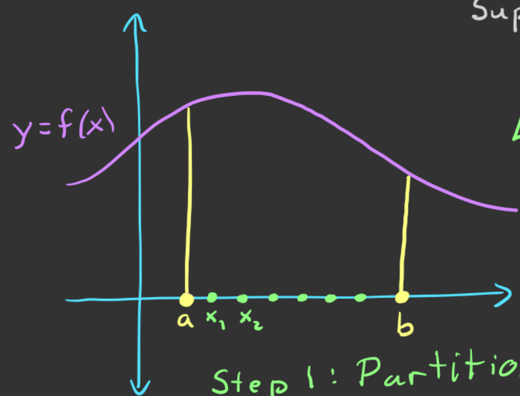
Analysis
Lesson 20
Riemann Sums and Integrals

This lesson has two parts, lots of classwork, and four homework problems.

Part I Riemann Sums Approximating Areas

Watch [Playlist RS-1to5]

Riemann Sums and Integrals

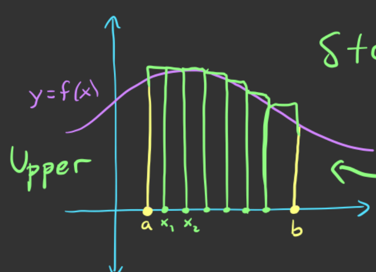


Suppose $f: \mathbb{R} \to \mathbb{R}^+$ is positive and uniformly continuous on $[a,b]$

Lets estimate the area between the x axis and the graph of $y=f(x)$ and $x=a$ and $x=b$

Step 1: Partition $[a,b]$ into N intervals

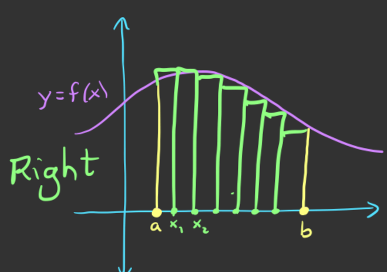
$$\underbrace{a=x_0<x_1<x_2<\dots<x_N=b}_{P} \text{ with } x_i-x_{i-1}<\delta_\varepsilon$$



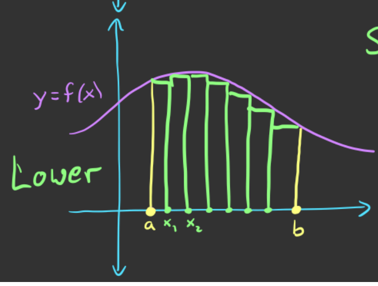
Step 2: Over estimate the area using

$$\text{Upper Sum}(P)=\sum_{i=1}^{N}\left(\max_{[x_{i-1},x_i]} f(x)\right)(x_i-x_{i-1})$$

$$\le \sum_{i=1}^{N}\left(f(x_i)+\varepsilon\right)(x_i-x_{i-1})$$



$$\le \underbrace{\sum_{i=1}^{N} f(x_i)(x_i-x_{i-1})}_{\text{Right Hand Sum}(P)} + \underbrace{\sum_{i=1}^{N}\varepsilon(x_i-x_{i-1})}_{\varepsilon(b-a)}$$



Step 3: Under estimate the area

$$\text{Lower Sum}(P)=\sum_{i=1}^{N}\left(\min_{[x_{i-1},x_i]} f(x)\right)(x_i-x_{i-1})$$

$$\ge \sum_{i=1}^{N}\left(f(x_i)-\varepsilon\right)(x_i-x_{i-1})$$

$$\geq \underbrace{\sum_{i=1}^{N} f(x_i)(x_i - x_{i-1})}_{\text{Right Hand Sum}(P)} - \underbrace{\sum_{i=1}^{N} \varepsilon(x_i - x_{i-1})}_{\varepsilon(b-a)}$$

Step 4: Combining the above we have

$$RHS(P) - \varepsilon(b-a) \leq \text{Lower Sum}(P) \leq \text{Area} \leq \text{Upper Sum}(P) \leq RHS(P) + \varepsilon(b-a)$$

Thus $\quad |\text{Area} - RHS(P)| \leq \varepsilon(b-a)$

So to estimate the area with error $< \varepsilon_{Area}$, take $\varepsilon = \frac{\varepsilon_{Area}}{b-a}$,

find δ_ε for your function.

take an evenly spaced partition

$$\frac{b-a}{N} < \delta_\varepsilon \quad \text{so} \quad N > \frac{b-a}{\delta_\varepsilon}$$

Let $x_0 = a \quad x_1 = a + \frac{b-a}{N} \quad x_2 = a + 2\frac{b-a}{N}$

$\ldots \quad x_j = a + j\frac{b-a}{N} \quad \ldots \quad x_N = a + N\left(\frac{b-a}{N}\right) = b$

Then $RHS(P)$ will be close enough!

Classwork: Estimate the area under $f(x)=x^2+4$ between $x=2$ and $x=5$ with error less than $.001$

Classwork: Estimate the area under $f(x)=x^2+4$ between $x=2$ and $x=5$ with error less than $.001=\varepsilon_{area}$

$\varepsilon = \frac{\varepsilon_{area}}{b-a} = ?$

What is δ_ε? ← Unif Contin

Use the theorem

$\delta_\varepsilon = \frac{\varepsilon}{M}$ where $M = \max_{[a,b]} |f'|$

What is N? $N > \frac{b-a}{\delta_\varepsilon}$

Find $x_0, x_1, \ldots x_j \ldots x_N$

RHS(P) = ?

Classwork: Estimate the area under $f(x)=x^2+4$ between $x=\overset{a}{2}$ and $x=\overset{b}{5}$ with error less than $.001=\varepsilon_{area}$

$$\varepsilon = \frac{\varepsilon_{area}}{b-a} = \frac{.001}{5-2} = \frac{.001}{3} = \frac{1}{3,000}$$

What is δ_ε? $\leftarrow$ Unif Contin

Use the theorem

$\delta_\varepsilon = \frac{\varepsilon}{M}$ where $M=\max\limits_{[a,b]}|f'|$

$$M = \max_{[2,5]} |f'(x)|$$

$$= \max_{[2,5]} |2x| = |2\cdot 5| = 10$$

increasing on $[2,5]$

$$\delta_\varepsilon = \frac{\varepsilon}{10} = \frac{(1/3,000)}{10} = \frac{1}{30,000}$$

What is N? $N > \frac{b-a}{\delta_\varepsilon} = \frac{5-2}{\left(\frac{1}{30,000}\right)} = 90,000$

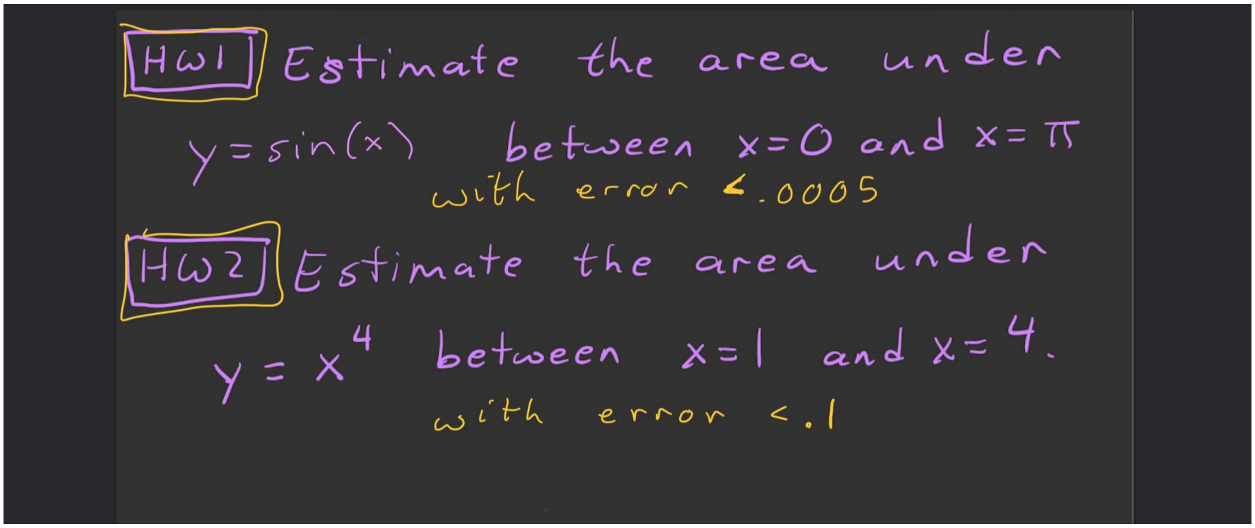
Find $x_0, x_1, \dots x_j \dots x_N$

$$x_0 = a = 2 \qquad x_1 = a + 1\left(\frac{b-a}{N}\right) = 2 + 1\left(\frac{5-2}{90,000}\right)$$

$$\dots \qquad x_j = 2 + j\left(\frac{5-2}{90,000}\right) = 2 + j\left(\frac{3}{90,000}\right)$$

$$x_N = 2 + N\left(\frac{3}{90,000}\right) = 2 + \cancel{90,000}\left(\frac{3}{\cancel{90,000}}\right) = 2+3 = 5 = b \checkmark$$

$$RHS(P) = \sum_{j=1}^{90,000} \left(x_j^2 + 4\right)\left(\frac{3}{90,000}\right)$$



Part II Riemann Integrals

Watch [**Playlist RS-6to10**]

$$\text{Lower Sum}(P) \le \text{Area} \le \text{Upper Sum}(P)$$

We already showed that if f is uniformly continuous

$\forall \varepsilon_{area} > 0 \ \exists N_{\varepsilon_{area}}$ and evenly spaced partition P_N s.t.

$$RHS(P_N) - \varepsilon_{area} \le LS(P_N) \le \text{Area} \le US(P_N) \le RHS(P_N) + \varepsilon_{area}$$

squeezed

$$\lim_{N\to\infty} RHS(P_N) = \text{Area}$$

$$\lim_{N\to\infty} \text{Lower Sum}(P_N) = \text{Area}$$

$$\lim_{N\to\infty} \text{Upper Sum}(P_N) = \text{Area}$$

Defn: A function $f:[a,b] \to \mathbb{R}$ is Riemann Integrable if

$$\sup_P \text{LowerSum}(P) = \inf_P \text{UpperSum}(P) \quad ✱$$

Thm: If f is uniformly continuous on $[a,b]$ then f is Riemann Integrable

Proof:

evenly spaced

(1) $\inf_P \text{UpperSum}(P) \le \text{UpperSum}(P_N)$

(2) $\le \text{RHS}(P_N) + \underline{\varepsilon(b-a)}$

(3) $\inf_P \text{UpperSum}(P) \le \lim_{N\to\infty} \text{RHS}(P_N) + 0$

(4) $\sup_P \text{LowerSum}(P) \ge \text{LowerSum}(P_N)$

(5) $\ge \text{RHS}(P_N) - \varepsilon(b-a)$

(6) $\sup_P \text{LowerSum} \ge \lim_{N\to\infty} \text{RHS}(P_N)$

(7) $\lim_{N\to\infty} \text{RHS}(P_N) \le \sup_{(P)} \text{LowerSum}(P) \le \inf_P \text{UpperSum}(P) \le \lim_{N\to\infty} \text{RHS}(P_N)$

(8) <u>All</u> <u>are</u> <u>equal</u>. So we have ✱.

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MAT320S21

Defn: The Riemann Integral of a Riemann Integrable function

$$\int_a^b f(x)\,dx = \inf_{P} \text{UpperSum}(P)$$

P is a partition of $[a,b]$

$$P = \{a = x_0 < x_1 < x_2 < \dots < x_N = b\}$$

$$\text{UpperSum}(P) = \sum_{j=1}^{N} \max_{[x_{j-1}, x_j]} f(x) \cdot (x_j - x_{j-1})$$

By defn of Riemann Integrable

$$\int_a^b f(x)\,dx = \sup_{P} \text{LowerSum}(P)$$

$$\text{LowerSum}(P) = \sum_{j=1}^{N} \min_{[x_{j-1}, x_j]} f(x) \cdot (x_j - x_{j-1})$$

Properties of the Riemann Integral

Thm
If $k \in \mathbb{R}$ and f is Riemann Integrable
then kf is Riemann Integrable
and $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$ *

Review before the proof
$\sup\{kz \mid z \in S\} = k\sup\{z \mid z\in S\}$ when $k \geq 0$.
$\sup\{kz \mid z\in S\} = k\inf\{z \mid z \in S\}$ when $k<0$

Proof has two cases: Case I $k<0$ Case II $k\geq 0$

Classwork!

Case I $k<0$:

① $\int_a^b kf(x)\,dx = \inf_{P \text{ of } [a,b]} \text{UpperSum}(kf, P)$ ① defn $\int$ and defn upper sum

$= \inf_{P \text{ of } [a,b]} \sum_{j=1}^{N} \max_{[x_{j-1},x_j]}(kf)\,(x_j - x_{j-1})$

② $= \inf_{P \text{ of } [a,b]} \sum_{j=1}^{N} k \min_{[x_{j-1},x_j]}(f)\,(x_j - x_{j-1})$ ② $k<0$ switches max + min

③ $= \inf_{P \text{ of } [a,b]} k \sum_{j=1}^{N} \min_{[x_{j-1},x_j]}(f)\,(x_j - x_{j-1})$ ③ factoring

(4) $= k \sup_{P \text{ of } [a,b]} \sum_{j=1}^{N} \min_{[x_{j-1}, x_j]}(f)\,(x_j - x_{j-1})$ (4) $k<0$ switches inf to sup

(5) $= k \sup_{P \text{ of } [a,b]} \text{Lower Sum}(f, P)$ (5) defn of lower sum

(6) $= k \int_a^b f(x)\,dx$ (6) by f is Riem Integrable.

Would also have to show
kf is Riemann Integrable.
Repeat the above with sup LowerSum(kf, P)
and also get $= k \int_a^b f(x)\,dx$.

Case II

$\boxed{\text{HW 3}}$ Prove Case II

$\boxed{\text{HW4}}$ Prove that $f(x) = C$ constant
is Riemann Integrable and
$\int_a^b C\,dx = C(b-a)$.

So far we only defined

$$\int_a^b f(x)\,dx \quad \text{when } a<b$$

Defn: $\displaystyle\int_a^b f(x)\,dx = 0$ when $a=b$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx \quad \text{when } a>b$$

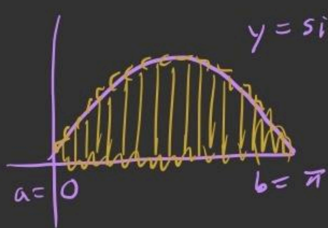
Check HW4 is still true when $a=b$ and when $a>b$.

Solutions to HW1 and HW2 are explained in [Video RS11] and [Video RS12]:

$\boxed{\text{HW1}}$ Estimate the area under

$y = \sin(x)$ between $x = 0$ and $x = \pi$

error $< .0005 = \frac{5}{10{,}000} = \frac{1}{2{,}000} = \mathcal{E}_{area}$



$$\mathcal{E} = \frac{\mathcal{E}_{area}}{b-a} = \frac{\left(\frac{1}{2{,}000}\right)}{\pi - 0} = \frac{1}{2000\pi}$$

What is $\delta_{\mathcal{E}}$?

$$M = \max_{[0,\pi]} \left| \frac{d}{dx}\sin(x) \right| = \max_{[0,\pi]} |\cos(x)| = 1$$

$$\delta_{\mathcal{E}} = \frac{\mathcal{E}}{M} = \frac{\mathcal{E}}{1} = \mathcal{E} = \frac{1}{2000\pi}$$

$$\frac{b-a}{N} < \delta_{\mathcal{E}} \Rightarrow N > \frac{b-a}{\delta_{\mathcal{E}}} = \frac{\pi}{\left(\frac{1}{2000\pi}\right)} = 2000\pi^2$$

$N = 2000 \cdot 4^2 = 32000$ using $4 > \pi$

What are $x_0 < x_1 < \dots < x_N$?

$$x_j = a + j\left(\frac{b-a}{N}\right) = \underbrace{0 + j\left(\frac{\pi - 0}{32{,}000}\right)}$$

$$\text{RHS} = \sum_{j=1}^{N} \sin(x_j)\left(\frac{\pi - 0}{32{,}000}\right)$$

Right Hand Sum

$\boxed{\text{HW2}}$ Estimate the area under $y = x^4$ between $x = 1$ and $x = 4$.

with error $< .1$

$$\varepsilon_{area} = .1 = \frac{1}{10} \qquad \varepsilon = \frac{\varepsilon_{area}}{b-a} = \frac{(1/10)}{4-1} = \frac{(1/10)}{3} = \frac{1}{30}$$

What is δ_ε? $\quad \delta_\varepsilon = \frac{\varepsilon}{M} \qquad \delta_\varepsilon = \frac{(1/30)}{256} = \frac{1}{7680}$

$$M = \max_{[1,4]} \left| \frac{d}{dx} x^4 \right| = \max_{[1,4]} |\underbrace{4x^3}_{\text{inc function}}| = |4 \cdot (4)^3| = 256$$

$\underline{2} \cdot \underline{2} \cdot \underline{2} \cdot \underline{2} \cdot \underline{2} \cdot \underline{2} \cdot \underline{2} \cdot \underline{2}$

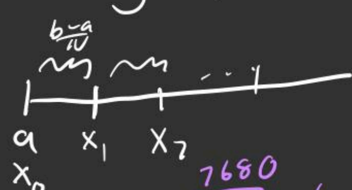
$30 \times 256 = 7680$

What is N?

$$\frac{b-a}{N} < \delta_\varepsilon \quad \text{so } N > \frac{b-a}{\delta_\varepsilon} = \frac{4-1}{\left(\frac{1}{7680}\right)} = 3(7680) = 23040$$

What are x_j?

$$x_j = a + j\left(\frac{b-a}{N}\right) = 1 + j\left(\frac{4-1}{23040}\right) = 1 + j\frac{3}{23040}$$



$$x_j = 1 + \frac{j}{7680}$$

$$RHS = \sum_{j=1}^{7680} (x_j)^4 \left(\frac{3}{23040}\right)$$