air}-m_g}{m_{air}+m_g}v_{air})$ is equal to the speed of the air 'touching' the 'starting gear' and m_{air} is the mass of the air 'touching' it, so that if you derive the constraint condition by isolating the other gear of a machine and the air 'touching' it into a physical system, you need to find out the quantitative relationship between the flow velocity of the air 'touching' the gear and the one of the air 'touching' the 'starting gear', which is impossible.

Secondly, by applying Conservation of momentum, we get:

$$p_{1i} + p_{2i} = p_{1f} + p_{2f}$$

Therefore $m_{air}v_{air} = m_{air}(v_{air} + v_g) + m_g v_g$

So let $\varphi(m_g, m_{air}) = -(m_{air}v_g + m_g v_g)$ be the constraint condition.

Next, as per Lagrange Multiplier Method, we have to suppose a Lagrange function: F(m_g, m_{air}, v_{air}) = f(m_g, m_{air}, v_{air}) + $\lambda\varphi(m_g, m_{air})$.

Finally, we have to solve an equation group:

$\{F'_x = f'_x(x, y, z) + \lambda\varphi'_x(x, y) = 0\ F'_y = f'_y(x, y, z) + \lambda\varphi'_y(x, y) = 0\ F'_z = f'_z(x, y, z) =$

$\therefore$

$\{f'_{m_g}(m_g, m_{air}, v_{air}) + \lambda\varphi'_{m_g}(m_g, m_{air}) = 0\ f'_{m_{air}}(m_g, m_{air}, v_{air}) + \lambda\varphi'_{m_{air}}(m_g, m_{air}) = 0\ f'$

$\therefore$

$\{2v_{air}[\frac{m_{air}(m_{air}+m_g)-2m_{air}m_g+m_g-m_{air}}{(m_g+m_{air})^3}] - \lambda v_g = 0\ 2m_g v_{air}[\frac{(m_g+m_{air})^2-2(m_{air}-m_g)(m_g+m_{air})}{(m_g+m_{air})^4}] - \lambda v_g = 0$

$\therefore$

$\{2v_{air}[\frac{m_{air}(m_{air}+m_g)-2m_{air}m_g+m_g-m_{air}}{(m_g+m_{air})^3}] - \lambda v_g = 0\ 2m_g v_{air}[\frac{(m_g+m_{air})^2-2(m_{air}-m_g)(m_g+m_{air})}{(m_g+m_{air})^4}] - \lambda v_g = 0$

$\therefore m_{air} = m_g$

Therefore, the minimal value of f(m_g, m_{air}, v_{air}) is 0, while there is no maximal value.

$\because \nabla f = \frac{\partial f}{\partial x}\vec{x} + \frac{\partial f}{\partial y}\vec{y} + \frac{\partial f}{\partial z}\vec{z}$

$\therefore \nabla g = \frac{\partial g}{\partial m_g}\vec{x} + \frac{\partial g}{\partial m_{air}}\vec{y} + \frac{\partial g}{\partial v_{air}}\vec{z}$

When $\nabla g = \frac{\partial g}{\partial m_g}\vec{x} + \frac{\partial g}{\partial m_{air}}\vec{y} + \frac{\partial g}{\partial v_{air}}\vec{z} = \vec{0}$

$\{\frac{\partial g}{\partial m_g} = 0\ \frac{\partial g}{\partial m_{air}} = 0\ \frac{\partial g}{\partial v_{air}} = 0$

$\therefore \{(-\ 2v_{air})[\frac{m_{air}}{(m_g+m_{air})^2}] = 0\ 2v_{air}[\frac{m_g}{(m_g+m_{air})^2}] = 0\ \frac{m_{air}-m_g}{m_{air}+m_g} = 0$

$\therefore m_{air} = m_g = 0$ which is impossible

We have known that this function does not have any stationary point. However we can also apply Lagrange Multiplier Method for obtaining the extrema:

Suppose F(m_g, m_{air}, v_{air}) = f(m_g, m_{air}, v_{air}) + λφ(m_g, m_{air})

Then we have to solve an equation group:

$\{F'_x = f'_x(x, y, z) + \lambda\varphi'_x(x, y) = 0\ F'_y = f'_y(x, y, z) + \lambda\varphi'_y(x, y) = 0\ F'_z = f'_z(x, y, z) =$

$\therefore$

$\{g'_{m_g}(m_g, m_{air}, v_{air}) + \lambda\varphi'_{m_g}(m_g, m_{air}) = 0\ g'_{m_{air}}(m_g, m_{air}, v_{air}) + \lambda\varphi'_{m_{air}}(m_g, m_{air}) = 0\ g'$

$\therefore$

$\{(-\ 2v_{air})[\frac{m_{air}}{(m_g+m_{air})^2}] - \lambda v_g = 0\ 2v_{air}[\frac{m_g}{(m_g+m_{air})^2}] - \lambda v_g = 0\ \frac{m_{air}-m_g}{m_{air}+m_g} = 0\ -\ (m_{air}v_g + m$

$\therefore \{v_g = 0\ \ m_{air} = m_g$

Therefore, the minimal value of g(m_g, m_{air}, v_{air}) is 0, while there is no maximal value.

But here comes the contradiction that, if we put m_{air} to m_g or m_g to m_{air} to minimalize g(m_g, m_{air}, v_{air}), then f(m_g, m_{air}, v_{air}) = 0, which makes the working efficacy of a machine 0, so we have to figure out one way to maximize f(m_g, m_{air}, v_{air}) and meanwhile minimalize g(m_g, m_{air}, v_{air}), which can’t be realized by finding out the maximal value of f(m_g, m_{air}, v_{air}) and the minimal value of g(m_g, m_{air}, v_{air}) to maximalize the working efficacy.

7. Function

As said before, the working efficacy is positively quantitatively related to f(m_g, m_{air},

v_{air}) and negatively quantitatively related to g(m_g, m_{air}, v_{air}), eliciting the idea that, suppose the working efficacy as w(f(m_g, m_{air}, v_{air}), g(m_g, m_{air}, v_{air})).

Here comes another problem: Should be the quantitative relationship curvilinear or linear?

We have to establish an expression that is definitely positively related to f(m_g, m_{air}, v_{air}) and negatively related to g(m_g, m_{air}, v_{air}) as the one of w(f(m_g, m_{air}, v_{air}), g(m_g, m_{air}, v_{air})).

If in the expression the relationship between f(m_g, m_{air}, v_{air}) and g(m_g, m_{air}, v_{air}) is curvilinear, then w(f(m_g, m_{air}, v_{air}), g(m_g, m_{air}, v_{air})) will not be as desired.

Thus it does not take any genius to figure out that the relationship should be linear.

So it's lawful to make the expression of w(f(m_g, m_{air}, v_{air}), g(m_g, m_{air}, v_{air})) as jf(m_g, m_{air}, v_{air}) + kg(m_g, m_{air}, v_{air}), where j is positive and k is negative.

And we are to apply gradient for this function to obtain the extrema.

$$\because \nabla f = \frac{\partial f}{\partial x}\vec{x} + \frac{\partial f}{\partial y}\vec{y} + \frac{\partial f}{\partial z}\vec{z}$$

$$\therefore \nabla f = \frac{\partial w}{\partial m_g}\vec{x} + \frac{\partial w}{\partial m_{air}}\vec{y} + \frac{\partial w}{\partial v_{air}}\vec{z}$$

When $\nabla f = \frac{\partial w}{\partial m_g}\vec{x} + \frac{\partial w}{\partial m_{air}}\vec{y} + \frac{\partial w}{\partial v_{air}}\vec{z} = \vec{0}$

$$\{\frac{\partial w}{\partial m_g} = 0 \ \frac{\partial w}{\partial m_{air}} = 0 \ \frac{\partial w}{\partial v_{air}} = 0$$

$\therefore$

$$\{2jv_{air}[\frac{m_{air}(m_{air}+m_g)-2m_{air}m_g+m_g-m_{air}}{(m_g+m_{air})^3}] + (-2kv_{air})[\frac{m_{air}}{(m_g+m_{air})^2}] = 0 \ 2jm_g v_{air}[\frac{(m_g+m_{air})^2-2(m_{air}-m_g)(m_g +}{(m_g+m_{air})^4}$$

Because j and k are not 0, $m_{air} = m_g$.

We can get $\{v_{air} = 0 \ m_{air} = m_g$.

So this function has infinite stationary points(as long as $m_{air} = m_g$ and $v_{air} = 0$, this point is a stationary point).

So after having tried all of the methods, sadly we only find out that the working efficacy of a machine does not have a maximal value while the minimum value of it is 0, which can be projected correctly intuitively.

But can we learn anything from the whole rigmarole?

8. Discussion

The figure composed of infinite points whose m_g-coordinates are equal to m_{air}-coordinates is a straight line that comes from the inside of this paper to the outside of it and intersects both the m_g-axis and the m_{air}-axis by -45°, as demonstrated by the figure 10.

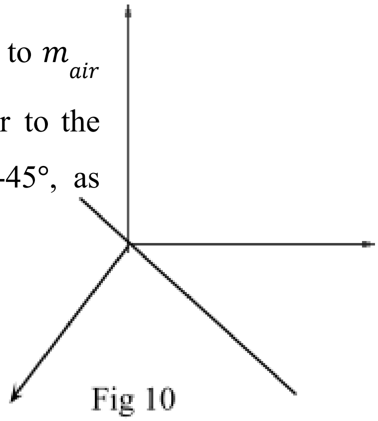


Fig 10

Since 0 is the minimal value of w(f(m_g, m_{air}, v_{air}), g(m_g, m_{air}, v_{air})) when $m_g = m_{air}$ and $v_{air} = 0$, the larger the distance from a point to the line is, the higher the value of w(f(m_g, m_{air}, v_{air}), g(m_g, m_{air}, v_{air})) is,

since when a function reaches its stationary points, the values of it are the extrema and in this case, when $m_g = m_{air}$ and $v_{air} = 0$, the value of w(f(m_g, m_{air}, v_{air}), g(m_g, m_{air}, v_{air})) is 0 as the minimal value of it.

To calculate the distance, firstly, we need to choose a point A that is not on the line, say, (m_g, m_{air}, v_{air}), and then find a point B on the line, say, (m_g', m_g', 0); Next, connect them with a line; Finally, vectorize the line, and let's name the vector as $\overrightarrow{AB}$ and the coordinate of it is $(m_g' - m_g, m_g' - m_{air}, -v_{air})$.

Any line has a direction vector,[11] and this line can't exempt from having one. For this line, the direction vector is (1,1,0),[12] since the function of line is $m_g - m_{air} = 0$, and let's name it as $\vec{d}$.

If $\overrightarrow{AB} \cdot \vec{d} = 0$, then the 'norm' of $\overrightarrow{AB}$ is just the distance.[13]

$\because\ \overrightarrow{AB} \cdot \vec{d} = 0$

$\therefore\ m_g' - m_g + m_g' - m_{air} = 0$[14]

$\therefore\ m_g' = \frac{m_g + m_{air}}{2}$

$\therefore\ \overrightarrow{AB} = (\frac{m_{air} - m_g}{2}, \frac{m_g - m_{air}}{2}, - v_{air})$

$\therefore\ |\overrightarrow{AB}| = \sqrt{(\frac{m_{air} - m_g}{2})^2 + (\frac{m_g - m_{air}}{2})^2 + v_{air}^2} = \sqrt{\frac{(m_{air} - m_g)^2}{2} + v_{air}^2}$

So, the larger $\sqrt{\frac{(m_{air} - m_g)^2}{2} + v_{air}^2}$ is, the higher the working efficacy of a machine is.

Therefore, we should suppose p(m_g, m_{air}, v_{air}) = $\sqrt{\frac{(m_{air} - m_g)^2}{2} + v_{air}^2}$, and when the value of this function is maximal, the working efficacy of a machine is maximal.

Firstly we should consider if gradient can be applied for obtaining the maximum value of it.

$\because\ \nabla f = \frac{\partial f}{\partial x}\vec{x} + \frac{\partial f}{\partial y}\vec{y} + \frac{\partial f}{\partial z}\vec{z}$

$\therefore \nabla p = \frac{\partial p}{\partial m_g}\vec{x} + \frac{\partial p}{\partial m_{air}}\vec{y} + \frac{\partial p}{\partial v_{air}}\vec{z}$

When $\nabla p = \frac{\partial p}{\partial m_g}\vec{x} + \frac{\partial p}{\partial m_{air}}\vec{y} + \frac{\partial p}{\partial v_{air}}\vec{z} = \vec{0}$

$\{\frac{\partial p}{\partial m_g} = 0\ \frac{\partial p}{\partial m_{air}} = 0\ \frac{\partial p}{\partial v_{air}} = 0$

$\therefore\ \{\frac{m_{air} - m_g}{\sqrt{\frac{(m_{air} - m_g)^2}{2} + v_{air}^2}} = 0\ \frac{m_g - m_{air}}{\sqrt{\frac{(m_{air} - m_g)^2}{2} + v_{air}^2}} = 0\ \frac{v_{air}}{\sqrt{\frac{(m_{air} - m_g)^2}{2} + v_{air}^2}} = 0$

$\therefore$ When $m_{air} = m_g$, v_{air}=0, the function reaches the stationary points of it.

We can only know the minimal value of the distance is 0. We still need to know the maximum value of it, if it is existent, and whether it is existent can be proven by applying Lagrange Multiplier Method, with the same constraint condition used above.

Suppose $F(m_g, m_{air}, v_{air}) = p(m_g, m_{air}, v_{air}) + \lambda\varphi(m_g, m_{air})$

Then we have to solve an equation group:

$$\{F'_x = p'_x(x, y, z) + \lambda\varphi'_x(x, y) = 0\ F'_y = p'_y(x, y, z) + \lambda\varphi'_y(x, y) = 0\ F'_z = p'_z(x, y, z) =$$

$\therefore$

$$\{\frac{m_{air} - m_g}{2\sqrt{\frac{(m_{air}-m_g)^2}{2} + v_{air}^2}} - \lambda v_g = 0\ \frac{m_g - m_{air}}{2\sqrt{\frac{(m_{air}-m_g)^2}{2} + v_{air}^2}} - \lambda v_g = 0\ \frac{v_{air}}{\sqrt{\frac{(m_{air}-m_g)^2}{2} + v_{air}^2}} = 0 - (m_{air} v_g + m_g$$

$$\therefore \{m_{air} = m_g\ v_{air} = 0\ v_g = 0$$

So unfortunately, it has been proven that this function does not have a maximal value.

It seems that we are inable to attain m_g, m_{air} and v_{air} that can generate the highest working efficacy of a machine, however, if we investigate into the function $p(m_g, m_{air}, v_{air}) = \sqrt{\frac{(m_{air}-m_g)^2}{2} + v_{air}^2}$, we can find out that the value of this function is positively related to $\frac{(m_{air}-m_g)^2}{2} + v_{air}^2$, since $\sqrt{x}$ is an increasing function,[15] and because $\frac{(m_{air}-m_g)^2}{2} + v_{air}^2$ is positively related to $|m_{air} - m_g| + v_{air}$, when we compare the working efficacies different values of m_g, m_{air}, v_{air} can generate, we can calculate $|m_{air} - m_g| + v_{air}$ and see which result of the values of m_g, m_{air}, v_{air} is higher, then the values whose result is higher can generate higher working efficacy.

We can exegete that, the higher the difference between m_{air} and m_g is, or the higher v_{air} is, the higher working efficacy the values of m_g, m_{air}, v_{air} can generate is, viz, the higher the difference between the mass of the air in a machine and the mass of the gears of it is, or the higher the flow velocity of the air of it is, the higher the working efficacy of it is, let's name, 'The Zhang's Law'.

9. Examples

1. Suppose there are two groups of gears whose mass is 10g and 20g respectively, which are in two machines whose mass of the air is 5g and 10g respectively and flow velocity of the air is 0.5m/s and 0.8m/s respectively. Which machine does have the higher working efficacy, if the numbers of the gears are equal and the deployments are also equal?

By applying ‘The Zhang’s Law’, the result of $|m_{air} - m_g| + v_{air}$ of the first machine is equal to $|0.005 - 0.01| + 0.5 = 0.505$, and the one of the second machine is equal to $|0.01 - 0.02| + 0.8 = 0.81$.

So as can be seen, the second machine has the higher working efficacy.

2. Suppose there are 3 gears. Which of the deployment 1 and 2 is the more efficient one? The deployment 2.

3. Suppose there are 4 gears. Which of the deployment 3 and 4 is the more efficient one? The working efficacies of the deployment 4 and the one 5 are equal.

4. Suppose there are 5 gears, which have 8 teeth and whose mass is 10g and volume is $1x10^{-6}m^3$. Which of the deployment 5, 6, 7, 8 and 9 is the most efficient one?

The working efficacy of the deployment 7 is definitely higher than the one of the deployment 6, when $0 < k\alpha < \frac{3}{2}$, the working efficacy of the deployment 5 is higher than the one of the deployment 8, and when $m_{g_1} > \frac{k\alpha}{10}$, the working efficacy of the deployment 7 is higher than the one of the deployment 5, and the working efficacy of the deploymewnt 5 is higher than the one of the deployment 9, therefore, the most efficient deployment is the deployment 7.

10. Conclusion

To sum up, the higher the difference between m_{air} and m_g is, or the higher v_{air} is, the higher working efficacy the values of m_g, m_{air}, v_{air} can generate is, viz, the higher the difference between the mass of the air in a machine and the mass of the gears of it is, or the higher the flow velocity of the air of it is, the higher the working efficacy of it is, let's name, 'The Zhang's Law'.

And for 3 gears, the deployment 2 is the more efficient deployment; For 4 gears, the working efficacies of the deployments 3 and 4 are equal; For 5 gears, To recap, the working efficacy of the deployment 7 is definitely higher than the one of the deployment 6. When $k\alpha \geq \frac{3}{2}$, the working efficacy of the deployment 7 is higher than the one of the deployment 5, and the one of the deployment 8 is higher than the one of the deployment 7, and when m_{g_1} > about 2.25045, the working efficacy of the deployment 8 is higher than the one of the deployment 9, and when $about\ 2.25045 > m_{g_1} > 0$, the working efficacy of the deployment 9 is higher than the one of the deployment 8, and when m_{g_1} is equal to about 2.25045, they are the same; when $0 < k\alpha < \frac{3}{2}$, the working efficacy of the deployment 5 is higher than the one of the deployment 8, and when $m_{g_1} > \frac{3}{20} + \frac{\sqrt{409}}{20}$, the working efficacy of the deployment 7 is higher than the one of the deployment 9, the working efficacy of the deployment 5 is higher than the one of the deployment 9 and the working efficacy of the deployment 7 is higher than the one of the deployment 5. And when $m_{g_1} < 0$, -$\frac{5m_{g_1}}{2} > 0$, but m_{g_1} can't be negative, so there is no range of values within which m_{g_1} lies so that the working efficacy of the deployment 7 is higher than the one of the deployment 9. When $m_{g_1} = \frac{3}{20} + \frac{\sqrt{409}}{20}$, the working efficacy of the deployment 7 is equal to the one of the deployment 9, and both the working efficacy of the deployment 7 and the one of the deployment 9 are higher than the one of the deployment 5.

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