Linear Algebra
Lesson 10
Linear Transformations
Linear Algebra MAT313 Fall 2022
Professor Sormani
Part I: Linear Transformations
Part II: Review of Complex Numbers

If it is after Oct 25: Skip this lesson. You will come back and complete it later in January.

You will cut and paste the photos of your notes and completed classwork in a googledoc entitled:

MAT313F22-lesson10-lastname-firstname

and share editing of that document with me sormanic@gmail.com. You will also include your homework and any corrections to your homework in this doc.

If you have a question, type QUESTION in your googledoc next to the point in your notes that has a question and email me with the subject MAT313 QUESTION. I will answer your question by inserting a photo into your googledoc or making an extra video.

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This lesson has two parts:
Part I: Linear Transformations
Part II: Review of Complex Numbers
It is an extra short lesson so that you can go straight to Lesson 11.
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Part I: Linear Transformations
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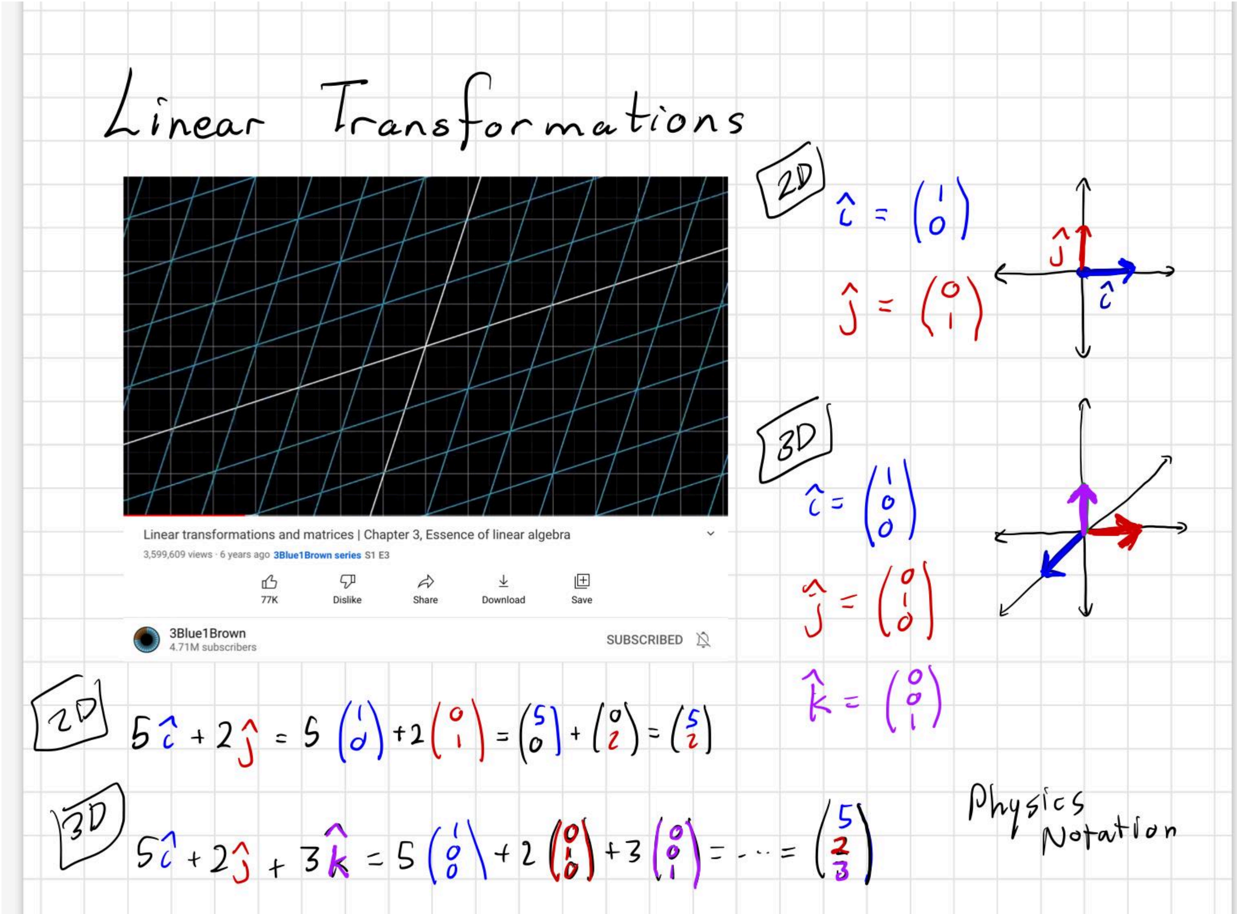
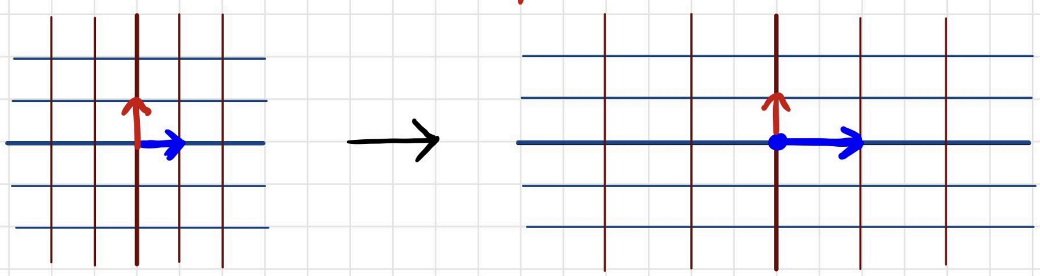
Watch 313F22-L10-1to9 Playlist



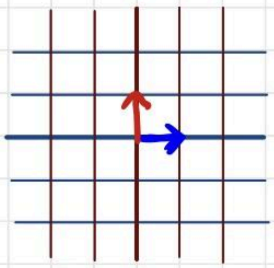
Watch the [Three Blue One Brown Video on Linear Tranformations]()

Classwork ①: Find the linear transformation matrix which scales the x direction by 2 and keeps the y direction fixed.



Solution Below:

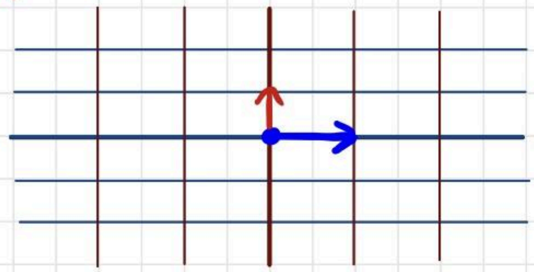
Classwork ①: Find the linear transformation matrix which scales the x direction by 2 and keeps the y direction fixed.



$$\begin{pmatrix}1\\0\end{pmatrix} \to \begin{pmatrix}2\\0\end{pmatrix}$$

$$\longrightarrow$$

$$\begin{pmatrix}0\\1\end{pmatrix} \to \begin{pmatrix}0\\1\end{pmatrix}$$



Find the matrix

$$\begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\\0\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}a & b\\c & d\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$\begin{pmatrix}a\cdot 1+b\cdot 0\\c\cdot 1+d\cdot 0\end{pmatrix} = \begin{pmatrix}2\\0\end{pmatrix} \qquad \begin{pmatrix}a\cdot 0+b\cdot 1\\c\cdot 0+d\cdot 1\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$$

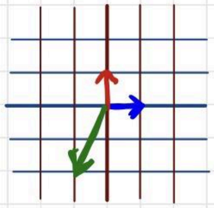
$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$\begin{pmatrix}a\\c\end{pmatrix} = \begin{pmatrix}2\\0\end{pmatrix} \qquad\qquad \begin{pmatrix}b\\d\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$$

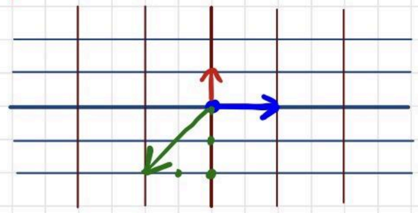
1st column of the matrix is $\begin{pmatrix}2\\0\end{pmatrix}$

2nd column of the matrix is $\begin{pmatrix}0\\1\end{pmatrix}$

The matrix for this linear transformation is $\begin{pmatrix}2 & 0\\0 & 1\end{pmatrix}$



$$\begin{pmatrix}1\\0\end{pmatrix} \to \begin{pmatrix}2\\0\end{pmatrix} \qquad \longrightarrow \qquad \begin{pmatrix}0\\1\end{pmatrix} \to \begin{pmatrix}0\\1\end{pmatrix}$$



Find the matrix

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\\0\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$\begin{pmatrix}a\cdot 1 + b\cdot 0\\ c\cdot 1 + d\cdot 0\end{pmatrix} = \begin{pmatrix}2\\0\end{pmatrix} \qquad \begin{pmatrix}a\cdot 0 + b\cdot 1\\ c\cdot 0 + d\cdot 1\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$\begin{pmatrix}a\\c\end{pmatrix} = \begin{pmatrix}2\\0\end{pmatrix} \qquad\qquad \begin{pmatrix}b\\d\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$$

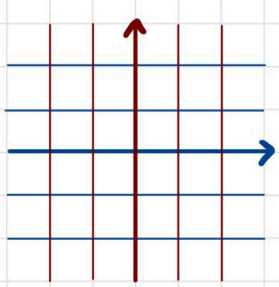
1st column of the matrix is $\begin{pmatrix}2\\0\end{pmatrix}$ — 2nd column of the matrix is $\begin{pmatrix}0\\1\end{pmatrix}$

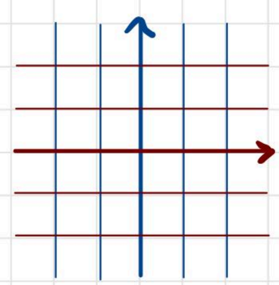
The matrix for this linear transformation is $\begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix}$

Where does the vector $\begin{pmatrix}-1\\-2\end{pmatrix}$ map to?

$$\begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}-1\\-2\end{pmatrix} = \begin{pmatrix}2\cdot(-1)+0(-2)\\ 0\cdot(-1)+1(-2)\end{pmatrix} = \begin{pmatrix}-2\\-2\end{pmatrix}$$

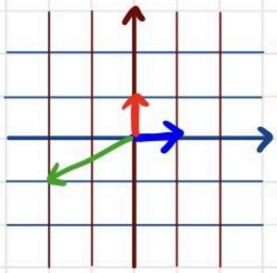
Classwork ②: Find the Linear Transformation Matrix which switches the x and y directions.





Solution Below:

Classwork ②: Find the Linear Transformation Matrix which switches the x and y directions.



input $\quad$ output

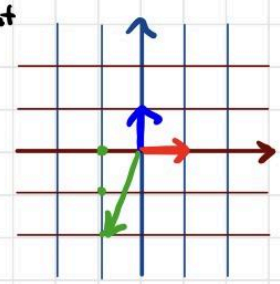
$\begin{pmatrix}1\\0\end{pmatrix} \longrightarrow \begin{pmatrix}0\\1\end{pmatrix}$

$\longrightarrow$

input $\quad$ output

$\begin{pmatrix}0\\1\end{pmatrix} \longrightarrow \begin{pmatrix}1\\0\end{pmatrix}$

$\begin{pmatrix}-2\\-1\end{pmatrix} \longrightarrow \begin{pmatrix}-1\\-2\end{pmatrix}$



Find the matrix

$$\begin{pmatrix}a & b\\c & d\end{pmatrix}\underbrace{\begin{pmatrix}1\\0\end{pmatrix}}_{\text{input}} = \underbrace{\begin{pmatrix}0\\1\end{pmatrix}}_{\text{output}} \quad \text{and} \quad \begin{pmatrix}a & b\\c & d\end{pmatrix}\underbrace{\begin{pmatrix}0\\1\end{pmatrix}}_{\text{input}} = \underbrace{\begin{pmatrix}1\\0\end{pmatrix}}_{\text{output}}$$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$\begin{pmatrix}a\cdot 1 + b\cdot 0\\c\cdot 1 + d\cdot 0\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$\begin{pmatrix}a\\c\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix} \qquad\qquad \begin{pmatrix}b\\d\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$$

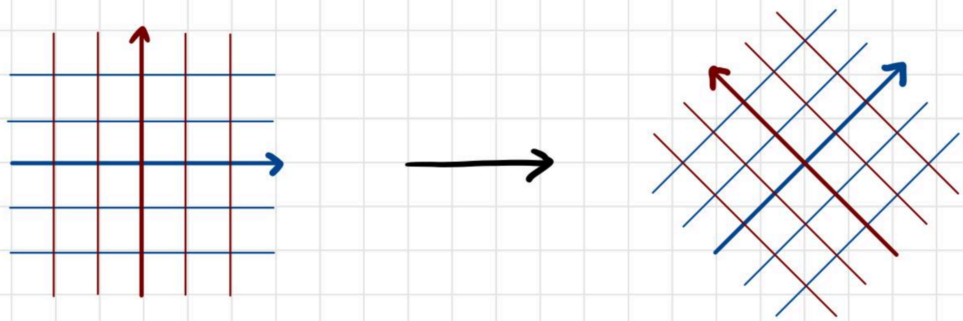
first column is the output for $\hat{i} = \begin{pmatrix}1\\0\end{pmatrix}$

second column is the output for $\hat{j} = \begin{pmatrix}0\\1\end{pmatrix}$

So the matrix is $\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$ switch matrix!

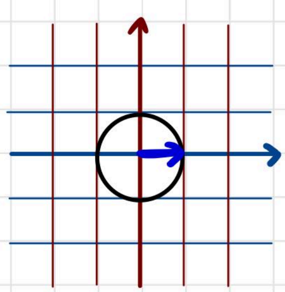
$$\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}-2\\-1\end{pmatrix} = \begin{pmatrix}0(-2)+1(-1)\\1(-2)+0(-1)\end{pmatrix} = \begin{pmatrix}-1\\-2\end{pmatrix} \quad \text{switched!}$$

Classwork (3) Find the Linear Transformation Matrix which rotates the plane by 45° ↺



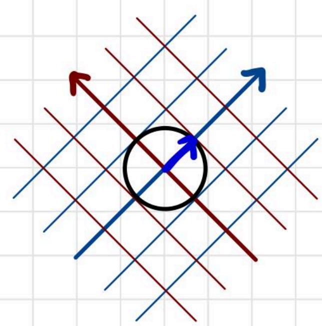
Solution Below:

Classwork (3) Find the Linear Transformation Matrix which rotates the plane by 45° ↺

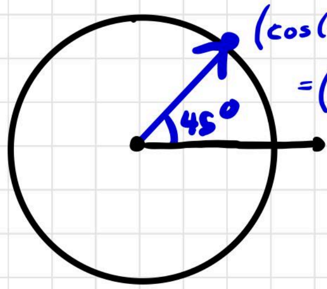


$$\begin{pmatrix}1\\0\end{pmatrix} \to \begin{pmatrix}\sqrt{2}/2\\ \sqrt{2}/2\end{pmatrix}$$

because:



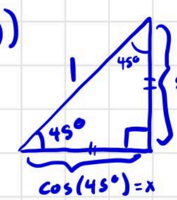
Recall the unit circle from trigonometry



$(\cos(45°), \sin(45°))$

$= \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

$$\begin{pmatrix}\sqrt{2}/2\\ \sqrt{2}/2\end{pmatrix}$$
output



$\sin(45°) = x$

$\cos(45°) = x$

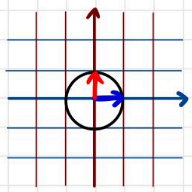
$x^2 + x^2 = 1^2$

$2x^2 = 1$

$x^2 = \frac{1}{2}$

$x = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

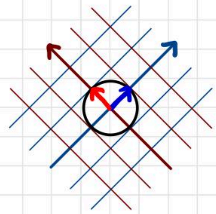
Classwork ③ Find the Linear Transformation Matrix which rotates the plane by 45°.



$$\begin{pmatrix}1\\0\end{pmatrix} \to \begin{pmatrix}\sqrt{2}/2\\ \sqrt{2}/2\end{pmatrix}$$

$$\longrightarrow$$

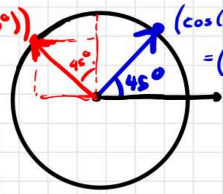
$$\begin{pmatrix}0\\1\end{pmatrix} \to \begin{pmatrix}-\sqrt{2}/2\\ \sqrt{2}/2\end{pmatrix}$$



Recall the unit circle from trigonometry

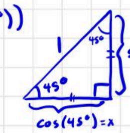
$(\cos(45°+90°), \sin(45°+90°))$
$= (-\sin(45°), \cos(45°))$
$= \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

$\begin{pmatrix}-\sqrt{2}/2\\ \sqrt{2}/2\end{pmatrix}$ output



$(\cos(45°), \sin(45°))$
$= \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

$\begin{pmatrix}\sqrt{2}/2\\ \sqrt{2}/2\end{pmatrix}$ output



$\sin(45°) = x$

$\cos(45°) = x$

$x^2 + x^2 = 1^2$
$2x^2 = 1$
$x^2 = \frac{1}{2}$
$x = \sqrt{\tfrac{1}{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

Find matrix:

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\sqrt{2}/2\\ \sqrt{2}/2\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-\sqrt{2}/2\\ \sqrt{2}/2\end{pmatrix}$$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$\begin{pmatrix}a\\c\end{pmatrix} = \begin{pmatrix}\sqrt{2}/2\\ \sqrt{2}/2\end{pmatrix} \qquad \begin{pmatrix}b\\d\end{pmatrix} = \begin{pmatrix}-\sqrt{2}/2\\ \sqrt{2}/2\end{pmatrix}$$

Our matrix is

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}\sqrt{2}/2 & -\sqrt{2}/2\\ \sqrt{2}/2 & \sqrt{2}/2\end{pmatrix}$$

In general
Rotation by θ
counterclockwise
has

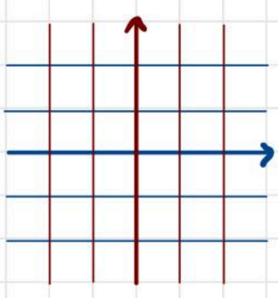
$$\begin{pmatrix}1\\0\end{pmatrix} \to \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}$$

$$\begin{pmatrix}0\\1\end{pmatrix} \to \begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}$$

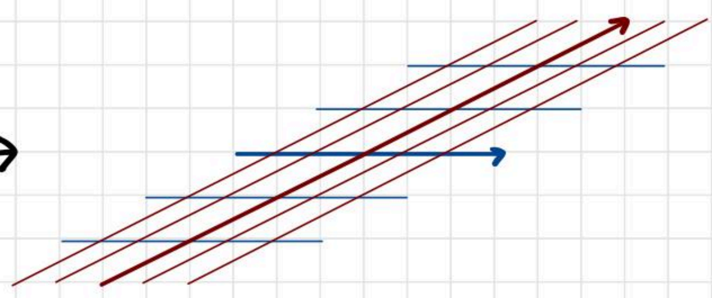
So the rotation matrix

$$Rot_\theta = \begin{pmatrix}\cos\theta & (-\sin\theta)\\ \sin\theta & \cos\theta\end{pmatrix}$$

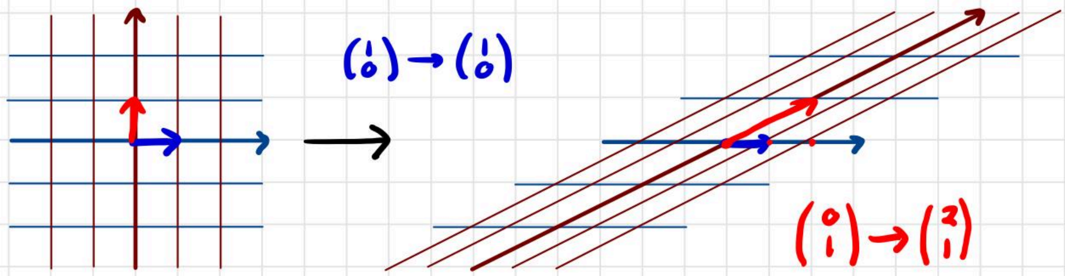
Classwork ④: Find a linear transformation which skews the plane:



$$\longrightarrow$$



Classwork (4): Find a linear transformation which skews the plane:



Find the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \overset{\text{output}}{\begin{pmatrix}1\\0\end{pmatrix}} \quad \text{and} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \overset{\text{output}}{\begin{pmatrix}2\\1\end{pmatrix}}$$

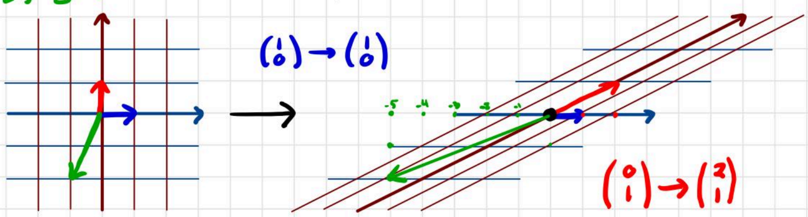
$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$\begin{pmatrix}a\\c\end{pmatrix} = \overset{\text{output}}{\begin{pmatrix}1\\0\end{pmatrix}} \qquad\qquad \begin{pmatrix}b\\d\end{pmatrix} = \overset{\text{output}}{\begin{pmatrix}2\\1\end{pmatrix}}$$

So our matrix is $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

Classwork ④: Find a linear transformation which skews the plane:

Where does $\begin{pmatrix}-1\\-2\end{pmatrix}$ go?



Applications! Computer Animation + Art!

Inside Art Programs there are matrices doing the skews!

Find the matrix

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \overset{\text{output}}{\begin{pmatrix}1\\0\end{pmatrix}} \quad \text{and} \quad \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \overset{\text{output}}{\begin{pmatrix}2\\1\end{pmatrix}}$$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

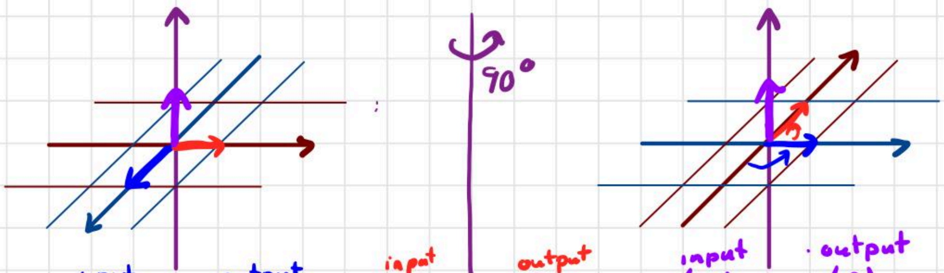
$$\begin{pmatrix}a\\c\end{pmatrix} = \overset{\text{output}}{\begin{pmatrix}1\\0\end{pmatrix}} \qquad\qquad \begin{pmatrix}b\\d\end{pmatrix} = \overset{\text{output}}{\begin{pmatrix}2\\1\end{pmatrix}}$$

So our matrix is $\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}$

$$\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}\begin{pmatrix}-1\\-2\end{pmatrix} = \begin{pmatrix}1(-1)+2(-2)\\ 0(-1)+1(-2)\end{pmatrix} = \begin{pmatrix}-1-4\\ -2\end{pmatrix} = \underset{\text{output}}{\begin{pmatrix}-5\\-2\end{pmatrix}}$$

Classwork (5): Find a 3D Linear Transformation which rotates about the z axis by 90°

Very Important for Computer Graphics and Robotics



input $\begin{pmatrix}1\\0\\0\end{pmatrix} \longrightarrow$ output $\begin{pmatrix}0\\1\\0\end{pmatrix}$ input $\begin{pmatrix}0\\1\\0\end{pmatrix} \longrightarrow$ output $\begin{pmatrix}-1\\0\\0\end{pmatrix}$ input $\begin{pmatrix}0\\0\\1\end{pmatrix} \longrightarrow$ output $\begin{pmatrix}0\\0\\1\end{pmatrix}$

Find a matrix such that

$$\begin{pmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix} \text{ and } \begin{pmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}-1\\0\\0\end{pmatrix} \text{ and } \begin{pmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

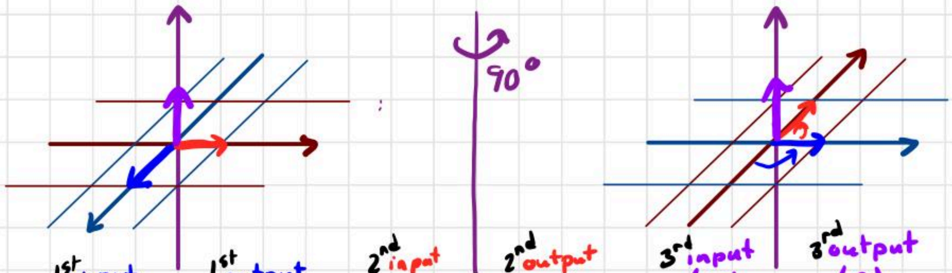
$$\Downarrow \qquad \Downarrow \qquad \Downarrow$$

$$\begin{pmatrix}a_{11}\cdot 1 + a_{12}\cdot 0 + a_{13}\cdot 0\\ a_{21}\cdot 1 + a_{22}\cdot 0 + a_{23}\cdot 0\\ a_{31}\cdot 1 + a_{32}\cdot 0 + a_{33}\cdot 0\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$\Downarrow$$

$$\underbrace{\begin{pmatrix}a_{11}\\ a_{21}\\ a_{31}\end{pmatrix}}_{1^{st}\text{ column}} = \underbrace{\begin{pmatrix}0\\1\\0\end{pmatrix}}_{1^{st}\text{ output}}$$

Very Important for Computer Graphics and Robotics



1st input $\begin{pmatrix}1\\0\\0\end{pmatrix} \longrightarrow$ 1st output $\begin{pmatrix}0\\1\\0\end{pmatrix}$ 2nd input $\begin{pmatrix}0\\1\\0\end{pmatrix} \longrightarrow$ 2nd output $\begin{pmatrix}-1\\0\\0\end{pmatrix}$ 3rd input $\begin{pmatrix}0\\0\\1\end{pmatrix} \longrightarrow$ 3rd output $\begin{pmatrix}0\\0\\1\end{pmatrix}$

Find a matrix such that

$$\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}0\\1\\0\end{pmatrix} \text{ and } \begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}=\begin{pmatrix}-1\\0\\0\end{pmatrix} \text{ and } \begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\Downarrow \qquad\qquad \Downarrow \qquad\qquad \Downarrow$$

$$\begin{pmatrix}a_{11}\cdot 1+a_{12}\cdot 0+a_{13}\cdot 0\\a_{21}\cdot 1+a_{22}\cdot 0+a_{23}\cdot 0\\a_{31}\cdot 1+a_{32}\cdot 0+a_{33}\cdot 0\end{pmatrix}=\begin{pmatrix}0\\1\\0\end{pmatrix} \qquad \begin{pmatrix}a_{11}0+a_{12}1+a_{13}0\\a_{21}0+a_{22}1+a_{23}0\\a_{31}0+a_{32}1+a_{33}0\end{pmatrix}=\begin{pmatrix}-1\\0\\0\end{pmatrix} \qquad \begin{pmatrix}a_{11}0+a_{12}0+a_{13}1\\a_{21}0+a_{22}0+a_{23}1\\a_{31}0+a_{32}0+a_{33}1\end{pmatrix}=\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\Downarrow \qquad\qquad \Downarrow \qquad\qquad \Downarrow$$

$$\underbrace{\begin{pmatrix}a_{11}\\a_{21}\\a_{31}\end{pmatrix}}_{\text{1st column}}=\underbrace{\begin{pmatrix}0\\1\\0\end{pmatrix}}_{\text{1st output}} \qquad \underbrace{\begin{pmatrix}a_{12}\\a_{22}\\a_{32}\end{pmatrix}}_{\text{2nd column}}=\underbrace{\begin{pmatrix}-1\\0\\0\end{pmatrix}}_{\text{2nd output}} \qquad \underbrace{\begin{pmatrix}a_{13}\\a_{23}\\a_{33}\end{pmatrix}}_{\text{3rd column}}=\underbrace{\begin{pmatrix}0\\0\\1\end{pmatrix}}_{\text{3rd output}}$$

So the matrix is $\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}=\begin{pmatrix}0&-1&0\\1&0&0\\0&0&1\end{pmatrix}$

HW1-HW4:

Homework

HW1: Find the linear transformation matrix which fixes the x direction and scales the y direction by 3.

Where does $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$ get mapped to?

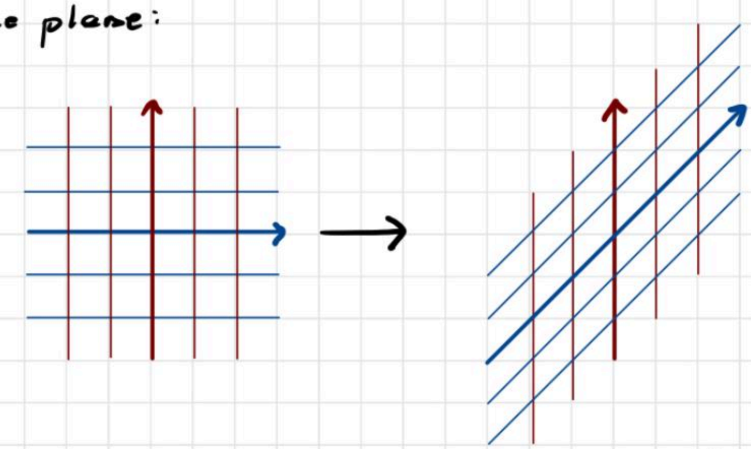
HW2: Find the linear tranformation matrix which fixes the x direction and scales the y direction by -1.

Where does $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$ get mapped to?

HW3: Find the linear transformation matrix which rotates the plane by 30° counter clockwise.

Where does $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$ get mapped to?

HW4: Find the linear transformation matrix which skews the plane:



Extra Credit: Find the 3D linear transformation matrix which rotates 45° about y axis.

Hint for the extra credit:

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Part II Complex Numbers
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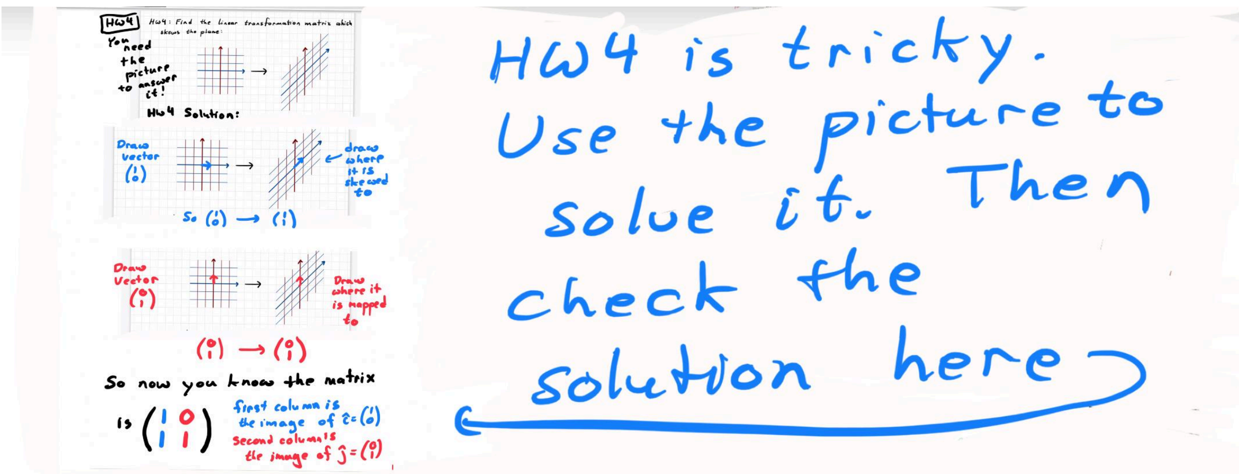
Read [Beezer Preliminaries on Complex numbers](), or watch Kahn Academy videos, and then practice at [IXL]() and complete Lesson 1 questions if you never did them before. Then complete the homework below.

HW5 below:
5a) (5+2i)+(3-4i)=
5b) (5+4i)-(6+7i)=
5c) (5+2i)(3+4i)
5d) (5+2i)(5-2i)=
5e) (5+2i)/(3+4i)=

You may continue to Lesson 11 without waiting for feedback.



Hint for the extra credit:
https://www.khanacademy.org/math/linear-algebra/matrix-transformations/lin-trans-examples/v/rotation-in-r3-around-the-x-axis