Analysis Lesson 11
Subsequences, liminf, limsup, and the Bolzano Weierstrass Theorem

Two parts and 9 HW

Part I Watch Subsequences Playlist and then do the four HW.

12:34 AM Sat Mar 6

MAT320S21

Axiomatic Geometry | MAT320S21 | Linear Algebra III | Scratchwork

Bolzano-Weierstrass Theorem

on the Real Line

Subsequences

liminf and limsup

A sequence is a set in which every point has been assigned a unique natural number, and every natural number has a point.

$$\{x_j \mid j \in \mathbb{N}\} = \{x_1, x_2, x_3, x_4, x_5, \ldots\}$$

we keep track of the order.

Set $\{1,2,3\} = \{3,2,1\}$ ← Contrast with Set!

A subsequence is a list of points from the original sequence in the same order.

Example: Even Subseq $= \{x_2, x_4, x_6 \dots\}$

Odd Subseq $= \{x_1, x_3, x_5, x_7 \dots\}$

Any selection $\{x_{j_1}, x_{j_2}, x_{j_3}, \dots\}$
j_k
$= \{x_{j_k} \mid k \in \mathbb{N}\}$

Even subseq $j_k = 2k$
$\{x_{2\cdot 1}, x_{2\cdot 2}, x_{2\cdot 3} \dots\} = \{x_2, x_4, x_6 \dots\}$

$\boxed{\text{HW1}}$ $j_k =$ what for the odd seq?
Want $j_1 = 1$ $j_2 = 3$ $j_3 = 5$ $j_4 = 7$ ...
$j_k =$ what formula depending on k?

Part1 HW1 above

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Does a subsequence converge?

$\{(-1)^j \mid j \in \mathbb{N}\} = \{(-1)^1, (-1)^2, (-1)^3, (-1)^4, \dots\}$

$= \{-1, +1, -1, +1, -1, +1 \dots\}$

we know this sequence does not converge

but does it has a subseq. that converges?

Yes, even subseq $= \{(-1)^{2k} \mid k \in \mathbb{N}\} = \{1, 1, 1 \dots\}$

$j_k = 2k$ $\qquad \lim_{k\to\infty} x_{j_k} = \lim_{k\to\infty} 1 = 1$

also odd subseq $= \{(-1)^{2k-1} \mid k \in \mathbb{N}\} = \{-1, -1, -1 \dots\}$

$j_k = 2k-1$ $\qquad \lim_{k\to\infty} x_{j_k} = \lim_{k\to\infty} -1 = -1$

Some sequences have a converging subsequence and different subsequences can have different limits.

$\boxed{HW2}$ $\{x_j \mid j \in \mathbb{N}\} = \left\{\frac{(-1)^j j}{j+1} \mid j \in \mathbb{N}\right\}$ No proof needed

Does it have a converging subseq?

What are the limits of odd + even subseq?

Part 1 HW2 above

Classwork: Prove the *even* subsequence of

$$\{x_j \mid j \in \mathbb{N}\} = \left\{\frac{(-1)^j j}{j+1} \;\middle|\; j \in \mathbb{N}\right\} \text{ converges to } 1.$$

Even subseq is $j_k = 2k$

$$\{x_{2k} \mid k \in \mathbb{N}\} = \left\{\frac{(-1)^{2k}(2k)}{(2k)+1} \;\middle|\; k \in \mathbb{N}\right\}$$

Show:

$$\forall \varepsilon > 0 \ \exists N_\varepsilon \text{ s.t. } \forall k \geq N_\varepsilon \ |x_{2k} - 1| < \varepsilon$$

$$\forall \varepsilon > 0 \ \exists N_\varepsilon \text{ s.t. } \forall k \geq N_\varepsilon \ \left|\frac{(-1)^{2k}(2k)}{(2k)+1} - 1\right| < \varepsilon$$

must use parenthesis around $(2k) = j$ or $(2k+1)$ for odd

$\forall \varepsilon > 0 \ \exists N_\varepsilon$ s.t. $\forall k \geq N_\varepsilon \ \left| \frac{(-1)^{2k}(2k)}{(2k)+1} - 1 \right| < \varepsilon$

Proof: ① Given any $\varepsilon > 0$ Choose $N_\varepsilon = \boxed{\quad}$

② For any $k \geq N_\varepsilon$

Justify downwards

Solve up for k

Finish this classwork!

final $\left| \frac{(-1)^{2k}(2k)}{(2k)+1} - 1 \right| < \varepsilon$

Part 1 HW3 finish classwork above

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Some sequences have no converging subsequences!

Example $x_j = j^2$

$\{ j^2 \mid j \in \mathbb{N} \} = \{ 1^2, 2^2, 3^2, 4^2 \ldots \}$

$= \{ 1, 4, 9, 16, \ldots \}$

This seq $\lim_{j \to \infty} j^2 = \infty$

We will prove later that this sequence has no conv. subseq.

Part 1 HW4 imitate the proof below to prove any subsequence of the sequence above also diverges to infinity (not required)

Thm: If $\lim_{j\to\infty} x_j = L$ then for any subsequence $\lim_{k\to\infty} x_{j_k} = L$ too.

Given: $x_j \to L$ $\quad \forall \varepsilon > 0\ \exists N_\varepsilon^x$ s.t $\forall j \geq N_\varepsilon^x$ $|x_j - L| < \varepsilon$

Show: $x_{j_k} \to L$ for any subseq $\quad \forall \varepsilon > 0\ \exists N'_\varepsilon$ s.t $\forall k \geq N'_\varepsilon$ $|x_{j_k} - L| < \varepsilon$

Proof:

Set up yourself

pause!

Given: $x_j \to L$ $\forall \varepsilon > 0\ \exists N_\varepsilon^x$ s.t $\forall j \ge N_\varepsilon^x$ $|x_j - L| < \varepsilon$

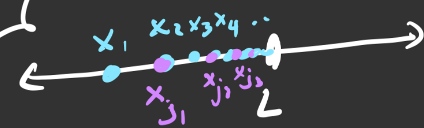
Show: $x_{j_k} \to L$ $\forall \varepsilon > 0\ \exists N_\varepsilon'$ s.t $\forall k \ge N_\varepsilon'$ $|x_{j_k} - L| < \varepsilon$

for any subseq

Proof: ① Given any $\varepsilon > 0$ Choose $N_\varepsilon' = \boxed{}$

② For any $k \ge N_\varepsilon'$ we have $j_k \ge k \ge N_\varepsilon'$ ② by $j_k \ge k$

thoughts



$1 \le j_1 < j_2 < j_3$ ... $2 \le j_2$ $3 \le j_3$ $4 \le j_4$ ...

$\boxed{k \le j_k}$

final $|x_{j_k} - L| < \varepsilon$

[redacted]

Proof: (1) Given any $\varepsilon > 0$ choose $N'_\varepsilon = \lceil N_\varepsilon \rceil$ (1) by given

(2) For any $k \ge N'_\varepsilon$ we have $j_k \ge k \ge N'_\varepsilon$ (2) by $j_k \ge k$

(3) $j_k \ge N_\varepsilon$ (3) by choice of N'_ε in step 1

final $|x_{j_k} - L| < \varepsilon$. final by given

QED.

We can apply this theorem to prove $\lim_{j \to \infty} (-1)^j$ Does Not Exist

It has no limit because it has two subseq w/ diff limits.

Part 2

Watch [limsup playlist]() and then do the five HW.

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SCREEN RECORDING now
Screen Recording video saved to Photos

Scratchwork MAT320S21

limsup and liminf

Given a sequence $\{x_j \mid j \in \mathbb{N}\}$ that has an upper bound, B,

$\forall j \in \mathbb{N} \quad x_j \le B$

$\sup\{x_j \mid j \in \mathbb{N}\}$ exists and $\le B$ by continuoum hypothesis.

$s_1 = \sup\{x_1, x_2, x_3, x_4 \dots\} \le B$

$s_2 = \sup\{x_2, x_3, x_4 \dots\} \le B$

which is bigger? $s_1 \ge s_2$

$s_3 = \sup\{x_3, x_4, x_5 \dots\} \le B$

$s_1 \ge s_2 \ge s_3$

$s_k = \sup\{x_k, x_{k+1}, x_{k+2}, \dots\} \le B$

sequence of sups

$s_1 \ge s_2 \ge s_3 \ge \dots \ge s_k \ge \dots$

if we assumes $\{x_j \mid j \in \mathbb{N}\}$ has a lower bound b

$s_1 \ge s_2 \ge \dots \ge s_k \ge \dots \ge b$

decreasing and bounded below

so by the monotone conv thm

final $\lim_{k\to\infty} s_k$ exists. final just

Defn $\limsup_{j\to\infty} x_j = \lim_{k\to\infty} s_k$

Thm: $\limsup_{j\to\infty} x_j$ exists if x_j is bounded above and below.

limsup

$\boxed{\text{HW1}}$ Prove this theorem using our work above but writing it in two columns with statements + justifications.

Not an ε-N proof

Part 2 HW1 is above

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Defn: $\liminf_{j\to\infty} x_j = \lim_{k\to\infty} \inf\{x_k, x_{k+1}, \ldots\}$

Thm: If $\{x_j \mid j \in \mathbb{N}\}$ is bounded from above and below then $\liminf_{j\to\infty} x_j$ exists.

Proof:

① $\{x_j \mid j \in \mathbb{N}\}$ is bounded	① given
② $\exists$ upper bound B $\forall j \in \mathbb{N}$ $x_j \le B$; $\exists$ lower bound b $\forall j \in \mathbb{N}$ $b \le x_j$	② defn bounded
③ $y_1 = \inf\{x_1, x_2, x_3 \ldots\}$ exists	③ by lower bound + continuum hyp.
④ $y_2 = \inf\{x_2, x_3, x_4 \ldots\}$ exists	④ "
⑤ $y_k = \inf\{x_k, x_{k+1} \ldots\}$ exists	⑤ "
⑥ $y_1 \le y_2 \le y_3 \le \ldots \le y_k \le \ldots$	⑥ If $A \supset B$ then $\inf A \le \inf B$
⑦ y_k is increasing "nondecreasing"	⑦ defn of increasing
⑧ $y_k \le x_k \le B$ $\{y_k\}$ bounded above	⑧ by defn of y_k in step 5 and step 2
⑨ $\lim_{k\to\infty} y_k$ exists	⑨ by monotone conv thm & steps 7 + 8
⑩ $\liminf_{j\to\infty} x_j = \lim_{k\to\infty} y_k$ exists	⑩ defn of liminf.

QED

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Write the first six terms of $\{(-1)^j \mid j \in \mathbb{N}\}$ ✓

find the limit if it exists ✓

find a converging subsequence if it has one ✓

find the liminf and lim sup ✓ Classwork

$x_1 = (-1)^1 = -1 \quad x_2 = (-1)^2 = 1 \quad x_3 = (-1)^3 = -1 \quad x_4 = 1 \quad x_5 = -1 \quad x_6 = 1$

no limit (alternating between -1 and 1)

because has two subseq that conv to diff limits

even sub $x_{2j} \to 1$ odd subseq $x_{2j-1} \to -1$

$$\liminf_{j \to \infty} x_j = \lim_{k \to \infty} \underbrace{\inf\{x_k, x_{k+1} \dots\}}_{y_k} = \lim_{k \to \infty} -1 = -1$$

$$y_1 = \inf\{x_1, x_2 \dots\} = \inf\{-1, 1, -1, 1 \dots\} = -1$$

$$y_2 = \inf\{x_2, x_3 \dots\} = \inf\{1, -1, 1, -1 \dots\} = -1$$

$$y_k = \inf\{x_k \dots\} = -1$$

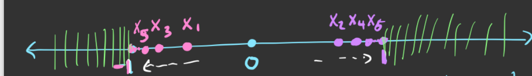
$$\limsup_{j \to \infty} x_j = \lim_{k \to \infty} \sup\{- - -\} = \lim_{k \to \infty} 1 = 1$$

$s_1 = \quad - \; - \; - \quad = 1$

$s_2 = \quad - \; - \; - \quad = 1$

Write the first six terms of $\left\{ \frac{(-1)^j j}{j+1} \mid j \in \mathbb{N} \right\}$ ✓
find the limit if it exists ✓
find a converging subsequence if it has one
find the liminf and lim sup. Classwork

$x_1 = \frac{-1}{2}$ $x_2 = \frac{2}{3}$ $x_3 = \frac{-3}{4}$ $x_4 = \frac{4}{5}$ $x_5 = \frac{-5}{6}$ $x_6 = \frac{6}{7}$



There is no limit because it has two subsequences with different limits

odds $\lim_{k\to\infty} x_{2k-1} = -1$ evens $\lim_{k\to\infty} x_{2k} = 1$

$$\liminf_{j\to\infty} x_j = \lim_{k\to\infty} \inf\{x_k, x_{k+1}, \ldots\} = \lim_{k\to\infty} -1 = -1$$

$$\limsup_{j\to\infty} x_j = \lim_{k\to\infty} \sup\{x_k, x_{k+1}, \ldots\} = \lim_{k\to\infty} +1 = +1$$

Observe

$$\liminf_{j\to\infty} x_j = -1 = \lim_{k\to\infty} x_{2k-1}$$

$$\limsup_{j\to\infty} x_j = +1 = \lim_{k\to\infty} x_{2k}$$

$\boxed{\text{HW2}}$ Write the first six terms of $\left\{ \frac{(-1)^j}{j} \,\middle|\, j \in \mathbb{N} \right\}$

Find the liminf and the limsup

Find the limits of the odd and even subseq.

$\boxed{\text{HW3}}$ Write the first 12 terms of $\left\{ \sin\left(\frac{j\pi}{2}\right) \,\middle|\, j \in \mathbb{N} \right\}$

Find the liminf and the limsup showing the details as above.

Find the limits of the subsequences:

$$\lim_{k\to\infty} x_{4k} \qquad \lim_{k\to\infty} x_{4k+1} \qquad \lim_{k\to\infty} x_{4k+2} \qquad \lim_{k\to\infty} x_{4k+3}$$

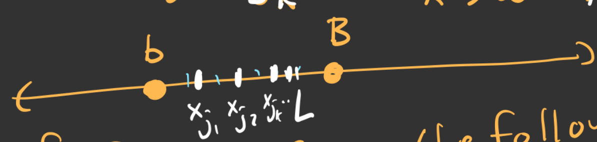
Which agree with the limsup and liminf?

Part 2 HW2 and HW3 are above

Bolzano-Weierstrass Theorem

Thm If $\{x_j \mid j \in \mathbb{N}\} \subset [b, B]$

then $\exists$ subseq x_{j_k} s.t. $\lim_{k\to\infty} x_{j_k} = L \in [b, B]$



The proof follows from the following three theorems.

Thm I: If $\{x_j \mid j \in \mathbb{N}\} \subset [b, B]$ then $\limsup_{j\to\infty} x_j$ exists $\in [b, B]$

Thm II If $\{y_k \mid k \in \mathbb{N}\} \subset [b, B]$ has a limit $y_k \to y$ then $\lim_{k\to\infty} y_k = y \in [b, B]$ (proof is HW)

Thm III If $\limsup_{j\to\infty} x_j = L$ then $\exists$ subseq $x_{j_k} \to L$.

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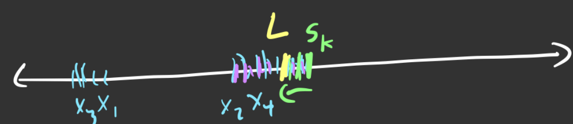


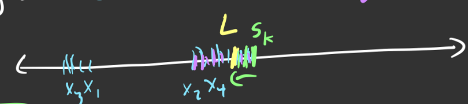
(1) $\lim_{k\to\infty} s_k = L$ where $s_k = \sup\{x_k, x_{k+1}, \ldots\}$ — (1) by given and defn limsup

(2) s_k is an upper bound for $\{x_k, x_{k+1}, \ldots\}$ and $s_k - \frac{1}{k}$ is not an upper bound — (2) by defn of sup

(3) $\exists\ x_{j_k} \in \{x_k, x_{k+1}, \ldots\}$ s.t. $s_k - \frac{1}{k} < x_{j_k}$ — (3) by defn of upper bound

~~Later~~ we can erase any x_{j_k} that are out of order.

Thm III If $\limsup_{j\to\infty} x_j = L$ then $\exists$ subseq $x_{j_k} \to L$.



(1) $\boxed{\lim_{k\to\infty} s_k = L}$ where $s_k = \sup\{x_k, x_{k+1}, \dots\}$ — (1) by given and defn limsup

(2) s_k is an upper bound for $\{x_k, x_{k+1}, \dots\}$ and $s_k - \frac{1}{k}$ is not an upper bound — (2) by defn of sup

(3) $\exists\ x_{j_k} \in \{x_k, x_{k+1}, \dots\}$ s.t. $\boxed{s_k - \frac{1}{k} < x_{j_k} \le s_k}$ — (3) by defn of upper bound

(4) $\boxed{\lim_{k\to\infty} s_k - \frac{1}{k}} = \lim_{k\to\infty} s_k - \lim_{k\to\infty} \frac{1}{k} = L - 0 = \boxed{L}$ — (4) $\lim_{k\to\infty} a_k - b_k = a_\infty - b_\infty$ and $\lim_{k\to\infty} \frac{1}{k} = 0$

(5) $\lim_{k\to\infty} x_{j_k} = L$ — (5) by sandwich theorem

QED

$\boxed{\text{HW}}$ Prove: If $\liminf_{j\to\infty} x_j = L$ then $\exists$ subseq $x_{j_k} \to L$.

Part 2 HW4 is above

Part 2 HW5 Find a sequence that has a subsequence converging to a limit which is not the limsup and also not the liminf.