B. Sc. Mathematics

Semester – VI

LINEAR PROGRAMMING

SANGHAMAM COLLEGE OF ARTS AND SCIENCE
ANNAMANGALAM , GINGEE -604210

DEPARTMENT OF MATHEMATICS

B.SC
(MATHAMATICS)

Section - I

Operations Research: Origin, Definition and scope.

Linear Programming: Formulation and solution of linear programming problems by graphical and simplex methods, Big - M and two-phase methods, Degeneracy, Duality in linear programming.

Section – II&III

Transportation Problems: Basic feasible solutions, Optimum solution by stepping stone and modified distribution methods, Unbalanced and degenerate problems, Transhipment problem. Assignment problems: Hungarian method, Unbalanced problem, Case of maximization, Travelling salesman and crew assignment problems.

Section – IV

Game Theory: Two-person zero sum game, Game with saddle points, the rule of dominance; Algebraic, Graphical and linear programming methods for solving mixed strategy games.

Note: The question paper of each course will consist of **five** Sections. Each of the sections **I to IV** will contain **two** questions and the students shall be asked to attempt **one** question from each. **Section- V** shall be **compulsory** and will contain **eight** short answer type questions without any internal choice covering the entire syllabus.

Books recommended:

1. H.A. Taha, Operation Research-An introduction, Printice Hall of India.
2. P.K. Gupta and D.S. Hira, Operations Research, S. Chand & Co.
3. S.D. Sharma, Operation Research, Kedar Nath Ram Nath Publications.
4. J.K. Sharma, Mathematical Model in Operation Research, Tata McGraw Hill.

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INTRODUCTION TO OPERATIONS RESEARCH

Structure

Introduction

Origin and Definitions of Operations Research

Scope of Operations Research

Advantages of Operations Research

Limitations of Operations Research

Convex Set

Check Your Progress

Summary

INTRODUCTION

Operations Research (O.R.) is a discipline that provides scientific methods for the purpose of solving real life problems that helps us in determining the best utilization of limited resources. Here we study about optimization techniques. In everyday life, we observe many situations of optimization around us. For example, suppose we want to maximize the profit or minimize the cost then maximization of the profit or minimization of cost is the optimization of profit/cost. In O.R., we obtain the optimal solution for decision making problems with the help of optimization techniques. This chapter contains origin, definitions and scope of Operations Research. In this unit, we also discuss the concept of convex sets.

Objectives. The objective of these contents is to get familiar reader with Operations Research. After studying this unit, reader should be able to define/describe the following concepts like:

What is Operations Research?

Origin of Operations Research.

Scope of operation research.

Convex Set.

ORIGIN AND DEFINITIONS OF OPERATIONS RESEARCH

Origin: Operations Research came into existence and gained prominence during the World War II in Britain with the establishment of team of scientists to study the strategic and tactical problems of various military operations. Scientists of different disciplines were part of this team, their research on military operation soon find applications in other fields also. Now, it was started applying in the fields of industry, trade, agriculture, planning and various other fields of economy and named as

'Operations Research'. Hence the scientific methods and techniques of Operations Research became equally useful for the planners, economists, administrators, irrigation or agricultural experts and statisticians etc. The use of Operations Research has not limited to the Britain only. Many countries of the world had started using O.R. India was one of the few first countries who started using O. R. Regional Research Laboratory located at Hyderabad was the first Operations Research unit established in India during 1949. With the opening of this unit Operations Research in India came into existence. At the same time one more unit was set up in Defence Science Laboratory. In 1955, Operations Research Society of India was formed. Today, O.R. became a professional discipline and studied as a popular subject in Management institutes and school of Mathematics.

Definitions:

Operations Research can be defined simply as combination of two words operation and research where operation means some action applied in any area of interest and research imply some organized process of getting and analysing information about the problem environment. However, many scientists or experts has been defined O.R. in various ways but the opinions about the definitions of it have been changed according to the growth of the subject. So before defining O.R. it is important to see few definitions of it.

1. O.R. is a scientific method of providing executive departments with a quantitative basis for decisions regarding the operations under their control.

-Morse and Kimbal (1946)

2. O.R. is a scientific method of providing executive with an analytical and objective basis for decisions.

-P.M.S. Blackett (1948)

3. O.R. is the application of scientific methods, techniques and tools to problems involving the operations of system so as to provide these in control of the operations with optimum solutions to the problem.

-Churchman, Acoff, Arnoff (1957)

4. O.R. is a management activity pursued in two complementary ways one-half by the free and bold exercise of commonsense untrammeled by any routine, and other half by the application of a

repertoire of well-established pre created methods and techniques.

-Jagjit Singh (1968)

On the basis of all above opinions, Operations Research can be defined in more general and comprehensive way as:

“Operation research is a branch of science which is concerned with the application of scientific methods and techniques to decision making problems and with establishing the optimal solutions".

SCOPE OF OPERATIONS RESEARCH

Scope of O.R. is very wide in today’s world as it provides better solution to various decision-making problems with great speed and efficiency. Areas where methods/models developed in Operations Research can be applied are given here under:

1. In Agriculture:

With the explosion of population and consequent shortage of food, every country is facing the problem of optimum allocation of land to various crops in accordance with the climatic conditions, optimum distribution of water from different resources. Problems of agriculture production under various restrictions can be solved by applications of Operations Research techniques.

2. **In Defence Operations**:

Since Second World War operation research have been used for Defence operations with the aim of obtaining maximum gains with minimum efforts.

3. In Finance:

In these modern times, government of every country or every organisation wants to introduce such type of planning/policies regarding their finance and accounting which optimize capital investment, determine optimal replacement strategies, apply cash flow analysis for long range capital investments, formulate credit policies, credit risk. Techniques developed in O.R. can be applied for attaining above said things.

4. In Marketing:

A Marketing Administrator has to face many problems like production selection, formulation of competitive strategies, distribution strategies, selection of advertising media with respect to cost and time, finding the optimal number of salesmen, finding optimum time to launch a product. All such problems can be overcome using Operations Research Techniques.

5. In Personnel Management:

Every organization wants to make selection of personnel on minimum salary. It needs to find the best combination of workers in different categories with respect to costs, skills, age and nature of jobs. It also needs to frame recruitment policies, assign jobs to machines or workers.

6. In LIC:

Operations Research Techniques can be fruitfully applied in LIC offices as it enables the policy makers to decide the premium rates for various modes of policies.

7. In Research and Development:

In determination of the areas of concentration of research and development. It also helps in project selection.

O.R. helps in solving many other problems faced by public as well as private sectors such as the ones in economic and social planning, management of natural resources, energy, housing pollution control, waiting lines and administrative problems, insurance policies and many more.

ADVANTAGES OF OPERATIONS RESEARCH

Following are certain advantages of Operations Research (OR):

- Operations Research helps decision –maker to take better and quicker decisions. It helps decision –maker to evaluate the risk and results of all the alternative decisions. So, it improves the quality of decisions and makes the decisions more effective.
- Operation Research helps, in preparing future managers as it provides in-depth knowledge about a particular action.
- Operations Research develop models, which provides logical and systematic approach for understanding, Solving and controlling a problem.
- Operations research reduces the chances of failure as it provides many alternatives for one problem, which helps the management to choose the best decision. Even managers can evaluate the risks associated with each solution and can decide whether they want to go with the solution or not.
- It helps users in optimum use of resources. For example, linear programming techniques in Operations Research suggest most effective methods and efficient ways of optimality.
- It helps in finding the limitations and scope of an activity.
- Using this information, he can measure the performance of employees and can compare it with the standard performance. It modifies mathematical solutions before these are applied. Managers may accept or modify the mathematical solutions obtained using Operations Research techniques.
- It helps suggest alternative solutions for the same optimum profit if the management wants so.

LIMITATIONS OF OPERATIONS RESEARCH

- Formulation of mathematical models may take into account all possible factors for defining a real- life problem and hence is difficult. As a result, the help of computers is required for the large number of cumbersome computations for such problems. This discourages small companies and other organisations from using O.R. techniques.
- Unquantifiable factors: Some problems may involve a large number of intangible factors such as human emotions, human relationship, etc. which cannot be quantified. Hence, the best solution cannot be determined for such problems because such factors have to be excluded.
- Dependence on experts: A specialist, who may be a mathematician or a statistician, is needed to understand the formulation of models, find solutions and recommend their implementation. Managers, who deal with such problems, may not have such specialisation. Managers, who deal with such problems, may not have such specialisation and hence the results may not be optimal.
- Model is abstraction of real-life situations and not the reality.
- Assumptions need to be made about the nature and importance of some factors in order to construct an Operation Research model.

- A reasonably good solution without the use of Operation Research may be preferred by the management as compared to a slightly better solution provided by using Operation Research since it is very expensive in terms of time and money.

In the next chapter onwards, we shall introduce various O.R. techniques for obtaining optimal and feasible solutions. Before studying these techniques, you must familiar with some important basic concepts like convex sets and basic feasible solutions. Now, we will discuss these concepts.

CONVEX SET

A region or a set K is convex if and only if for any two points on the set K, the line segment connecting these points lies entirely in K. Mathematically, . $(x_1, y_1) \in K$.

$,0 \leq \lambda \leq 1$

$(\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda) y_2) \in K$

Where $(\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2)$ gives all the points which lie on the line segment joining (x_1, y_1) and (x_2, y_2).

Example of a Convex Set

Example.

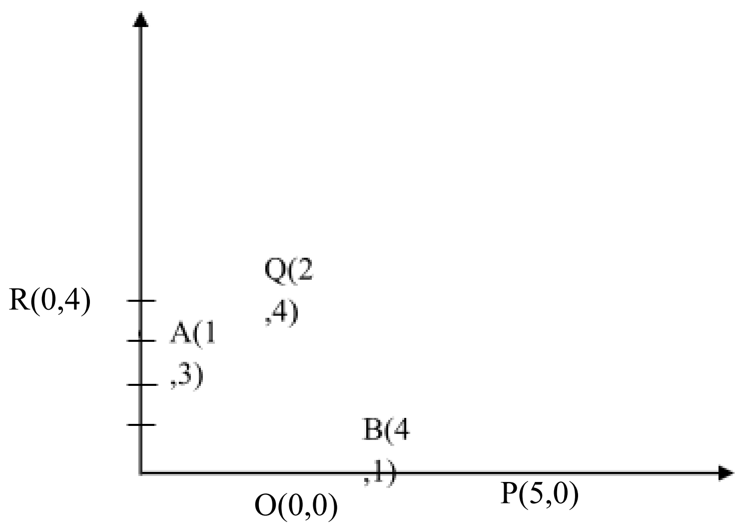


Fig. 1.1

Consider the region enclosed by OPQR. Let us denote it by K. It is convex as the line segment joining any two points in this region lies wholly within it. As an example, let us take two points A (1, 3) and B (4, 1).

Then all points on the line segment joining A and B are given by

$(\lambda(1) + (1-\lambda)4, \lambda(3) + (1-\lambda)(1)) = (\lambda + 4 - 4\lambda, 3\lambda + 1 - \lambda)$

$= (4 - 3\lambda, 1 + 2\lambda), 0 \leq \lambda \leq 1.$

Here $\lambda = 0$ gives the point Q (4, 1) and $\lambda = 1$ gives the point P (1, 3). Other points on the line segment AB are given by

$= (4 - 3\lambda, 1 + 2\lambda)$, where $0 < \lambda < 1$

For example, let us take λ=0.1, 0.3, 0.5, 0.7, 0.9; then the corresponding points (after substituting the values of λ = 0.1, 0.3, 0.5, 0.7, 0.9) are;

(4 – 3(0.1), 1 + 2(0.1)), (4-3$\times$ 0.3, 1 + 2 $\times$ 0.3), (4- 3$\times$ 0.5, 1 + 2 $\times$ 0.5), (4-3$\times$ 0.7, 1 + 2 $\times$ 0.7), (4-3$\times$ 0.9, 1 + 2 $\times$ 0.9)

i.e., (3.7,1.2), (3.1 ,1.6), (2.5,2), (1.9, 2.4), (1.3, 2.8)

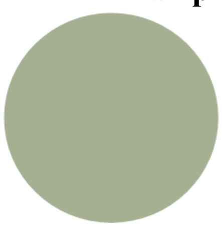
All these points clearly lie on the line and also in the region K.

Similarly, all other points on the line segment AB also lie inside the region K. Hence, the line segment AB lies in K.

Therefore, K is convex in this example.

Example of Non-Convex Set

Example.



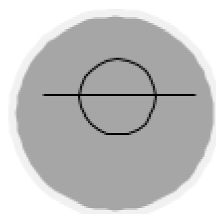


Fig. 1.2

Consider the shaded region in Fig. 1.2, clearly the line segment joining two points do not lie wholly in the region and hence this is an example of non-convex set.

Example. Show that the set T = {(x, y): $x^2 + y^2 \leq 1$} is a convex set.

Solution: Let us take any two points A (x_1, y_1) and B (x_2, y_2) in Fig. 1.3 such that:
$x_1^2 + y_1^2 \leq 1$, And$x_2^2 + y_2^2 \leq 1$.

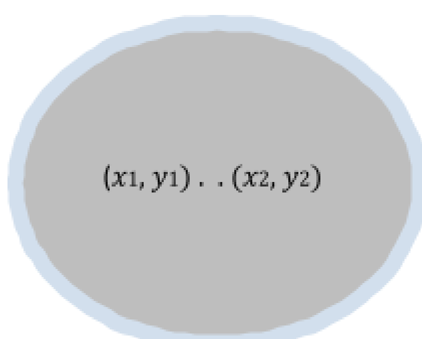


Fig. 1.3

Now, the line segment joining A and B is the set

$\{\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2 : 0 \leq \lambda \leq 1\}$.

Let $u_1 = \lambda x_1 + (1-\lambda)x_2$, $u_2 = \lambda y_1 + (1-\lambda)y_2$

Therefore, all points on the line segment AB are given by (u_1, u_2).

Now, the line segment AB lies wholly in T if

$$u_1^2 + u_2^2 \le 1$$

Since $u_1^2 + u_2^2 = [\lambda x_1 + (1-\lambda)x_2]^2 + [\lambda y_1 + (1-\lambda)y_2]^2$

$=\lambda^2 x_1^2 + (1-\lambda)^2 x_2^2 + 2\lambda(1-\lambda)x_1x_2 + \lambda^2 y_1^2 + (1-\lambda)^2 y_2^2 + 2\lambda(1-\lambda)y_1y_2$

$= \lambda^2[x_1^2 + y_1^2] + (1-\lambda)^2[x_2^2 + y_2^2] + 2\lambda(1-\lambda)[x_1x_2 + y_1y_2]$

We have

$u_1^2 + u_2^2 \le \lambda^2 + (1-\lambda)^2 + 2\lambda(1-\lambda)[x_1x_2 + y_1y_2]$ ...(1)

Now consider $(x_1x_2 + y_1y_2)^2 = x_1^2x_2^2 + y_1^2y_2^2 + 2x_1x_2y_1y_2$

$= x_1^2x_2^2 + y_1^2y_2^2 + x_1^2y_2^2 + x_2^2y_1^2 - x_1^2y_2^2 - x_2^2y_1^2 + 2x_1x_2y_1y_2$

$= (x_1^2 + y_1^2)(x_2^2 + y_2^2) - x_1y_2(x_1y_2 - x_2y_1) - x_2y_1(x_2y_1 - x_1y_2)$

$= (x_1^2 + y_1^2)(x_2^2 + y_2^2) - (x_2y_1 - x_1y_2)^2$

$\le (x_2y_1 - x_1y_2) \le 1$

$\Rightarrow (x_1x_2 + y_1y_2) \le 1$

∴ From (1) and (2), we have

$$u_1^2 + u_2^2 \le \lambda^2 + (1-\lambda)^2 + 2\lambda(1-\lambda)$$

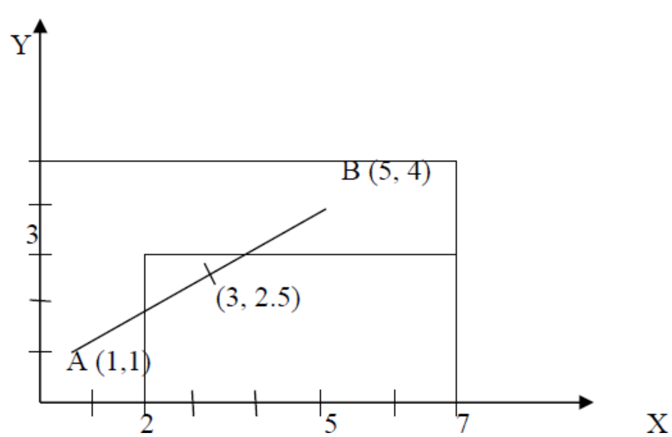
Or $u_1^2 + u_2^2 \le [\lambda + (1-\lambda)]^2$

$$\Rightarrow u_1^2 + u_2^2 \le 1$$

∴ T is convex set.

Example. Show that the set T = {(x, y): 0 ≤ y ≤ 5 when 0 ≤ x ≤ 2 and 3 ≤ y ≤ 5 when 2 ≤ x ≤ 7} is not a convex set.

Solution: Let us take two points A (1, 1) and B (5, 4) in the given region S.



Now, all the points on the line segment AB are given as

$\{ \lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2, \quad 0 \le \lambda \le 1\}$

$= \{ \lambda(1) + (1-\lambda)5, \lambda(1) + (1-\lambda)4, 0 \le \lambda \le 1\}$

$= \{5\text{-}4\lambda, 4 - 3\lambda, 0 \le \lambda \le 1\}$

Let us take one of these points and put $\lambda = \frac{1}{2}$. So, the point is

$(5-4\times\frac{1}{2}, 4-3\times\frac{1}{2}) = (5-2, 4-\frac{3}{2}) = (3, 2.5)$

This point is on the line, but does not belong to the given set T since for y = 2.5, x should lie between 0 and 2 but here it is 3. Therefore, the given set is not convex.

Extreme Points of a convex set

A point (x, y) in a convex set K is called an extreme point if it is not possible to locate two distinct points in or on K so that the line joining them will include (x, y).

Mathematically, a point (x, y) is an extreme point of a convex set if it cannot be expressed as a convex combination of any two points (x_1, y_1) and (x_2, y_2) [for $(x_1, y_1) \neq (x_2, y_2)$] in the set such that

$x = \lambda x_1 + (1-\lambda)x_2$ and $y = \lambda y_1 + (1-\lambda)y_2$, $0<\lambda < 1$

Remark:

i) The vertices of the polygons, which are convex sets, are the extreme points.
ii) Every point on the circumference of the region containing the portion in and on the circle is an extreme point.

IMPORTANT DEFINITIONS Solution

A set of values of the decision variables which satisfy the constraints of the given LPP is said to be a solution of that LPP.

Feasible Solution

A solution in which values of decision variables satisfy all the constraints and non-negativity conditions of an LPP simultaneously is known as feasible solution.

Infeasible Solution

A solution in which values of decision variables do not satisfy all the constraints and non-negativity conditions of an LPP simultaneously is known as infeasible solution.

Basic solution

Suppose there are m equations representing constraints (limited available resources) containing m + n variables in an allocation problem. The solution obtained by setting any n variables equal to zero and solving for the remaining m variables and the remaining n variables are non – basic variables.

The maximum number of possible basic solutions is given by the formula C^{m+n}_{n}

For example, if there are 2 equations in 3 variables, then the maximum number of possible basic solutions is

$$C^3_2 = \frac{3!}{2!\,(3-2)!} = 3.$$

Basic Feasible Solution

A basic solution for which all the basic variables are non – negative is called the basic feasible solution.

Further basic feasible solution are of two types:

Degenerate Solution

A basic feasible solution is known as degenerate if value of at least one basic variable iszero.

Non-Degenerate Solution

A basic feasible solution is known as non- degenerate if value of all basic variables are non-zero and positive.

Optimum Basic Feasible Solution

A basic feasible solution which optimizes i.e. maximise or minimise the objective function value of the given LPP is called optimum basic feasible solution.

Unbounded Solution

A solution in which value of the objective function of the given LPP increase/decrease indefinitely is called an unbounded solution.

Example. Determine all the basic feasible solutions of the equations: $2x_1 + 6x_2 + 2x_3 + x_4 = 3$

$6x_1 + 4x_2 + 4x_3 + 6x_4 = 2$

Solution: Here the maximum number of possible basic solutions is

$$C^4_2 = \frac{4!}{2!\,2!} = \frac{4 \times 3 \times 2}{2 \times 2} = 6$$

We obtain as follows:

Setting $x_1 = 0, x_2 = 0$, we have $2x_3 + x_4 = 3$ and $4x_3 + 6x_4 = 2$

$$\Rightarrow x_3 = 2, x_4 = -1$$

Setting $x_1 = 0, x_3 = 0$, we have $6x_2 + x_4 = 3$ and $4x_2 + 6x_4 = 2$

$$\Rightarrow x_2 = \frac{1}{2}, x_4 = 0$$

Setting $x_1 = 0, x_4 = 0$, we have $6x_3 + 2x_3 = 3$ and $4x_2 + 4x_3 = 2$

$$\Rightarrow x_2 = \frac{1}{2}, x_3 = 0$$

Setting $x_2 = 0, x_3 = 0$, we have $2x_1 + x_4 = 3$ and $6x_1 + 6x_4 = 2$

$$\Rightarrow x_1 = \frac{8}{3}, x_4 = -\frac{7}{3}$$

Setting $x_2 = 0, x_4 = 0$, we have $2x_1 + 2x_3 = 3$ and $6x_1 + 4x_4 = 2$

$$\Rightarrow x_1 = -2, x_4 = \frac{7}{2}$$

Setting $x_3 = 0, x_4 = 0$, we have $2x_1 + 6x_2 = 3$ and $6x_1 + 4x_2 = 2$

$$\Rightarrow x_1 = 0, x_2 = \frac{1}{2}$$

Thus, all basic solutions of the given system of equations are;

$(0, 0, 2, -1)$, $(0, \frac{1}{2}, 0, 0)$, $(0, \frac{1}{2}, 0, 0)$, $(\frac{8}{3}, 0, 0, -\frac{7}{3})$, $(-2, 0, \frac{7}{2}, 0)$, $(0, \frac{1}{2}, 0, 0)$.

Here $(0, \frac{1}{2}, 0, 0)$ is repeated thrice and hence the basic solutions are

$(0, 0, 2, -1)$, $(0, \frac{1}{2}, 0, 0)$, $(\frac{8}{3}, 0, 0, -\frac{7}{3})$, $(-2, 0, \frac{7}{2}, 0)$.

Of these solutions, the basic feasible solution is $(0, \frac{1}{2}, 0, 0)$ as in the other solutions, all decision variables are not positive.

Exercises.

1 Find all basic solutions for the system of simultaneous equations:

$$2x_1 + 3x_2 + 4x_3 = 5 \quad \text{and} \quad 3x_1 + 4x_2 + 5x_3 = 6.$$

(Hint. The maximum number of possible basic solutions is

$$C^3_2 = \frac{3!}{2!\,(3-2)!} = 3.$$

And no feasible solution)

2 Determine all the basic feasible solutions of the system of equations:

$3x_1 + 5x_2 + x_3 = 15$and $5x_1 + 2x_2 + x_4 = 10.$

Further, discuss that whether the solution is degenerate or non-degenerate.

(Hint. The maximum number of possible basic solutions will be $C^4 = 4!$

$= 4\times3\times2 = 6.$ 2 2!2! 2×2

And the possible solutions are

(0, 0, 15, 10), (0, 3, 0, 4), (0, 5, -10, 0), (5, 0, 0, -15), (2, 0, 9, 0) and $(\frac{50}{19}, \frac{45}{19}, 0, 0)$.)

CHECK YOUR PROGRESS

1 Explain the concept and scope of Operations Research.

2 Explain the advantages and disadvantages of Operations Research.

3 Define the followings:

(i) Feasible solution

(ii) Degenerate solution

(iii) Extreme points of a convex set.

SUMMARY

In this chapter, we have introduced Operations Research, its scope, advantages and limitations. We have observed that Operations Research is a very powerful method of getting the best out of limited resources. It finds applications in almost every field. Here, we explain concept of convex sets which is another important concept. We study feasible solution, basic solution, and basic feasible solution of a system of equations less in number than the number of decision variables. Such solutions are required to be obtained for finding out optimal solution of the given LPP.

2

LINEAR PROGRAMMING PROBLEMS

Structure

Introduction

Linear Programming Problem (LPP)

Mathematical Formulation of LPP

Graphical Method

Canonical and Standard Form of LPP

Check Your Progress.

Summary

INTRODUCTION

In 1947, George Dantzig and his associates, while working with the US Air Force during World War II, developed this technique, primarily for solving military logistics problems. They observed that a large number of military programming/planning problems could be formulated as maximizing/minimizing a linear form of profit/cost function whose variable were restricted to values satisfying a system of linear constraints. In chapter 1, We have already discussed the concept of optimization and explained the basic feasible solution of linear programming problem.

In this chapter, we study linear programming problems (LPP), their mathematical formulation, objective function concept and graphical method. We use graphical method mainly for solving problems involving two variables. Linear programming can be applied to a variety of problems such as production, transportation, advertising and problems in public and private organizations, e.g., business, industry, hospitals, libraries as also in education. In order to solve linear programming problems, we need to convert them into a canonical or standard form.

Objectives. The objective of these contents is to provide some important concepts/results to the reader like:

- Linear programming problems, it's applications and limitations.
- Mathematical formulation of linear programming problem.
- Graphical Method
- Canonical and Standard form of an LPP

LINEAR PROGRAMMING PROBLEM (LPP)

We have already familiar with the concept of optimization. A mathematical programming is an optimization technique by which the maximum or minimum value of a function is determined under certain conditions. Mathematical programming in which constraints are expressed as linear equalities / inequalities is called linear programming.

We first introduce three basic components essential for the development of LP theory.

Decision Variables: The variables in terms of which the problem is defined.

Objective Function: A function which is to be maximised or minimised subject to the given constraints/limitations.

Constraints: There are always certain limitations on the use of resources that limit the degree to which an objective can be achieved. These limitations are known as constraints or restrictions. Constraints must be represented as linear equalities or inequalities in terms of decision variables.

Every organisation, big or small wants to find the best allocation of resources in order to optimize the objective function. We can use linear programming only if the following conditions are satisfied:

i. Objective function should be well defined
ii. Objective function can be expressed as a linear function of the decision variables.
iii. There should be finite number of constraints and can be expressed as linear equalities or inequalities in terms of variables.
iv. Decision variables should be non- negative.

Advantages and Limitations of an LPP

Linear programming methods are used in many fields including business and industry by almost all their departments such as production, marketing, finance etc. it's some advantages are

- Helps in attaining the optimum use of resources i.e. maximise profit and minimise costs
- Improve the quality of decisions

There are many more advantages. In spite of having many advantages and wide areas of applications, there are some limitations as well. Following are certain limitations of linear programming:

- We can apply linear programming method only if relationships are linear.

- While solving LPP, it is possible that we will get non-integral values even for those decision variables which have only integral values.
- Constraints in the linear programming methods are written assuming all parameters are known and should be constant. However, in real problems, sometimes these are neither known nor constant.
- LP deals with the problems having single objective, whereas in real-life, there are many situations where we have to achieve multi-objectives.

MATHEMATICAL FORMULATION OF LPP

In our daily life, there are many real-life situations where LP problems may arise and for using LPP methods/techniques to find a solution of such situations, it becomes necessary to present the given word problem into mathematical form correctly. The steps of mathematical formulation of LPP are summarized as follows:

i) Identify the decision variable of the given problem.
ii) Formulate the objective function, which is to be maximised or minimised, as a linear function of the decision variables.
iii) Formulate the constraints and express them as linear inequalities or equalities in terms of decision variables.
iv) Introduce non-negative restrictions as negative values of the decision variables do not have any valid physical interpretation.

The steps of mathematical formulation of LPP are explained with the help of an example.

Example. A small-scale industry manufactures two products P and Q which are processed in a machine shop and assembly shop. Product P requires 2 hours of work in a machine shop and 4 hours of work in the assembly shop to manufacture while product Q requires 3 hours of work in the machine shop and 2 hours of work in the assembly shop. In one day, the industry cannot use more than 16 hours of machine shop and 22 hours of assembly shop. It earns a profit of rupees 3 per unit of product P and rupees. 4 per unit of product Q. Give a mathematical formulation of the problem as to maximise profit.

Solution: let x and y be the number of units of product P and Q, which are to be produced. Here x and y are the decision variables. Suppose Z is the profit function.

Since one unit of product P and one unit of product Q gives the profit of rupees 3 and rupees 4, respectively, the objective function is

Maximize $Z = 3x + 4y$

The requirement and availability in hours of each of the shops for manufacturing the products are tabulated as follows:

	Machine Shop	**Assembly Shop**	**Profit**
Product P	2 hours	4 hours	Rs.3 per unit
Product Q	3 hours	2 hours	Rs.4 per unit
Available hours per day	16 hours	22 hours	

Total hours of machine shop required for both types of product = 2x+3y

Total hours of assembly shop required for both types of product =4x+2y

Hence, the constraints as per the limited available resources are:

$2x+3y \le 16$ and $4x+2y \le 22$

Since the number of units produced for both P and Q cannot be negative, the non-negative restrictions are: $x \ge 0$, $y \ge 0$

Thus, the mathematical formulation of the given problem is:

Maximise $Z = 3x + 4y$

Subject to the constraints

$2x + 3y \le 16$

$4x + 2y \le 22$

And non-negative restrictions

$x \ge 0$, $y \ge 0$

Exercise. A company produces two types of items P and Q that require gold and silver. Each unit pf type P requires 4g silver and 1g gold while that od type Q requires 1g silver and 3g gold. The company produces 8g silver and 9g gold. If each unit of type P brings a profit of rupees 44 and that of type Q rupees 55, determine the number of units of each type that the company should produce to maximise the profit.

Answer. Let Z be the profit function. The mathematical formulation of the given problem is

Max. $Z = 44x + 55y$

Subject to the constraints:

$4x + y \le 8$,

$x + 3y \le 9$, $x \ge 0$, $y \ge 0$.

GRAPHICAL METHOD

The graphical method is used to solve simple linear programming problems having two decision variables. For solving LPPs involving more than two decision variables, we use another method called simplex method. We discuss it in chapter 3.

The steps of graphical method for solving an LPP are as follows:

1. **Plot the graph corresponding to the given constraints.**
2. **Determine the region for each given constraint.**
3. **Determine the feasible region.**
4. **Determine corner/extreme points.**
5. **Examine corner/extreme points.**

The following example explains steps of Graphical method.

Example.

The graphs are plotted for the equalities corresponding to the given inequalities for constraints as well as restrictions. That is, we first draw the straight lines. For example, suppose one of the given inequalities is $2x+3y \le 6$. Then, we first plot the graph for the equation $2x+3y = 6$, which is straight line. For this, we take any two points on it join them.

For example, for $x = 0$, $y = 6/3 = 2$ and for $y = 0$, $x = 6/2 = 3$. So, we get the straight line shown in fig. 2.1.

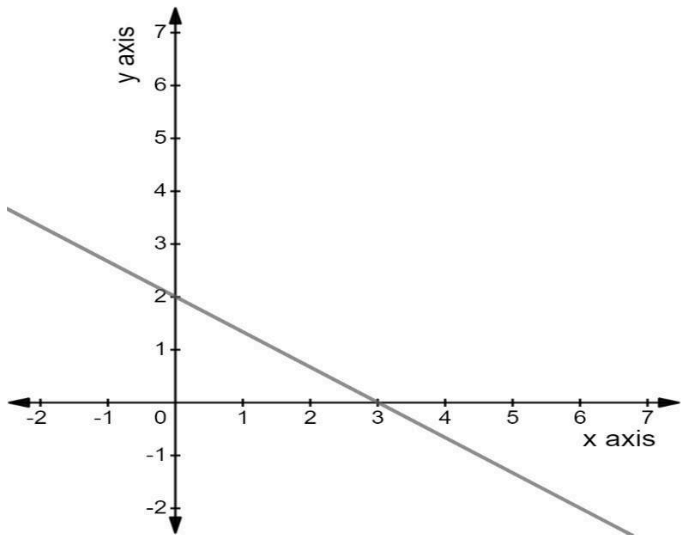


Fig.2.1

First, we determine the region corresponding to each inequality. Let us consider the inequality $2x+3y \le 6$ again. We can find the region on the graph satisfied by this inequality by substituting $x = 0$ and $y = 0$ in it. We get

$$2(0) + 3(0) \le 6 \Rightarrow 0 \le 6$$

which is correct. So, it is the region containing the point (0, 0). Hence, half plane shown in Fig. 2.2 by shading the region starting from the line towards the point (0, 0) is the graph of the given inequality.

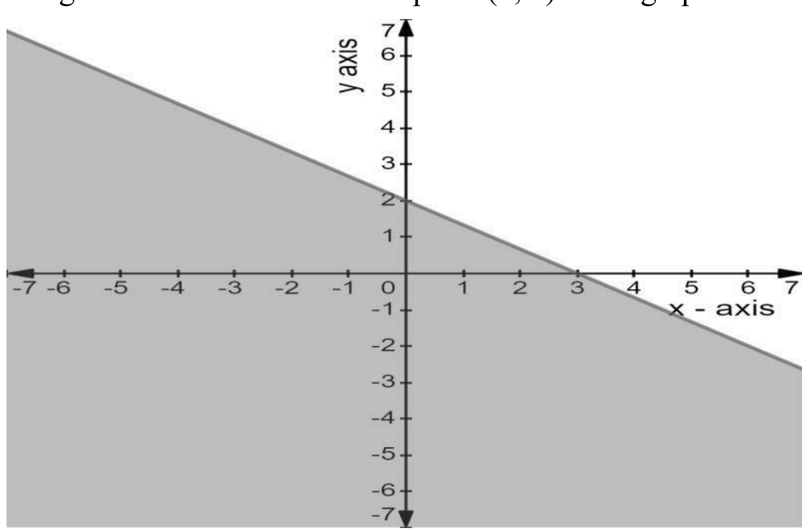


Fig.2.2

Had the given inequality been 2x+3y≥6, then we would have shaded the region on the opposite side of the line. This is because on putting x=0, y=0 in the inequality, we get

$2(0) + 3(0) \geq 6 \Rightarrow \quad 0 \geq 6$

which is not correct. Thus, the point (0, 0) does not satisfy the inequality and hence does not lie in the region. Thus, graph for the inequality 2x-3y ≥ 6 would, therefore, be as shown in Fig. 2.3.

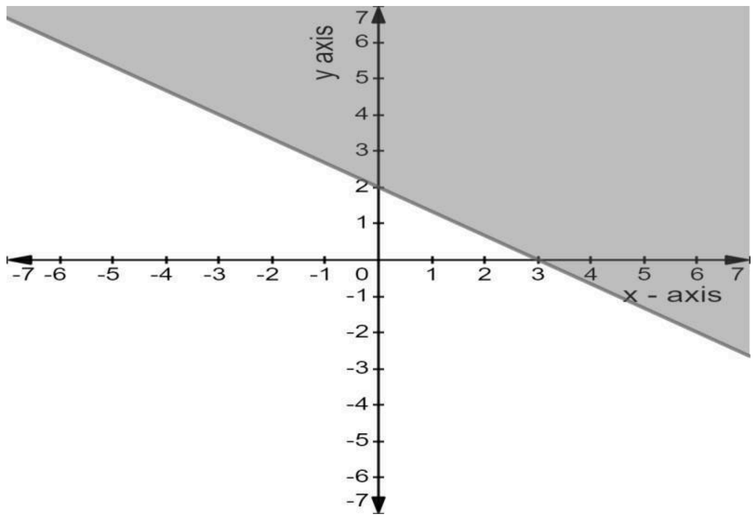


Fig.2.3

In this example, we have used the point (0, 0) to determine which half plane corresponds to the given inequality. However, you can take any other point. But using (0, 0) is far easier. If the right-hand side of the given inequality is zero, using the point (0, 0) in it is meaningless. For example, suppose the given inequality is 2x-3y≥0.The plot of 2x-3y=0 is given in fig. 2.4. It is straight line passing through the origin. Using the point (0, 0) in the inequality 2x-3y ≥ 0, we get 2(0)-3(0) ≥ 0, i.e., 0 ≥ 0. So, we cannot decide which half plane is the region of the given inequality. Therefore, in this case, we use any other point, say (2, 0), on putting x=2, y=0 in the given inequality, we get

$2(2)-3(0) \geq 0$ and $4 \geq 0$

which is true. Therefore, the half plane containing (2, 0) is the required region as shown in Fig.2.4. 2x-3y=0 or 3y=2x or y=2x/3

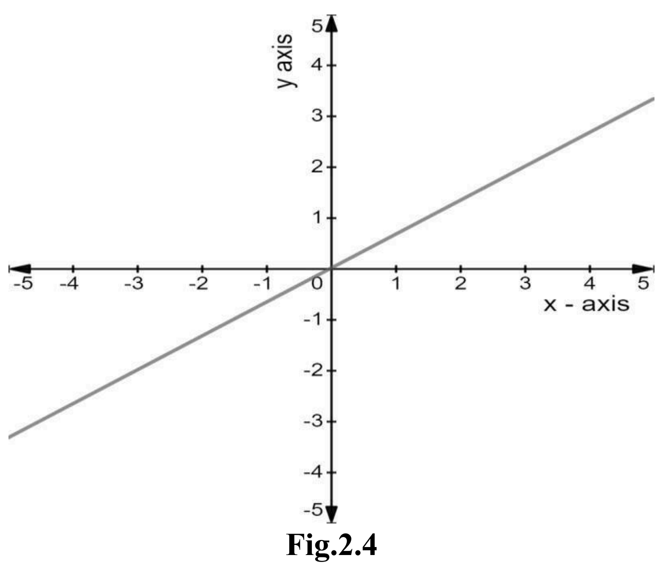


Fig.2.4

After determining the regions for each inequality, we find their common region. This is the region where all the given inequalities and non-negative restrictions are satisfied. This common region is known as feasible region or the solution set or polygonal convex set.

Now, we determine each of the corner points (vertices) of the polygon obtained in step 3. This is done either by plotting graphs on graph paper or by solving the two equations of the lines intersecting at that point.

Evaluate the value of the objective function at each corner point and determine the extreme point of the feasible region that has optimum objective function value.

According to the feasible region, we have different types of solution.

a) If the feasible region is bounded. The maximum of the values obtained for the objective function at the corner points is the optimum value when the objective function is of maximization form. The point corresponding to this maximum value give the required values of the decision variables. The minimum of the values obtained for the objective function at the corner points is the optimum value when the objective function is of minimization form. The point corresponding to this minimum value give the required values of the decision variables.

b) If the feasible region is not bounded. Then either there are additional hidden conditions which can be used to bound the region or there is no solution to the problem.

c) If the same optimum value exists at two of the vertices, then there are multiple solutions to the problem. Suppose these are two points (x_1, y_1) and (x_2, y_2). Then other solutions are given by the point as follow:

[First ordinate of first point × t + First ordinate of second point × (1- t) ,

Second ordinate of first point × t + Second ordinate of second point × (1 – t)]

where t is any real number lying between 0 and 1.

For example, let the objective function be Z = 3x – y and let A (2,1) and B (3,4) be the points which give the same optimum value of the objective function, i.e., Z = 5. Then other solutions which give the same value of the objective function are:

(2 × t + 3 × (1 – t), 1 × t + 4 × (1 –

t)) Or (2t + 3 – 3t, t + 4 – 4t)

Or (3 – t, 4 – 3t), $0 \leq t \leq 1$

Here t = 0 gives the point (3, 4), which is the point B and t = 1 gives the point (2, 1), which is the point A. The real values of t between 0 and 1 give other points which give the same optimum solution. One such point other than A and B is

$\left(3 - \frac{1}{2}, 4 - 3 \times \frac{1}{2}\right)$, $t = \frac{1}{2}$, i.e. , $\left(\frac{5}{2}, \frac{5}{2}\right)$ –

You can verify that Z = 5 at the point $\left(\frac{5}{2}, \frac{5}{2}\right)$.

Example. A company produces two types of items P and Q that require gold and silver. Each unit pf type P requires 4g silver and 1g gold while that od type Q requires 1g silver and 3g gold. The company produces 8g silver and 9g gold. If each unit of type P brings a profit of rupees 44 and that

of type Q rupees 55, determine the number of units pf each type that the company should produce to maximise the profit. What is the maximum profit?

Solution. Let x be the number of units of type P to be produced and y be the number of units of type Q to be produced. It is given that:

	Silver	Gold	Profit
Type P	4g	1g	Rs.44 per unit
Type Q	1g	3g	Rs.55 per unit
Available (at the most)	8g	9g	

Let Z be the profit function. The mathematical formulation of the given problem

is Max. $Z = 44x + 55y$

Subject to the constraints:

$4x + y \leq$

$8, \; x + 3y \leq$

$9, \; x \geq 0, \; y$

$\geq 0.$

First of all, we plot the graphs for the equations:

$4x + y = 8, \; x + 3y = 9 \; , \; x = 0 \; , \; y = 0$.

Since these equations are of straight lines, only two points are sufficient to plot the graphs (see fig.2.5). For the line $4x + y = 8$, we take the following two points:

X	0	2
Y	8	0

Similarly, for the line $x + 3y = 9$, we take

X	0	9
Y	3	0

Now, plotting the above lines, we get fig.2.5.

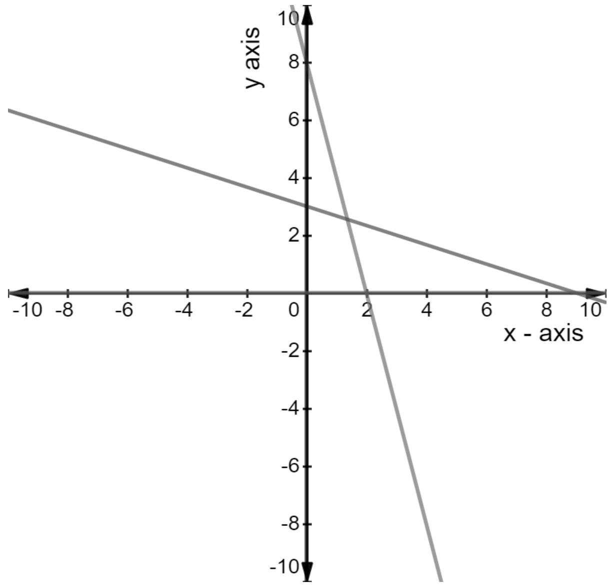


Fig.2.5

Note that x = 0 is the y axis and y = 0 is the x axis.

For plotting the graph of the inequality $4x + y \leq 8$ we put (0, 0) In it. We get $0 \leq 8$, which is true. Therefore, starting from the line $4x + y \leq 8$, we shall shade towards origin. Similarly, for the graph $x + 3y \leq 9$, we shall shade towards origin. For the graph $x \geq 0$. we shall shade towards right side of x = 0 and for the graph $y \geq 0$, the region above y = 0 will be shaded.

Thus, the regions for the inequalities are shown in fig.2.6.

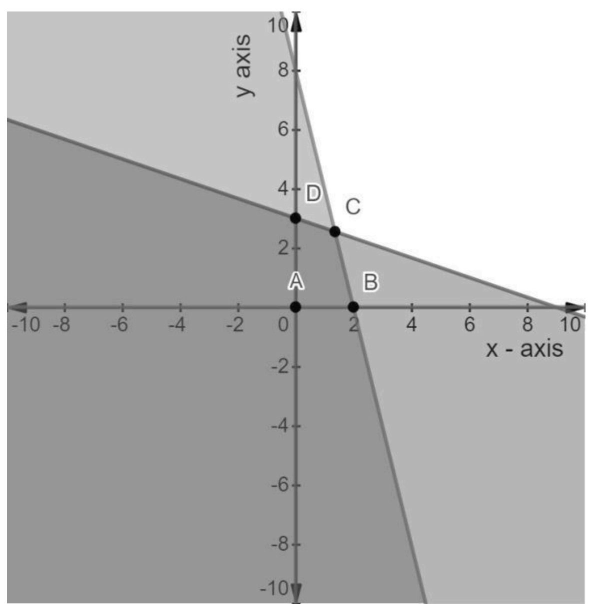


Fig.2.6.

You can see from the figure 2.6 that the coordinates of A, B, and D are (0, 0), (2, 0) and (0,3) respectively. The coordinates of the point C are obtained by solving the equations $4x + y = 8$ and $x + 3y = 9$ as it is the point of intersection of the two lines represented by them. the solution of equations $4x + y = 8$ and $x + 3y$
$= 9$ is given as

$x = \frac{15}{11}$ and $y = \frac{28}{11}$

So the vertices of ABCD are A(0,0) , B(2,0) , C($\frac{15}{11}$, $\frac{28}{11}$) and D (0,3).

We now obtain the values of Z = 44x + 55y at each of the vertices of ABCD as follows:

At A (0,0) , Z = 44(0) + 55(0) = 0

At B(2,0) , Z = 44(2) + 55(0) = 88

At C($\frac{15}{11}$, $\frac{28}{11}$), Z = 44($\frac{15}{11}$) + 55($\frac{28}{11}$) = 60 + 140 = 200

At D (0,3), Z = 44(0) + 55(3) = 165

Thus, the value of Z is maximum at C ($\frac{15}{11}$, $\frac{28}{11}$) and the optimum solution is Max.Z = 200 when $x = \frac{15}{11}$ and $y = \frac{28}{11}$.

Exercises.

1. **Maximise Z = 6X + 3Y subjects to the constraints 2X + 5Y ≤ 120, 4X + 2Y ≤ 80, X ≥ 0 , Y ≥ 0.**
2. **Maximise $z = 3x_1 + 2x_2$ subjects to the constraints $x_1 - x_2 \leq 1$, $x_1 + x_2 \geq 3$, $x_1 \geq 0$, $x_2 \geq 0$**
3. **Maximise $Z = x_1 + x_2$ subjects to the constraints $x_1 + x_2 \leq 1$, •$3x_1 + x_2 \geq 3$, $x_1 \geq 0$, $x_2 \geq 0$**
4. **A company manufactures two products X and Y, each of which requires three types of processing. The length of time for processing each unit and the profit per unit are given in the following table:**

	Product X (hr/unit)	**Product Y (hr/unit)**	**Available capacity per day (hr)**
Process 1	**12**	**12**	**840**
Process 2	**3**	**6**	**300**
Process 3	**8**	**4**	**480**
Profit per unit	**5**	**7**	

How many units of each product should the company manufacture per day in order to maximise profit?

5. **A company produces soft drinks and has a contract requiring that a minimum of 80 units of chemical A and 60 units of chemical B go into each bottle of the drink. The chemicals are available in a prepared mix from two different suppliers. The supplier X_1 has a mix of 4 units of A and 2 units of B that costs rupees 10, and the supplier X_2 has a mix of 1 unit of A and 1 unit of B that costs rupees 4. How many mixes from the company X_1 and company X_2 should the company purchase to honour contract requirement and yet minimise cost?**

CANONICAL AND STANDARD FORM OF AN LPP

After formulating a linear programming problem, our next step is to solve it. You have learnt that linear programming problems can be represented as problems of maximisation or minimisation with constraints such as ≤ , =, ≥ . In order to develop a standard procedure for solving LPPs, we need to convert them into well - known form. We now discuss the General LPP along with these two forms. The canonical form is especially used in the duality theory and the standard form is used to develop the general procedure for solving any linear programming problem. In order to understand these forms, you also need to learn about slack and surplus variable.

General Linear Programming Problem

Let us formulate the general linear programming problem. Let Z be a linear function of **n** basic variables X_1**,**X_2, X_3,..., X_n, which is to be maximised (or minimised). We write the problem as

Maximise (or minimise) Z = $C_1X_1 + C_2X_2 + C_3X_3 + ... + C_nX_n$ **(1)**

where C_1, C_2, C_3, , C_n are known constant termed as **cost coefficients** of basic variables.

Let (a_{ij}) be an m×n real matrix of m×n constants a_{ij}'s and let $\{b_1, b_2, ..., b_m\}$ be a set of constants such that

$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n$ ≤ or = or , ≥ b_1

$a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n$ ≤ or = or , ≥ b_2

. . . . **.... (2)**

. . . .

. . . .

$a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n$ ≤ or = or, ≥ b_m

and $x_j \geq 0$ for all j = 1 ,2, 3, ..., n **(3)**

The linear function Z in equation (1) is called the objective function. The set of inequalities given in (2) is called constraints of a general LPP and the set of inequalities given in (3) are known as non – negative restrictions of a general LPP.

Slack and Surplus Variables

In general, if any linear programming problem, we have a constraint of the type

$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n \leq b_1$ where $b_1 \geq 0$

Then this inequality can be converted into an equation by adding one non – negative variable s_1 to the left-hand side. This new variable is called a slack variable and the constraints are transformed into the following equation:

$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n + s_1 = b_1$ where $s_1 \geq 0$, $b_1 \geq 0$

Thus, a non – negative variable subtracted from the left – hand side of less than or equal to (≤) type of a constraint that converts it into an equation is called a slack variable**.** The values of this variable can be interpreted as the amount of unused resource.

Similarly, if in any linear programming problem, we have a constraint of the type

$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n \geq b_1$

Then this inequality can be converted into an equation by subtracting one non –negative variable s_1 from the left-hand side. This new variable is called a surplus variable**.** The value if this variable can be interpreted as the amount over and above the required minimum level.

Canonical Form

The characteristics of the canonical form are:

(a). Objective function should be of maximization form. If it is given in minimization form, it should be converted into maximization form.

(b). All the constraints should be of "≤ " type, except for non- negative restrictions. Inequality of " ≥ " type, if any, should be changed to an inequality of the " ≤ " type.

(c). All variables should be non-negative. If a given variable is unrestricted in sign (i.e., positive, negative or zero), it can be written as a difference of two non-negative variables. Suppose x is unrestricted in sign, then x can be written as x= x'- x'' where $x' \geq 0, x'' \geq 0$.

Example. Express the following LPP in Canonical form:

Minimize Z = $2x_1 + x_2 + 4x_3$

Subject to the constraints:

$-2x_1 + 4x_2 \leq 4$

$x_1 + 2x_2 + x_3 \geq 5$

$2x_1 + 3x_3 \leq 2$

$x_1, x_2 \geq 0$ and x_3 is unrestricted in sign

Solution: Here, the objective function is of the minimisation form. We rewrite it in the maximisation form as follows:

Minimise Z = $2x_1 + x_2 + 4x_3$

Thus, we have to maximise -Z = $-2x_1 - x_2 - 4x_3$. So the problem becomes,

Maximise $Z' = -2x_1 - x_2 - 4x_3$. where Z' = -Z

Now the second constraints is of the type "≥". Hence to convert it into type "≤", we multiply the inequality by -1 and write

$-x_1 - 2x_2 - x_3 \leq -5$

Other constraints are already in the desired form. But $x_3$is unrestricted in sign. So, we write
$x_3 = x'_3 - x''_3$, where $x'_3 \geq 0$, $x''_3 \geq 0$.

The canonical form of the given problem, therefore, is

Maximise $Z' = -2x_1 - x_2 - 4(x'_3 - x''_3)$, where $Z' = -Z$

Subject to the constraints:

$-2x_1 + 4x_2 \leq 4$

$-x_1 - 2x_2 - (x'_3 - x''_3) \leq -5$

$2x_1 + 3(x'_3 - x''_3) \leq 2$

$x_1 \geq 0, x_2 \geq 0, x'_3 \geq 0, x''_3 \geq 0$

Standard Form:

The characteristics of the Standard Form are:

1) The objective function should be in the maximization form as we explained in canonical form.

2) The right-side element of each constraint should be non- negative. If it is negative, we multiply the inequality by -1.

3) All constraints should be expressed in the form of equations, except for the non- negative restrictions by augmenting slack or surplus variables.

Example. Express the following LPP in the standard form:

Minimise $Z = 2x_1 + x_2 + 4x_3$

subject to the constraints:

$-2x_1 + 4x_2 \leq 4$

$x_1 + 2x_2 + x_3 \geq 5$

$2x_1 + 3x_3 \leq -2$

$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.$

Solution: The objective function should be of maximization form, i.e. we have to

Maximise $Z' = -2x_1 - x_2 - 4x_3$, where $Z' = -Z$

subject to the constraints:

$-2x_1 + 4x_2 \leq 4$

$x_1 + 2x_2 + x_3 \geq 5$

$-2x_1 - 3x_3 \leq 2$[∵ Right side should be non- negative]

$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.$

Now the inequalities are to be converted to equations. Note that the first and third inequalities are of the type “less than or equal to (≤)”.

Therefore, a slack variable is to be added to the left side of each of these inequalities. The second inequality is of the type “more than or equal to (≥) ” . So, a surplus variable is to be subtracted from the left side of this inequality.

Thus, a standard form of the given LPP is

Max.Z’ = = $-2x_1 - x_2 - 4x_3 + 0s_1 + 0s_2 + 0s_3$, where Z’ = -Z

subject to the constraints:

$-2x_1 + 4x_2 + s_1 = 4$

$x_1 + 2x_2 + x_3 - s_2 = 5$

$-2x_1 - 3x_3 + s_3 = 2$

$x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$, $s_1 \geq 0$, $s_2 \geq 0$, $s_3 \geq 0$.

Exercise.

1. Express the following LPP in

   (i) Canonical form

   (ii) Standard Form

Minimise Z = $x_1 - 2x_2 + x_3$

subjects to the constraints:

$2x_1 + 3x_2 + 4x_3 \geq -4$

$3x_1 + 5x_2 + 2x_3 \geq 7$

$x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$.

2. Max. Z = 5x + 7y subjects

to the constraints:

12x + 12y ≤ 840 => x + y ≤ 70

3x + 6y ≤ 300 => x + 2y ≤ 100

8x + 4y ≤ 480 => 2x + y ≤ 120

x ≥ 0 , y ≥ 0

Answer.

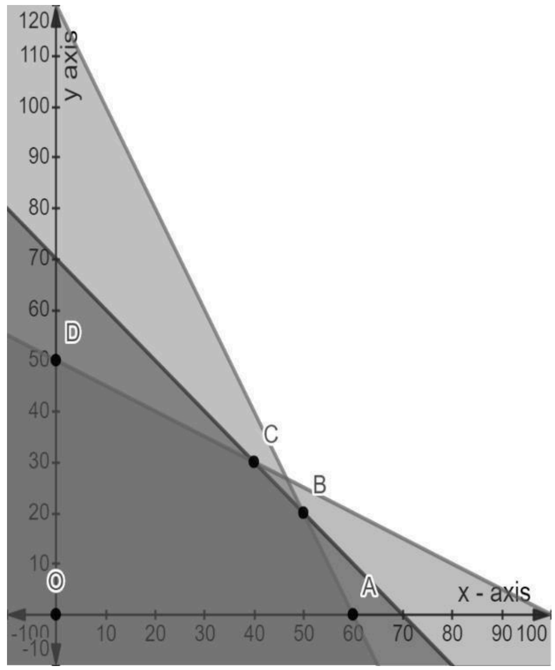


Fig.2.8.

The maximum value of Z is 410 at C(40,30) , i.e., at x = 40 , y = 30.

3. Minimise z = $10x_1 + 4x_2$
subjects to the constraints

$4x_1 + x_2 \geq 80$

$2x_1 + x_2 \geq 60$

$x_1 \geq 0$, $x_2 \geq 0$

Answer.

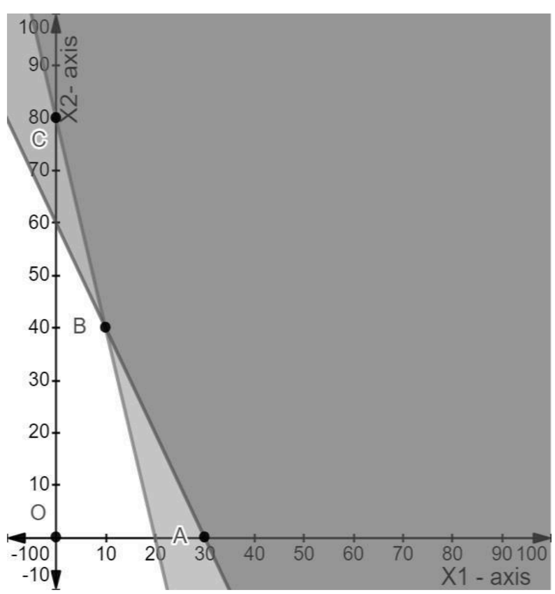


Fig.2.9.

The maximum value of Z is 260 at B (10,40), i.e. x_1 = 10 and x_2 = 40.

Note: Had the objective function been of maximization form, the problem would have the unbounded solution. This is because the values of x_1 and x_2 could be increased beyond any limit, which would result in higher value of Z with no upper bound.

CHECK YOUR PROGRESS

1. What is linear programming? What are its major advantages and limitations?
2. What is meant by feasible solution of an LP problem?

SUMMARY

In this chapter, we introduced the concept of LPP and explain how these are formulated mathematically. We define objective function and graphical method of obtaining optimum value which is used to solve linear programming problems having two decision variables. Here, we study about feasible region/ solution set in detail. Also, we discuss canonical and standard form of an LPP as to solve LPP we need to convert them into a canonical or standard form.

3

SIMPLEX METHOD AND DUALITY IN LINEAR PROGRAMMING

Structure

Introduction

Simplex Method

Artificial Variable Techniques

Big-M Method

Two-Phase Method

Degeneracy

Duality in Linear Programming

Check Your Progress

Summary

INTRODUCTION

In chapter 2, we have studied the graphical method of solving linear programming problems and learnt how to express a linear programming problem in canonical and standard forms. As we know that the graphical method can be used only to solve the problems involving two decision variables and most of the real-life problems when mathematically formulated have more than two variables. For more than two decision variables, methods based on the concept of slack or surplus variables are used.

In this chapter, first we shall discuss the Simplex method for solving the linear programming problems involving more than two decision variables. After learning the procedure of simplex method, we discuss artificial variable techniques (Big-M Method and Two-Phase Method) for solving LPP and in last we shall discuss the concept of degeneracy in linear programming.

Objectives. The objective of these contents is to provide some important concepts/methods to the reader like:

- Simplex Method.
- Artificial Variable Techniques.
- Big-M and Two-Phase Method for Solving LPP Involving Artificial Variable(s).
- Degeneracy.

SIMPLEX METHOD

Simplex method was developed by G B Dantzig in 1947. The method is an iterative or step by step procedure by which one can obtained a new basic feasible solution from a given initial basic feasible solution. In this method, the value of the objective function improves with each solution and the optimum solution is achieved in a finite number of steps.

Suppose we have to optimize (maximiseor minimise) Z, a linear function of n basic variables $X_1, X_2, \ldots, X_n$. The LPP is written as:

Maximise Z = $C_1X_1 + C_2X_2 + \cdots + C_nX_n$ …

(1) subject to the constraints:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq b_2$$
$$\begin{matrix} . & . & . & . \\ . & . & . & . \end{matrix} \qquad (2)$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq b_m$$

and $x_j \geq 0$ for all j= 1,2, …, n … (3)

where the constants $C_1, C_2, \ldots, C_n$ are the cost coefficients of decision variables. Let (a_{ij}) be m×n real matrix and $\{b_1, b_2, \ldots, b_m\}$ be a set of constants.

The linear function Z gives in equation (1) is called the objective function. The set of inequalities gives in equation (2) is called constraints of LPP and the inequalities gives in equation (3) are known as non-negative restrictions of LPP (which means that all x_j values are non –negative)

Let us explain the step by step procedure for solving the LPP by the Simplex method.

Step 1: Convert the LPP into standard form by adding slack variables

We convert the given LPP into standard form by adding slack variables $s_1, s_2, \ldots, s_m$.

Maximise $Z = C_1X_1 + C_2X_2 + \cdots + C_nX_n + 0s_1 + 0s_2 + \cdots + 0s_m$ … (4) subject to the given constraints:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + s_1 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + s_2 = b_2$$
$$\vdots \qquad \ldots (5)$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n + s_m = b_m$$

and $x_j \geq 0$ *and* $s_i \geq 0$ for all i =1, 2,…, m and j= 1,2, …, n … (6)

Step 2: Construct the initial simplex table.

The initial simplex table is formed as follows:

$C_j \rightarrow$			C_1	C_2	$..C_r$	0	0…	
Basic variables	Profit/unit (C_B)	Quantity	X_1	X_2	$..X_r$	S_1	S_2..	Replacement Ratios
S_1 S_2 . . .	0 0 . . .	b_1 b_2 . . .	a_{11} a_{21} . . .	a_{12} a_{22} . . .	$..a_{1r}$ $..a_{2r}$ . . .	1 0 . . .	0.. 1.. . . .	
	$z_j \rightarrow$ Z $c_j - z_j \rightarrow$		C_BX_1	C_BX_2	$..C_BX_r$	C_BS_1	C_BS_2	

Table 1: Initial Simplex Table

Step 3: Test for optimality

Calculate the values of $c_j - z_j$, the nature of the solution could be any one of the following:

1. If all $c_j - z_j \leq 0$, the solution under test is optimal solution.
2. If at least one value of $c_j - z_j$is positive and corresponding to the most positive $c_j - z_j$,all the elements of the column $X_j$are negative or zero, the solution under test is an unbounded solution.
3. Suppose at least one value of $c_j - z_j$is positive. Suppose the most positive value is, say $c_3 - z_3$and at least one entry in the column of X_3 is positive. Then the solution under test is not optimal.

Step 4: Select incoming variable to enter the basis

If the solution is not optimal then we look for most positive entry and it could be any of $(c_j - z_j)$. In this case, we proceed as follows to obtain the optimal solution:

1. Let X_r be the variable which corresponds to the most positive value of $c_j - z_j$. This variable is called the incoming variable.
2. The column to be entered is called the key or pivot column.

Step 5: Test for feasibility (variable to leave the basis)

After finding the incoming variable, we determine the outgoing variable. For this we proceed as follow:

1. We divide the value of the Quantity (Qty) column be the corresponding positive values in the column X_r. These ratios are called Replacement Ratios (RR). Note that we do not consider the negative values in the column of X_r for calculating RR. Then we select the minimum RR. The basic variable corresponding to this value of the RR is called the outgoing variable. It is called outgoing variable because it is removed (goes out) from the next simplex table. The row selected in this manner is called key or pivot row.
2. The element that lies at the intersection of the key row and key column is called the key element or leading element or pivot element.

Step 6: Finding the new solution

1. We convert the key element to unity by dividing all entries in the row by the key element itself.
2. In the next step, we would like that the values of all other elements in the column corresponding to key element are zero. For this we carry out suitable operations on each row using the row containing the key element.

Step 7: Repeat the procedure

Go to step 3 and repeat the procedure until all entries in the $c_j - z_j$ are either negative or zero i.e. we repeat the procedure until either an optimal solution is obtained or there is an indication of unbounded solution.

The following example illustrate the simplex method:

Example. Maximise Z=$3x_1 + 2x_2$

subject to the constraints:

$$x_1 + x_2 \leq 4$$

$$x_1 - x_2 \leq 2$$

$$x_1 \geq 0, x_2 \geq 0$$

Solution: Step 1: First we convert the given LPP into standard form by adding slack variables s_1 and s_2.

Maximize Z = $3x_1 + 2x_2 + 0s_1 + 0s_2$

subject to the constraints:

$$x_1 + x_2 + s_1 = 4$$

$$x_1 - x_2 + s_2 = 2$$

$$x_1, x_2, s_1, s_2 \geq 0$$

Step 2: We now construct the initial simplex table (Table 2: Initial Simplex Table)

$c_j \rightarrow$			3	2	0	0
Basic variables	Profit/unit (C_B)	Quantity	x_1	x_2	s_1	s_2
s_1	0	4	1	1	1	0
s_2	0	2	1	-1	0	1
	Z=	$z_j \rightarrow$ $c_j - z_j \rightarrow$				

Note: Columns corresponding to the basic variables in initial simplex tables in the simplex method form an identity matrix. For example, in Table 2, the basic variables are s_1 and s_2 and their coefficients in the constraints form identity matrix.

The value of Z in column of Table2 is obtained from the equation

$$Z=\sum c_j b_j, j=1, 2$$

The values of $z_j's$ are obtained from the equation

$$z_j = \sum^m c_i a_{ij},$$ where m is the number of rows. In this example m=2

Next, we obtain $c_j - z_j$, known as net-evaluations. These are 3,2,0,0.

Thus, the table takes the following form (Table3):

Table 3: Simplex Table

$c_j \rightarrow$			3	2	0	0
Basic variables	Profit/unit (C_B)	Quantity	x_1	x_2	s_1	s_2
s_1	0	4	1	1	1	0
s_2	0	2	1	-1	0	1
	Z=0	$z_j \rightarrow$ $c_j - z_j \rightarrow$	0 3	0 2	0 0	0 0

Step 3: Test for optimality

Since all $c_j - z_j \geq 0$, therefore the current solution is not optimal.

Step 4: Now we select incoming and outgoing variable**.**

For this, we select the most positive value of $c_j - z_j$, which is 3 in this case.

This corresponds to the variable x_1 which becomes the incoming variable. We shall enter it as a basic variable in the next simplex table.

Step 5: One of the variables s_1, s_2 will now be the outgoing variable which would be replaced by x_1**.** To find out which one of these variables ($s_1 or\ s_2$) is the outgoing variable, we determine the replacement ratio (RR). Recall that RR for any row is obtained by dividing the value of Quantity column by the corresponding value of the Incoming variable for that particular row. For example, consider the first row in Table 3. The value of Quantity in the first row is 4 and the value of the incoming variable x_1 in this row is 1.

Therefore, RR for the first row is $\frac{4}{1}$. For the second row, Quantity is 2 and value of x_1 is 1. Hence, RR is $\frac{2}{1}$. RR is shown in the last column of the next simplex table (Table4).

The outgoing variable is the variable for which RR minimum. In Table 4, RR in the second row is minimum. It corresponds to s_2 and hence s_2 is the outgoing variable.

The element which lies at the intersection of the column of incoming variable (x_1) and the row of the outgoing variable (s_2) is called the key element. Here it is 1 and is enclosed by the rectangle as shown in Table4.

Incoming variable Key element

$c_j \rightarrow$			3	2	0	0	
Basic variables	Profit/unit (C_B)	Quantity	x_1	x_2	s_1	s_2	R.R.
s_1	0	4	1	1	1	0	4/1=4
(outgoing Variable)s_2	0	2	1	-1	0	1	2/1=2
	Z=0	$z_j \rightarrow$	0	0	0	0	
		$c_j - z_j \rightarrow$	3	2	0	0	

The objective function Z=0 at $x_1 = 0\ and\ x_2 = 0$ (since $x_1\ and\ x_2$ do not appear in the column of basic variables, these are non-basic variables. The values of non-basic variables are taken as zero). This is the initial solution, which we shall improve.

Step 6: Now, we form the next simplex table to find the adjacent vertex, i.e., the improved solution. The steps for forming this table are explained below:

a) The initial simplex table (Table 4) has revealed that x_1 is an incoming variable which will enter in place of s_2, the outgoing variable. The cost coefficient of $s_2$will also be replaced by the cost coefficient of x_1 in “Profit / Unit” column. In this case its value is 3. Therefore, now the simplex table takes the form of Table 5. Other entries of the table will be filled up as explained in point (b).

b) Since x_1 has entered as a basic variable, the coefficients of x_1 along with s_1 should form an identity matrix; i.e., the column corresponding to x_1 should be $\binom{0}{1}$. Thus, we have to make the key element unity and the other element zero. Note that it is already unity in this case (Table 4). Had it been any number other than unity, we would have divided the row containing leading element by the leading element itself, excluding the elements of the column “Profit / Unit”. So, the table takes the form

Table 5: Simplex Table

$c_j \rightarrow$			3	2	0	0
Basic variables	Profit/unit (C_B)	Quantity	x_1	x_2	s_1	s_2
s_1	0	4	1	1	1	0
x_1	3	2	1	-1	0	1

c) Now, we have to make the other element in the column of key element (x_1) zero. In this case, itsvalue is 1 ($a_{11} = 1$). For this, we multiply the row of the key element (excluding profit / unit) by negative of the element a_{11} (in this case) and add it to the first row.

This row operation is shown below:

First row of Table 7→	4	1	1	1	0
Second row of Table 7→ (on multiplying by -1)	-2	-1	1	0	-1
Sum of the rows	2	0	2	1	-1

Thus, the sum of the rows is the new first row (excluding profit/ unit), which replaces the first row of Table5. We get Table 6 as follows:

Table 6: Simplex Table

$c_j \rightarrow$			3	2	0	0
Basic variables	Profit/unit (C_B)	Quantity	x_1	x_2	s_1	s_2
s_1	0	2	0	2	1	-1
x_1	3	2	1	-1	0	1

Note: Now the matrix for the basic variables s_1 and x_1 in the Table 6 is the identity matrix.

d) Next, we calculate z_j and $c_j - z_j$. The resulting simplex table is given below:

Table 7: Simplex Table

$c_j \rightarrow$			3	2	0	0	
Basic variables	Profit/unit (C_B)	Quantity	x_1	x_2	s_1	s_2	RR
$\leftarrow s_1$	0	2	0	2	1	-1	2/2=1←
x_1	3	2	1	-1	0	1	-
	Z=$0\times 2 + 3 \times 2 = 6$	$z_j \rightarrow$	$0 \times 0 + 3 \times 1 = 3$	$0 \times 2 + 3 \times (-1) = -3$	$0 \times 1 + 3 \times 0 = 0$	$0 \times (-1) + 3 \times 1 = 3$	
		$c_j - z_j \rightarrow$	0	5 t	0	-3	

Here Z=6 at $x_1 = 2$ (see the value in the Quantity column) and x_2=0 (x_2 being non-basic variable).

In Table 7, the incoming variable is x_2 corresponding to the most positive value of $c_j - z_j = 5$. The key element is 2. We find the RR by dividing the elements in the Quantity column by the corresponding elements in the column of the incoming variable x_2 and ignore the negative or zero values. So, in this case, there will be only one RR and that will be considered as minimum RR. This implies that s_1 is the outgoing variable. If none of the elements in the column of incoming variable is positive, then the given LPP has an unbounded solution and we will stop there.

Step 7: Repeat the procedure

1. For the next simplex table, x_2 will enter in place of s_1 as a basic variable and accordingly we shall write the cost coefficient of x_2 in the LPP as the value for the column of profit / unit, i.e.,2. The element 2 enclosed in a rectangle in Table 7 is the key element. So, we shall divide the row containing the key element by the key element itself, i.e., by 2, excluding the values in the column of profit / unit. Thus, we get Table 8:

Table 8: Simplex Table

$c_j \rightarrow$			3	2	0	0
Basic variables	Profit/unit (C_B)	Quantity	x_1	x_2	s_1	s_2
x_2	2	1	0	1	1/2	-1/2
x_1	3	2	1	-1	0	1

Now the coefficients of x_2 and x_1 have to form an identity matrix, i.e., the column corresponding to x_2 should be $\binom{1}{0}$. We have already made the key element unity. Now, to make the second element , (i.e., - 1) in its column as zero, we just add the first row corresponding to x_2, to the second row of Table 8 excluding the values in the column of profit / unit as follows:

First row of Table 10→	1	0	1	1/2	− 1/2
Second row of Table 10→ (on multiplying by 1)	2	1	-1	0	1
Sum of the rows	3	1	0	1/2	1/2

We write the sum as above as the second row, excluded profit / unit as shown in Table 9. Then we obtain Z, z_j, and $c_j - z_j$ as explained in Step 1(iii) and (iv) and also in Step 2(c) . The complete resulting simplex table is given below:

Table 9: Simplex Table

$c_j \rightarrow$			3	2	0	0	
Basic variables	Profit / Unit	Quantity	x_1	x_2	s_1	s_2	RR
x_2	2	1	0	1	1/2	-1/2	
x_1	3	3	1	0	1/2	1/2	
	Z=2× $1 + 3 \times 3 = 11$	$z_j \rightarrow$	$2\times 0 + 3 \times 1 = 3$	$2\times 1 + 3 \times 0 = 2$	$2\times 1/2 + 3 \times 1/2 = 5/2$	$2\times (-1/2) + 3 \times 1/2 = 1/2$	
		$c_j - z_j \rightarrow$	0	0 t	-5/2	-1/2	

Now, in Table 9 none of the net-evaluation, i.e., the values of $c_j - z_j$ are positive. Therefore, the optimum solution is attained for $x_1 = 3$ and $x_2 = 1$, the values of x_1 and x_2 in the Quantity column of Table 11.

At this stage, it is necessary to check whether any of the non-basic variables (other than those appearing in the first column, i.e., the column with caption "Basic Variables", i.e., x_1 and x_2 in Table 9) has value 0 in the net-evaluation row. If "yes", then the LPP has multiple optimum solutions. If "not", then we stop here concluding that the unique solution. In this example, it is given by

Max. Z= 11 at $x_1 = 3$ and $x_2 = 1$ (see the values in the Quantity column)

Note: It should be noted that in the initial simplex table (Table 2), $c_j - z_j$ is same as c_j. Also $c_j - z_j$ corresponding to the column of unit matrix are always zero. So, there is no need to calculate them.

We discuss the case of multiple optimum solutions in the next example.

Example. Max. Z= 6x+3y Subject to

the constraints:

$2x + 5y \leq 120$

$2x + y \leq 40$

$x \geq 0, y \geq 0$

Solution: Rewriting the given LPP in the standard form, we have

$$\text{Max. } Z = 6x + 3y + 0s_1 + 0s_2$$

subject to the constraints:

$$2x + 5y + s_1 = 120$$

$$2x + y + s_2 = 40$$

$$x, y, s_1, s_2 \geq 0$$

We form the initial simplex table (Table1) as explained in Example 1:

Table 1: Initial Simplex Table

$c_j \rightarrow$			6	3	0	0	
Basic variables	Profit / Unit	Quantity	x	y	s_1	s_2	RR
s_1	0	120	2	5	1	0	120/2=60
$\leftarrow s_2$	0	40	2	1	0	1	40/2=20
	Z=0	$z_j \rightarrow$	0	0	0	0	
		$c_j - z_j \rightarrow$	6 t	3	0	0	

Note from Table 1 that $c_j - z_j$=6 is maximum for x and RR is minimum for s_2. Therefore, x is the incoming variable and s_2 is the outgoing variable. Then in the second simplex table, x will enter in place of s_2 as a basic variable. Profit / unit will be written accordingly. The key element is 2 and enclosed by the rectangle in Table 1. To make the key element unity, we divide the second row in Table 1 by 2 excluding the values of the column of profit / unit. So, we get Table 2 as follows:

Table 2: Simplex Table

$c_j \rightarrow$			6	3	0	0
Basic variables	Profit / Unit	Quantity	x	y	s_1	s_2
s_1	0	120	2	5	1	0
X	6	20	1	1/2	0	1/2

Now, we have to make the other element in column of x zero so that the coefficients of s_1 and x form the identity matrix. So, we multiply the second row of Table 2 (excluding the elements in the column of profit /unit) by the negative of the element (a_{ij}) in the row of s_1 and column of x in Table 1, i.e., by -2 and then add it to the first row of the Table 2 as follows

First row of Table 2→	120	2	5	1	0
Second row of Table 2→ (on multiplying by -2)	-40	-2	-1	0	-1
Sum of the rows	80	0	4	1	-1

We also calculate Z, z_j, *and* $c_j - z_j$. So, the resulting completed simplex table is as follows:

Table 3: Simplex Table

$c_j \rightarrow$			6	3	0	0	
Basic variables	Profit / Unit	Quantity	x	y	s_1	s_2	RR
s_1	0	80	0	4	1	-1	
X	6	20	1	1/2	0	1/2	
	Z=120	$z_j \rightarrow$	6	3	0	3	
		$c_j - z_j \rightarrow$	0	0	0	-3	

As none of the net-evaluations is positive, the optimum solution is attained. The optimum solution is Max. Z = 120 when x = 20 (see the value in the Quantity column),

Y = 0 (since y is non-basic variable)

But we also note that the non-basic variable y has zero value in its net-evaluation row in Table 3. Therefore, the given LPP has multiple optimal solutions. To find another optimal solution, let us find another vertex at which Max. Z = 120.

So, another simplex table has to be formed. Here, instead of selecting the column corresponding to the most positive element in the net-evaluation row, we select the column of non-basic variable which has zero in the net-evaluation row, i.e., the column of y shown in Table 4.

Table 4: Simplex Table

$c_j \rightarrow$			6	3	0	0	
Basic variables	Profit / Unit	Quantity	x	y	s_1	s_2	RR
$\leftarrow s_1$	0	80	0	4	1	-1	80/4=20←
X	6	20	1	1/2	0	1/2	20/(1/2)=40
	Z=120	$z_j \rightarrow$	6	3	0	3	
		$c_j - z_j \rightarrow$	0	0 t	0	-3	

Note that in Table 4, the key element is 4 and the minimum RR corresponds to s_1. So, s_1 is the outgoing variable and y is the incoming variable. We form the next simplex table (Table 5) following the steps explained for forming Table 3. The resulting simplex table (Table 5) is given as follows:

Table 5: Simplex Table

$c_j \rightarrow$			6	3	0	0	
Basic variables	Profit / Unit	Quantity	x	y	s_1	s_2	RR
Y	3	20	0	1	1/4	-1/4	
X	6	10	1	0	-1/8	5/8	
	Z=120	$z_j \rightarrow$	6	3	0	3	
		$c_j - z_j \rightarrow$	0	0	0	-3	

You should verify all entries of Table 5 before studying further. Note that for forming Table 5, we have first divided the first row by the value of key element to make the key element unity.

Then we obtain the second row of Table 5 as follows:

Second row of Table 4→	20	1	1/2	0	1/2
First row of Table 5→ (on multiplying by -1/2)	-10	0	-1 /2	-1/8	1/8
Sum of the rows	10	1	0	-1/8	5/8

From the above simplex table (Table 5), we find that Max. Z = 120 at (10,20) also (see the value of x = 10, y = 20 in the Quantity column).

So, we have two vertices at which the maximum value of Z is the same, i.e., 120. So, the other solutions of the LPP are obtained as follows:

First ordinate of other solutions = t × (First ordinate of first /vertex)

+ (1-t)× (First ordinate of first / vertex)

Second ordinate of other solutions = t × (Second ordinate of first /vertex)

+ (1-t)× (Second ordinate of first / vertex)

So, the other solutions are given as

First ordinate = t $\times 10 + (1 - t) \times 20 = 10t + 20 - 20t = 20 - 10t$

Second ordinate = t $\times 20 + (1 - t) \times 0 = 20t$

The other solutions are (20-10t, 20t), $0 \leq t \leq 1$.

Exercises.

Solve the following LPPs by the Simplex method:

1. Maximise Z = $2x_1 + 4x_2$

subject to the constraints:

$$x_1 + 2x_2 \leq 5$$
$$x_1 + x_2 \leq 4$$
$$x_1 \geq 0, \qquad x_2 \geq 0$$

Answer. Max. Z = 10 at the two vertices $(0, 5/2)$ and (3, 1).Max. Z = 10at many other points also which are given as (3 – 3t, 1+$3t/2$), $0 \leq t \leq 1$.

2. Maximise Z = $100x_1 + 60x_2 + 40x_3$

Subject to the constraints:

$$x_1 + x_2 + x_3 \leq 100 \; 10x_1$$
$$+ 4x_2 + 5x_3 \leq 500$$
$$x_1 + x_2 + 3x_3 \leq 150, x_1, \; x_2, x_3 \; \geq 0$$

Answer. Z=22000/3 at $x_1 = 100/3$, $x_2 = 200/3$,$x_3 = 0$

ARTIFICIAL VARIABLE TECHNIQUES

After converting the given LPP into standard form, sometimes we observe that some of the variables in the standard form are surplus variables and the corresponding column vectors do not provide unit vectors for the initial basis and hence the column which form unit (identity) matrix are missing in the initial simplex table. Such a situation is usually observed if the constraints equation(s) is (are) of the type "≥". In order to have unit vectors in the basis, we introduce new type of variable (s) known as artificial variable(s). These are added to act as basic variable. They have no physical meaning and are used only to initiate the solution so that the simplex procedure may be adopted as usual till the optimal solution is obtained. The artificial variables are eliminated from the simplex table as and when they become non- basic variables. This technique is called the artificial variable technique for solving LPP. There are two methods for solving LPPs involving artificial variables.

i) Big-M or Method of Penalties or Charne's Method

ii) Two-Phase Method

BIG-M METHOD

Big-M Method is used for removing artificial variable(s) from the basis. It is also known as Penalty method or Charne's Method. In this method, the objective function coefficients impose a huge and henceunacceptable penalty. In case of maximisation, the objective function is modified by adding $-MA_1$, where M is arbitrary large and A_1 is an artificial variable. If there are two artificial variables $A_1$and A_2, then $-MA_1-MA_2$ is added to the objective function. Similar treatment is done for more artificial variables.

The logic behind taking the coefficient as –M is that we should never get the net-evaluation positive in the column of the artificial variable, i.e., the artificial variable should not enter again as a basic variable. M is very big and hence adding $-MA_1$ is the penalty to the objective function. Hence this method is called penalty method. Though –M is big penalty, it does not affect the objective function. This is because the value of artificial variable should come out to be zero so that $-MA_1$ becomes zero. If the artificial variable remains as a basic variable till the final simplex table, then its value in the quantity column should be zero for the solution of LPP to exist. Otherwise, if in the final simplex table, an artificial variable appears as a basic variable and is non zero, the LPP does not possess any feasible solution.

Example: Maximise Z = $x_1 + 2x_2$

Subject to the constraints:

$x_1 - x_2 \geq 3$

$2x_1 + x_2 \leq 10$

$x_1 \geq 0, x_2 \geq 0$

Solution: First we convert the given LPP in the standard form as follows:

Max Z = $x_1 + 2x_2 + 0s_1 + 0s_2$

Subject to the constraints

$x_1 - x_2 - s_1 = 3$

$2x_1 + x_2 + s_2 = 10$

$x_1 \geq 0, x_2 \geq 0, s_1 \geq 0, s_2 \geq 0$

Now, let us try to form simplex table as follows:

Table 1: Simplex Table

		$c_j \rightarrow$	1	2	0	0
Basic variable	Profit/ unit	Qty	x_1	x_2	s_1	s_2
			1	-1	-1	0
			2	1	0	1

We cannot form the initial simplex table with the variables x_1, x_2, s_1, s_2 as there is only one variable s_2 that has a column of unit matrix. Therefore, one more variable, i.e., artificial variable A_1(say) needs to be introduced in the first constraints to get another column of unit matrix. Its coefficient in the objective function will be taken as –M.

Thus, the objective function is

Max. Z = $x_1 + 2x_2 + 0s_1 + 0s_2 - MA_1$

Subject to the constraints

$x_1 - x_2 - s_1 + A_1 = 3$

$2x_1 + x_2 + s_2 = 10$

$x_1 \geq 0,\ x_2 \geq 0, s_1 \geq 0,\ s_2 \geq 0$

The initial simplex table, therefore, is as follows (Table 2: Initial Simplex table)

		cj	1	2	0	0	-M	
Basic variable	Profit/ unit	Qty	x_1	x_2	s_1	s_2	A_1	RR
←A_1	-M	3	1	-1	-1	0	1	3/1=3←
s_2	0	10	2	1	0	1	0	10/2=5
	Z= -3M	zj → cj-zj →	-M 1+M t	M 2-M	M -M	0 0	-M 0	

Here, since M is big, 1+M is most positive and x_1 is the incoming variable. The least replacement ratio is (3/1) which corresponds to A_1 and hence it is the outgoing variable. Thus, the resulting simplex table is as follows:

Table 3: Simplex Table

		cj→	1	2	0	0	-M	
Basic variable	Profit/unit	Qty	x_1	x_2	s_1	s_2	A_1	RR
x_1	1	3	1	-1	-1	0	×	
← s_2	0	4	0	3	2	1	×	4/3←
	Z=3	zj→ cj-zj→	1 0	-1 3	-1 1	0 0	×	

We obtain the second row (excluding profit/unit) in table 3 as follows:

First row of the second simplex table (table3) → -6 -2 2 2 0 ×

(After multiplying by -2)

Second row of first simplex table (table2) →10 2 1 0 1 ×

second **row of the third simplex table →** **4** **0** **3** **2** **1** ×

Once the artificial variable is removed from the basic variable, there is no need to do any computational work for it.

Thus,we get the resulting simplex table.

Table 4: Simplex Table

		$c_j \rightarrow$	1	2	0	0	-M	
Basic variables	Profit/unit	Qty	x_1	x_2	s_1	s_2	A_1	
x_1	1	13/3	1	0	-1/3	1/3	×	
x_2	2	4/3	0	1	2/3	1/3	×	
	Z=21/3 =7	$z_j \rightarrow$ $c_j-z_j \rightarrow$	1 0	2 0	1 -1	1 -1	× ×	

The first row (excluding profit/unit) of Table4 is obtained as follows:

Second row of third simplex table (Table 4) → 4/3 0 1 2/3 1/3 ×

(After multiplying by 1)

First row of second simplex table → 3 1 -1 -1 0 ×

Sum of rows	**→ 13/3**	**1**	**0**	**-1/3**	**1/3**	**×**

Since none of the net-evaluation is positive, the optimum solution is attained and is given by

Max. Z=7 at $x_1 = 13/3$ and $x_2 = 4/3$.

Exercises.

1. Minimize $Z = 4x_1 + 2x_2$

Subject to the constraints

$$3x_1 + x_2 \geq 27$$

$$x_1 + x_2 \geq 21$$

$$x_1 + 2x_2 \geq 30$$

$$x_1, x_2 \geq 0$$

Answer. Max. $Z' = -48$, i.e. Max. $-Z = -48$i.e. Min. Z = 48 when $x_1 = 3$, $x_2 = 18$

2. Maximise $Z = x_1 + 2x_2$

Subject to the constraints:

$$x_1 + x_2 \leq 4$$

$$x_1 + x_2 \geq 6$$

$$x_1, x_2 \geq 0$$

Answer. No feasible solution.

3. Maximise Z = $10x_1 + 2x_2$

Subject to the constraints:

$-x_1 + x_2 \leq 2$

$x_1 + x_2 \geq 4$

$x_1, x_2 \geq 0$

Answer. Unbounded solution.

TWO-PHASE METHOD

The two-phase method provides an alternate procedure for removing artificial variables from the basis by which one can not only get initial basic feasible solution but also eliminate redundant equation existing among the constraints. It also terminates the iteration if a feasible solution of the problem is absent, for this the method is divided into two phases of iterations.

In the first phase, the process of eliminating artificial variable is performed so that we get a basic feasible solution of the LPP and the second phase is used to get the optimal solution. Since the process of finding the solution of an LPP is completed in two phases, this is called the two-phase method.

Remark: The basic feasible solution (if it exists) obtained at the end of phase I is used to start phase

II. Rules for applying two-phase method are as follows:

1. Assign a cost '-1' to each artificial variable and a cost '0' to all other variables (in place of their original cost) in the objective function.
2. Solve the auxiliary problem by simplex method until either of the following three properties arise:
   i. Max. Z* < 0 and at least one artificial vector appears in the optimum basis at a positive level. In this case not proceed to Phase II.
   ii. Max. Z* = 0 and at least one artificial vector appears in the optimum basis at zero level. In this case proceed to phase II.
   iii. Max. Z* = 0 and no artificial vector appears in the optimum basis. In this case also proceed to phase II.

The method is well explained by the following examples.

Example. Use two-phase simplex method to solve the problem: Min. Z = $x_1 - 2x_2 - 3x_3$

Subject to the constraints:

$-2x_1 + x_2 + 3x_3 = 2$

$2x_1 + 3x_2 + 4x_3 = 1$

$x_1, x_2, x_3 \geq 0$

Solution. Max. Z = $-x_1 + 2x_2 + 3x_3$

Subject to the constraints:

$$-2x_1 + x_2 + 3x_3 + a_1 = 2$$

$$2x_1 + 3x_2 + 4x_3 + a_2 = 1$$

$$x_1, x_2, x_3, a_1, a_2 \geq 0$$

Phase I: Auxiliary linear programming problem is

Max. Z* = $0x_1 + 0x_2 + 0x_3 - 1a_1 - 1a_2$

Subject to the constraints:

$$-2x_1 + x_2 + 3x_3 + a_1 = 2$$

$$2x_1 + 3x_2 + 4x_3 + a_2 = 1$$

$$x_1, x_2, x_3, a_1, a_2 \geq 0$$

		C_J	0	0	0	-1	-1	
Basic variable	C_B	X_B	x_1	x_2	x_3	A_1	A_2	Minimum Ratio
a_1	-1	2	-2	1	3	1	0	$2/3$
$\leftarrow a_2$	-1	1	2	3	4	0	1	¼ ←
		Z* = -3	0	-4	-7 t	0	0	$\leftarrow \Delta_j$
a_1	-1	$5/4$	$-7/2$	$-5/4$	0	1	$-3/4$	
x_3	0	¼	¼	¾	1	0	¼	
		$Z^* = -5/4$	$7/4$	$5/4$	0	0	$7/4$	$\leftarrow \Delta_j \geq 0$

Since all $\Delta_j \geq 0$ an optimum basic feasible solution to the auxiliary Linear Programming Problem has been attained. But at the same time Max. Z* is negative.

Here, there is no need to enter phase-II.

Note: Here in place of $c_j - z_j$, we take $-(c_j - z_j)$ as Δ_j so corresponding changes imposed.

Example. Solve the problem:

Min. Z= $\frac{15}{4}x_1 - 3x_2$

Subject to the constraints:

$3x_1 - x_2 - x_3 \geq 3$

$x_1 - x_2 + x_3 \geq 2$

And $x_1, x_2, x_3 \geq 0$

Solution. Min. Z = $\frac{15}{4}x_1 - 3x_2$

Subject to the constraints:

$3x_1 - x_2 - x_3 - s_1 + a_1 = 3$

$x_1 - x_2 + x_3 - s_2 + a_2 = 2$

And $x_1, x_2, x_3, a_1, a_2, s_1, s_2 \geq 0$

Phase: I

Max. $Z^* = 0x_1 + 0x_2 + 0x_3 + 0s_1 + 0s_2 - 1a_1 - 1a_2$

Subject to the constraints:

$3x_1 - x_2 - x_3 - s_1 + a_1 = 3$

$x_1 - x_2 + x_3 - s_2 + a_2 = 2$

And $x_1, x_2, x_3, a_1, a_2, s_1, s_2 \geq 0$

$S_1, S_2 \rightarrow$ Surplus

$a_1, a_2 \rightarrow$ Artificial variables.

Construct the simplex table:

		$C_j \rightarrow$	0	0	0	0	0	-1	-1	
Basic variable	C_B	X_B	x_1	x_2	x_3	s_1	s_2	A_1	A_2	Minimum Ratio
$\leftarrow a_1$	-1	3	3	-1	-1	-1	0	1	0	1$\leftarrow$
a_2	-1	2	1	-1	1	0	-1	0	1	2
		$Z^*=-5$	-4 t	2	0	1	1	0	0	$\leftarrow \Delta_j$
$\rightarrow x_1$	0	1	1	$-1/3$	$-1/3$	$-1/3$	0	$1/3$	0	
$\leftarrow a_2$	-1	1	0	$-2/3$	$4/3$	$1/3$	-1	$-1/3$	1	$3/4 \leftarrow$
		$Z^*=-1$	0	$2/3$	$-4/3$ t	$-1/3$	1	$4/3$	0	$\leftarrow \Delta_j$
x_1	0	$5/4$	1	$-1/2$	0	$-1/4$	$-1/4$	$1/4$	$1/4$	
$\rightarrow x_3$	0	$3/4$	0	$-1/2$	1	$1/4$	$-3/4$	$-1/4$	$3/4$	
		$Z^* = 0$	0	0	0	0	0	1	1	$\leftarrow \Delta_j \geq 0$

Since all $\Delta_j \geq 0$ and no artificial variable appears in the basis, an optimum solution to the auxiliary problems has been attained.

Phase II:

In this phase, we consider

Max. $Z' = \frac{-15}{2}x_1 + 3x_2 + 0x_3 + 0s_1 + 0s_2$

Now, apply simplex method in the usual manner:

		$C_j \rightarrow$	$-15/2$	3	0	0	0	
Basic variable	C_B	X_B	x_1	x_2	x_3	s_1	s_2	Minimum Ratio
x_1	$-15/2$	$5/4$	1	$-½$	0	$-¼$	$-¼$	
x_3	0	¾	0	$-½$	1	¼	¾	
		$Z' = -75/8$	0	¾	0	$15/8$	$15/8$	$\leftarrow \Delta_j \geq 0$

Since all $\Delta_j \geq 0$, an optimum basic feasible solution is:

$x_1 = 5/4$, $x_2 = 0$, $x_3 = ¾$, Min. Z = $75/8$

3.5.3. Example: Max. Z = $5x_1 + 8x_2$

Subjected to the constraints

$3x_1 + 2x_2 \geq 3$

$x_1 + 4x_2 \geq 4$

$x_1 + x_2 \leq 5$

And $x_1, x_2 \geq 0$

Solution: Max. Z = $5x_1 + 8x_2$

Subjected to the constraints

$$3x_1 + 2x_2 - s_1 + a_1 = 3$$

$$x_1 + 4x_2 - s_2 + a_2 = 4$$

$$x_1 + x_2 + s_3 = 5$$

And $x_1, x_2, x_3, s_1, s_2, a_1, a_2 \geq 0$

Phase: I

Max. Z = $0x_1 + 0x_2 + 0x_3 + 0s_1 + 0s_2 - 1a_1 - 1a_2$

Subjected to the constraints

$$3x_1 + 2x_2 - s_1 + a_1 = 3$$

$$x_1 + 4x_2 - s_2 + a_2 = 4$$

$$x_1 + x_2 + s_3 = 5$$

And $x_1, x_2, x_3, s_1, s_2, a_1, a_2 \geq 0$

		$C_j \rightarrow$	0	0	-1	-1	0	
Basic variable	C_B	X_B	x_1	x_2	A_1	A_2	s_3	Minimum Ratio
a_1	-1	3	3	2	1	0	0	$3/2$
$\leftarrow a_2$	-1	4	1	4	0	1	0	$1 \leftarrow$
s_3	0	5	1	1	0	0	1	5
		Z = -7	-4	-6 t	0	0	0	$\leftarrow \Delta_j$
a_1	-1	1	$5/2$	0	1	$-1/2$	0	$2/5 \leftarrow$
$\rightarrow x_2$	0	1	$1/4$	1	0	$1/2$	0	4
s_3	0	4	$3/4$	0	0	$-1/4$	1	$16/3$
		Z = -1	$-5/2$ t	0	0	$3/2$	0	$\leftarrow \Delta_j$
$\rightarrow x_1$	0	$2/5$	1	0	$2/5$	-5	0	
x_2	0	$9/10$	0	1	$-1/10$	$3/2$	0	
s_3	0	$27/10$	0	0	$-3/10$	$7/2$	1	
		Z = 0	0	0	1	1	0	$\leftarrow \Delta_j \geq 0$

Since all $\Delta_j \geq 0$ and no artificial variable appears in the basis, an optimum solution to the auxiliary problem has been attained.

Phase: II

In this phase, we consider

Max. Z = $5x_1 + 8x_2 + 0s_3$

		$C_j \rightarrow$	5	8	0	
Basic variable	C_B	X_B	x_1	x_2	s_3	Minimum Ratio
x_1	5	$2/5$	1	0	0	
x_2	8	$9/10$	0	1	0	
s_3	0	$27/10$	0	0	1	
			0	0	0	$\leftarrow \Delta_j \geq 0$

$x_1 = 2/5, x_2 = 9/10$ and Max. Z = 46

3.5.4. Exercise:

1. Solve the problem:

Max. Z = $2x_1 + 3x_2 + 5x_3$

Subject to the constraints

$3x_1 + 10x_2 + 5x_3 \leq 15$

$x_1 +$

$2x_2 + x_3$

≥ 4 $33x_1$

$- 10x_2 +$

$9x_3 \leq 33$

And $x_1, x_2, x_3 \geq 0$

Answer. There do not exist any feasible solution, because artificial variable is not removed.

2. Max. Z = $5x_1 - 2x_2 + 3x_3$

Subject to the constraints

$2x_1 + 2x_2 - x_3 \geq 2$

$3x_1 - 4x_2 \leq 3$

$x_2 + 3x_3 \leq 5$ And $x_1, x_2, x_3 \geq 0$

Answer. $x_1 = {}^{23}/_{3}$, $x_2 = 5, x_3 = 0, Max. Z = {}^{85}/_{3}$.

3. Solve the problem:

Min. Z = $x_1 + x_2$

Subject to the constraints

$2x_1 + x_2 \geq 4$

$x_1 + 7x_2 \geq 7$

And $x_1, x_2 \geq 0$

Answer. $x_1 = {}^{21}/_{18}$, $x_2 = {}^{10}/_{18}$, $Max. Z = {}^{31}/_{13}$

4. Solve the problem:

Max. Z = $2x_1 + x_2 + ¼\, x_3$

Subject to the constraints

$4x_1 + 6x_2 + 3x_3 \leq 8$

$3x_1 - 6x_2 - 4x_3 \leq 1$

$2x_1 + 3x_2 - 5x_3 \geq 4$

And $x_1, x_2, x_3 \geq 0$

Answer. $x_1 = {}^{9}/_{7}$, $x_2 = {}^{10}/_{21}$, $x_3 = 0, Max. Z = {}^{64}/_{21}$

5. Solve the problem:

Max. Z = $2x_1 + x_2$

Subject to the constraints

$5x_1 + 10x_2 - 3x_3 = 8$

$x_1 + x_2 + x_4 = 1$

And $x_1, x_2, x_3, x_4 \geq 0$

Answer. $x_1 = 0, x_2 = {}^4/_5 , x_3 = 0, x_4 = {}^1/_5 , Max. Z = {}^4/_5$

DEGENERACY

The process of obtaining a degenerate basic feasible solution in an LPP is called degeneracy. Degeneracy is revealed when a basic feasible solution contains a smaller number of non-zero variables than the number of independent constraints when at the initial stage while forming the initial simplex table if the values of some basic variables are zero and if at any iteration there is a tie in the minimum replacement ratios (RR)
i.e. RR is not unique. In case of tie, if we select any of the rows arbitrarily, it may be possible that the subsequent iterations may not produce improvements in the value of objective function. This concept is known as cycling. In this case, we have to go back and choose another row.

However, one can avoid the situation of cycling which arises due to degeneracy in LPP by adopting the following procedure:

Determine the non-negative ratios of the first column of the unit matrix (and not the quantity column) to the entries of the entering variable. Then choose the minimum of the values occurring at the places of tie. If we find tie again, then we compute the ratios of the second column of the unit matrix to the entry of the entering variable. We continue the process till the ratios do not break the tie.

EXAMPLE. Maximise Z = $3x_1 + 9x_1$

Subject to the constraints:

$x_1 + x_2 \leq 8$

$x_1 + 2x_2 \leq 4$

$x_1, x_2 \geq 0$

Solution. The standard form of the LPP is

Maximise Z = $3x_1 + 9x_2 + 0s_1 + 0s_2$

Subject to the constraints:

$x_1 + 4x_2 + s_1 = 8$

$$x_1 + 2x_2 + s_2 = 4$$

$$x_1, x_2, s_1, s_2 \geq 0$$

The initial simplex table for the given LPP is as follows:

Table 1: Initial Simplex Table

		$c_j \rightarrow$	3	9	0	0	
Basic variable	Qty		x_1	x_2	s_1	s_2	R.R.
s_1	0	8	1	4	1	0	8/4=2
s_2	0	4	1	2	0	1	4/2=2
	Z = 0	$z_j \rightarrow$ $c_j - z_j \rightarrow$	0 3	0 9 t	0 0	0 0	

Clearly, from Table 1, there is tie in the minimum replacement ratio as it is 2 in each of the two rows. Now, to avoid cycling, we should not select any row arbitrarily. We will proceed as follows:

We divide the non-negative ratios of the first column of identify matrix, i.e.

$\binom{1}{0}$ by the entries of the entering variable, i.e. $\binom{4}{2}$.

So, we have replacement ratios as $\binom{1/4}{0/2}$, i.e. $\binom{4}{2}$. So here we have found the minimum RR to be 0 and the tie has been broken. Therefore, we select the second row. Thus, the completed initial simplex table is as follows:

Table 2: Complete Initial Simplex Table

	$c_j \rightarrow$		3	9	0	0	
Basic Variables	Profit/unit	Qty	x_1	x_2	s_1	s_2	R.R.
s_1	0	8	1	4	1	0	2, ¼
← s_2	0	4	1	[2]	0	1	2, 0←
	Z = 0	$z_j \rightarrow$ $c_j - z_j \rightarrow$	0 3	0 9	0 0	0 0	

↑

Now we apply simply our simplex method procedure to obtain the optimum value of the objective function. It will be Max. Z =18 at $x_1 = 0$, $x_2 = 2$.

DUALITY IN LINEAR PROGRAMMING

The concept of duality is very important in linear programming theory as duality implies that each linear programming problem can be analyzed in two different ways but having equivalent solutions in other words every LPP has associated with another LPP where the original problem is called primal and the other one is called its dual. The optimal solution of the primal reveals information about the optimal solution of the dual and vice-versa. This fact is very useful as sometimes dual of an LPP is easier to solve than its primal.

Note: In general, either problem can be considered as primal and the remaining one its dual.

Remark: The original problem is usually referred to as primal.

Steps for Converting Any Primal into Its Dual

Step I: First convert the objective function to maximization form.

Step II: If a constraint has inequality sign ≥ , then convert the inequality sign ≥ into ≤ on multiply both sides by -1.

Step III: If a constraint has an equality sign (=) then, it is replaced by two constraints, one constraint having inequality sign ≥ and other one having ≤ and hence corresponding to one equality, we obtain two inequalities.

Step IV: Every unrestricted variable is replaced by the difference of two non-negative variables.

In this way, we get the standard form of the given primal. Now the dual of the given problem is obtained by:

Step V: Transposing the rows and columns of constraints coefficients. Also transposing the coefficients ($c_1, c_2, \ldots, c_n$) of the objective function and the right-side constants

($b_1, b_2, \ldots, b_m$). Changing the inequalities from ≤ to ≥ sign and minimizing the objective function instead of maximizing it.

Note: If we have an equality constraint in primal then its corresponding dual variables must be unrestricted in sign.

Result 1. The dual of the dual is primal.

Result 2. If either the primal or the dual problem has an unbounded solution, the other one has no feasible solution.

Result 3. If either the primal or the dual problem has a finite optimal solution, the other problem also has finite optimal solution. The optimal value of the objective functions of the two problems are equal.

The steps of obtaining dual of an LPP are explained with the help of an example

Example. Find the dual of the following LPP: Min. Z = $2x_2 + 5x_3$

Subject to the constraints:

$$x_1 + x_2 \geq 2$$

$$2x_1 + x_2 + 6x_3 \leq 6$$

$$x_1 - x_2 + 3x_3 = 4$$

And

$$x_1, x_2, x_3 \geq 0$$

Solution: Step: I

Change the objective function of minimization to maximization.

i.e. Max. $Z^* = -2x_2 - 5x_3$ where $Z^* = -Z$

Step: II

The inequality $x_1 + x_2 \geq 2$ can be written as $-x_1 - x_2 \leq -2$

Step: III

The equation $x_1 - x_2 + 3x_3 = 4$ can be expressed as a pair of inequalities:

$x_1 - x_2 + 3x_3 \leq 4$

$x_1 - x_2 + 3x_3 \geq 4$

Or

$-x_1 + x_2 - 3x_3 \leq -4$

$x_1 - x_2 + 3x_3 \leq 4$

Step: IV

Thus, original problem now becomes of the standard primal forms:

$$Z^* = 0x_1 - 2x_2 - 5x_3$$

Subject to the constraints

$-x_1 - x_2 \leq -2$

$2x_1 + x_2 + 6x_3 \leq 6$

$-x_1 + x_2 - 3x_3 \leq -4$

$x_1 - x_2 +$

Step: V

Thus, the required dual is:

$3x_3 \le 4$ $\qquad x_1, x_2, x_3 \ge 0$

And

Min. Z* = $-2w_1 + 6w_2 + 4w_3 - 4w_4$

Subject to the constraints

$-w_1 + 2w_2 + w_3 - w_4 \ge 0$

$-w_1 + w_2 - w_3 + w_4 \ge -2$

$6w_2 + 3w_3 - 3w_4 \ge -5$

And $w_1, w_2, w_3, w_4 \ge 0$

Example. Write the dual of the following primal: Minimum Z = $3x_1 - 2x_2 + 4x_3$

Subject to the constraints:

$3x_1 + 5x_2 + 4x_3 \ge 7$

$6x_1 - x_2 + 3x_3 \ge 4$

$7x_1 + 2x_2 - 3x_3 \le 10$

$x_1 - 2x_2 + 5x_3 \ge 3$

$4x_1 + 7x_2 - 2x_3 \ge 2$

And $x_1, x_2, x_3 \ge 0$

Solution. The original primal can be written in the standard primal form as:

Max. Z* = $-3x_1 + 2x_2 - 4x_3$

Subject to the constraints:

$$-3x_1 - 5x_2 - 4x_3 \leq -7$$
$$-6\,x_1 - x_2 - 3x_3 \leq -4$$
$$7x_1 - 2x_2 - x_3 \leq 10$$
$$-x_1 + 2x_2 - 5x_3 \leq -3$$
$$-\,4\,x_1 - 7\,x_2 + 2x_3 \leq -2$$

And $x_1, x_2, x_3 \geq 0$

Dual primal becomes

Min. Z* = $-7w_1 - 4w_2 + 10w_3 - 3w_4 - 2w_5$

Subject to the constraints

$$-3w_1 - 6w_2 + 7w_3 - w_4 - 4w_5 \geq -3$$
$$-5w_1 - w_2 - 2w_3 + 2w_4 - 7w_5 \geq 2$$
$$4w_1 - 3w_2 - 3w_3 - 5w_4 + 2w_5 \geq -4$$

And $w_1, w_2, w_3, w_4, w_5 \geq 0$

Example. Given the dual Linear Programming Problem Min. Z = $2x_1 + 3x_2 + 4x_3$

Subject to the constraints:

$$2x_1 + 3x_2 + 5x_3 \geq 2$$
$$3x_1 + x_2 + 7x_3 = 3$$
$$x_1 + 4x_2 + 6x_3 \leq 5$$

And $x_1, x_2, \geq 0$, x_3 is unrestricted.

Solution. Since the variable x_3 is unrestricted in sign, the given LPP can be transform into standard primal form by substituting $x_3 = x'_3 - x''_3$ where $x'_3 \geq 0$, $x''_3 \geq 0$

∴ Standard primal becomes:

Max. Z* = $-2x_1 - 3x_2 - 4(x'_3 - x''_3)$

Subject to the constrain

$$-2x_1 - 3x_2 - 5(x'_3 - x''_3) \leq -2$$
$$3x_1 + x_2 + 7(x'_3 - x''_3) \leq 3$$
$$-3x_1 - x_2 - 7(x'_3 - x''_3) \leq -3$$
$$x_1 + 4x_2 + 6(x'_3 - x''_3) \leq 5$$

And $x_1 , x_2 , x'_3 , x''_3 \geq 0$

∴ Required dual is

Min. $Z^* = -2w_1 + 3(w_2' - x_2'') + 5w_3$

Subject to the constraints

$$-2w_1 + 5(w_2' - x_2'') + w_3 \geq -2$$
$$-3w_1 + (w_2' - x_2'') + 4w_3 \geq -3$$
$$-5w_1 + 7(w_2' - x_2'') + 6w_3 \geq -4$$
$$5w_1 - 7(w_2' - x_2'') - 6w_3 \geq 4$$

And $w_1, w_2', w_2'', w_3 \geq 0$

Again, we write

Min. $Z^* = -2w_1 + 3w_2 + 5w_3$

Subject to the constraints

$$-2w_1 + 5w_2 + w_3 \geq -2$$
$$-3w_1 + w_2 + 4w_3 \geq -3$$

$5w_1 - 7w_2 - 6w_3 = 4$ And $w_1, w_2 \geq 0$ and w_3 is unrestricted.

Exercises.

1. Write the dual of the following Linear Programming

Problem Max. $Z = 2x_1 + 3x_2 + x_3$

Subject to the constraints:

$4x_1 + 3x_2 + 4x_3 = 6$

$x_1 - 2x_2 + 35 = 4$

And $x_1, x_2, x_3 \geq 0$

Answer. Min. $Z = 6(w_1 - w_2) + 4(w_3 - w_4)$

Subject to the constraint

$4(w_1 - w_2) + (w_3 - w_4) \geq 2$

$3(w_1 - w_2) + 2(w_3 - w_4) \geq 3$

$(w_1 - w_2) + 5(w_3 - w_4) \geq 1$

And $w_1, w_2, w_3 \geq 0$

2. Write the dual of the following Linear Programming

Problem Max. $Z = 3x_1 - 2x_2$

Subject to the constraints:

$x_1 \leq 4$

$x_2 \leq 6$

$x_1 + x_2 \leq 5$

$-x_2 \leq -1$

And $x_1, x_2, x_3 \geq 0$

Answer. Min. Z*= $4w_1 + 6w_2 + 5w_3 - 1w_4$

Subject to the constraints:

$w_1 + w_3 \geq 3$

$w_2 + w_3 - w_4 \geq -2$ And $w_1, w_2, w_3, w_4 \geq 0$

3. Min. Z = $x_1 + x_2 + x_3$

Subject to the constraints:

$x_1 - 3x_2 + 4x_3 = 5$

$x_1 - 2x_2 \leq 3$

$2x_1 - x_3 \geq 4$

And $x_1, x_2, \geq 0$, x_3 is unrestricted

Answer. Min. Z* = $5w_1 + 3w_2 - 4w_3$

Subject to the constraints

$w_1 + w_2 - 2w_3 \geq -1$

$-3w_1 - 2w_2 \geq -1$

$4w_1 + 0w_2 - w_3 \geq -1$

$-4w_1 + 0w_2 + w_3 \geq 1$ And w_2 , $w_3 \geq 0$ and w_3 is unrestricted.

4. Max. Z = $3x_1 + x_2 + x_3 - x_4$

Subject to the constraints:

$x_1 - 5x_2 + 3x_3 + 4x_4 \leq 5$

$x_1 + x_2 = -1$

$x_3 - x_4 \geq -5$

And $x_1, x_2, x_3, x_4 \geq 0$

Answer. Min. Z* = $5w_1 - w_2 + 5w_3$

Subject to the constraints

$w_1 + w_2 \geq 3$

$5w_1 + w_2 \geq 1$

$3w_1 - w_3 \geq 1$

$4w_1 + w_3 \geq -1$

And w_1 , $w_3 \geq 0$ and w_2 is unrestricted.

CHECK FOR PROGRESS

1. Describe simplex method of solving linear programming problem.
2. Explain the term 'Artificial variable' and its use in linear programming.
3. What do you mean by two-phase method in linear programming problem?
4. Write a note on the use of Big-M method in solving linear programming problem.
5. Explain the concept of degeneracy in simplex method.
6. Explain the concept of duality.
7. Discuss relationship between primal and its dual.

SUMMARY

In this chapter, we study simplex method which can be used for solving LPP involving more than two variables. Simplex method is an iterative method as it examines the extreme points in a systematic manner, repeating the same set of steps until an optimal solution is reached. We have introduced artificial variables which play an important role to solve an LPP involving greater-than-or-equal to constraints and/or equality constraints.

Here, we discuss two important methods namely Big-M and Two-phase method for solving LPP involving artificial variables. In this chapter, we introduce the concept of degeneracy which occurs in a linear programming problem when in the simplex table one or more basic variable has value zero or RR is not unique. In the last, we explain duality concept of LPPs which is very useful in science and engineering, game theory, economics etc.

4

TRANSPORTATION PROBLEM

Structure

Introduction

Mathematical Formulation of the Transportation Problem

Methods of Finding Initial Basic Feasible Solution

Methods of Finding Optimal Solution

Unbalanced Transportation Problem

Degeneracy

Check Your Progress

Summary

INTRODUCTION

Transportation problem is a special kind of Linear Programming Problem (LPP) in which the objective is to transport goods from a set of sources/origins to a set of destinations in such a manner that the total transportation or shipping cost is minimized. To achieve this objective, we must know about some parameters such as the quantity of available supplies, the quantity demanded and the costs of shipping a unit from various origins to various destinations. Solving LPPs by simplex method discussed in chapter 3 involves a large number of variables and constraints and takes a long time to solve it. So, in this chapter, we shall discuss the methods, which are specifically applied for solving transportation problems. Firstly, we shall explain how to formulate mathematically a transportation problem and methods of finding initial basic feasible solution. After finding the initial basic feasible solution, we discuss how optimality test is performed by applying the Stepping Stone Method or Modified Distribution Method (MODI)to find whether the obtained feasible solution is optimal or not. In the last, we shall explain the case of unbalanced transportation problems and degeneracy.

Objectives

After studying this chapter, the reader should be able to:

- Define a transportation problem.
- Obtain the basic feasible solution of a transportation problem using North – West Corner Rule, Least Cost and Vogel's Approximation methods.
- State the conditions for performing optimality test.
- Explain the algorithm of the Stepping Stone and Modified Distribution (MODI) methods of obtaining the optimal solution of a transportation problem.
- Solve the transportation problems for special cases such as unbalanced transportation problem, case of degeneracy.

MATHEMATICAL FORMULATION OF THE TRANSPORTATION PROBLEM

Let there be m origins/ sources of supply $O_1, O_2, \ldots, O_i \ldots O_m$ and n destinations$D_1, D_2, \ldots, D_j \ldots D_n$. The total number of the capacities of all m origins is assumed to be equal to the total number of the requirements of all n destinations. Let C_{ij} be the cost of shipping one unit from origin i to destination j. Let a_i be the capacity/ availability of items at origin i and b_j, the requirement/demand of the destination j. Then this transportation problem can be expressed in a tabular form as follows:

Origin	Destinations						Availability/ capacity
	D_1	D_1	…	D_j	…	D_n	
O_1	C_{11}	C_{12}	…	C_{1j}	…	C_{1n}	a_1
O_2	C_{21}	C_{22}	…	C_{2j}	…	C_{2j}	a_2
	.	.		.		.	.
	.	.		.		.	.
	.	.		.		.	.
O_i	C_{i1}	C_{i2}	…	C_{ij}	…	C_{in}	a_i
	.	.		.		.	.
	.	.		.		.	.
	.	.		.		.	.
O_m	C_{m1}	C_{m2}	…	C_{mj}	…	C_{mn}	a_m
Requirement/ Demand	b_1	b_2	…	b_j	…	b_n	Total

The condition for the existence of a feasible solution to a transportation problem is give as

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

The above equation tells us that the total requirement/demand equals the total capacity. If it is not so, a dummy origin or destination is created to balance the total capacity and requirement.

Now let x_{ij} be the number of units to be transported from origin i to destination j and C_{ij} the corresponding cost of transportation. Then the total transportation cost is $\sum_{i=1}^{m} \cdot \sum_{j=1}^{n} C_{ij} x_{ij}$

Subject to the constraints:

$$\sum_{j=1}^{n} x_{1j} = a_1 , \sum_{j=1}^{n} x_{2j} = a_2 , \dots , \sum_{j=1}^{n} x_{mj} = a_m \quad \dots\dots(3)$$

$$\sum_{i=1}^{m} x_{i1} = b_1 , \sum_{i=1}^{m} x_{i2} = b_2 , \dots , \sum_{j=1}^{n} x_{in} = b_n$$

And $x_{ij} \geq 0$ for all i = 1, 2, 3... m and j = 1, 2... n.

The Simplex method is regarded as the most generalized method to solve this. However, the solution is very lengthy and takes a long time to solve it since a large number of decision variables and artificial variables are involved. It is far simpler to solve it by transportation method as compared to the Simplex method. In the transportation method, we first obtain the initial basic feasible solution and then perform the optimality test.

Note: A transportation problem is said to be balanced if it satisfies the condition $\sum^{m} a_i = \sum^{n} b_j$.

METHODS OF FINDING INITIAL BASIC FEASIBLE SOUTION

There are several methods to obtain initial basic feasible solution. Here, we shall discuss the following methods to determine the initial basic feasible solution:

- **(i) North-West Corner Rule**
- **(ii) Least Cost Method**
- **(iii) Vogel's Approximation Method (Penalty or Regret Method)**

Vogel's Approximation method generally gives a solution closer to the optimum solution. Hence, it is preferred to the other two methods.

North – West Corner Rule

The North – West Corner Rule (NWC)is a simple and efficient method to obtain initial basic feasible solution. Itcan be summarized as follows:

Step 1: Start with cell (1, 1) at the north-west corner (upper left-hand corner) of the transportation matrix and allocate as much as possible there.

Step 2: Here, we have three cases

(a) If the quantity needed at First Destination (b_1) is less than the quantity available at First Origin (a_i), we allocate a quantity equal to the requirement at First Destination to the cell (1, 1). At this stage, Column 1 is exhausted, so we cross it out. Since the requirement b_1 is fulfilled, we reduce the availability $a_1 \; by \; b_1$ and proceed to north-west corner of the resulting matrix, i.e., cell (1,2).

(b) If the quantity needed at First Destination (b_1) is greater than the quantity available at First Origin(a_1), allocate a quantity equal to the quantity available at First Plant/Origin (a_1) $to \; cell$ (1, 1).At this stage, Row 1 is exhausted, so we cross it out and proceed to north- west corner of the resulting matrix, i.e., cell (2, 1).

(c) If the quantity needed at First Destination (b_1) is equal to the quantity available at First Origin (a_1), we allocate a quantity equal to the requirement at First Destination (or the quantity available at First Origin). At this stage, both column 1 as well as Row 1 is exhausted. We cross them out and proceed to the north-west corner of the resulting matrix, i.e., cell (2, 2).

Step 3: We continue the procedure, until we reach the south – east corner of the original matrix.

Example. Find the basic feasible solution of the given transportation problem by applying North – West Corner rule:

Warehouse / Factory	D	E	F	G	Capacity
A	42	48	38	37	160
B	40	49	52	51	150
C	39	38	40	43	190
Requirement	80	90	110	220	500

Solution. We start from the North – West corner, i.e., the Factory A and Warehouse D. The quantity needed at the First Warehouse (Warehouse D) is 80, which is less than the quantity available (160) at the First Factory A. Therefore, a quantity equal to the warehouse D is to be allocated to the cell (A, D). Thus, the requirement of Warehouse D is met by Factory A. So, we cross out column 1 and reduce the capacity of Factory A by 80. Then we go to cell (A, E), which is North – West corner of the resulting matrix.

Now, the quantity needed at the second Warehouse (Warehouse E) is 90, which is greater than the quantity available (80) at the First Factory A. Therefore, we allocate a quantity equal to the capacity at Factory A, i.e., 80 to the cell (A, E). The requirement of Warehouse E is reduced to 10. The capacity of Factory A is exhausted and has to be removed from the matrix. Therefore, we cross out row 1 and proceed to cell (B, E).

Now, the quantity needed at the second Warehouse (Warehouse E) is 10, which is less than the quantity available at the Second Factory B, which is 150. Therefore, the quantity 10 equal to the requirement at Warehouse E is allocated to the cell (B, E). Hence, the requirement of Warehouse E is met and we cross out column 1. We reduce the capacity of Factory B by 10 and proceed to cell (B, F).

Again, the quantity needed at the Third Warehouse (Warehouse F) is 110. It is less than the quantity available at the Second Factory (Factory B), which is140. Therefore, a quantity equal to the requirement at Warehouse F is allocated to the cell (B, F). Since the requirement of Warehouse F is met, we cross out Column 1 and reduce the capacity of Factory B by 110. Then we proceed to cell (B, G).Now, the quantity needed at the Fourth Warehouse (Warehouse G) is 220, which is greater than the quantity available at the Second Factory (Factory B).Therefore, we allocate the quantity equal to the capacity of Factory B to the cell (B, G) so that the capacity of Factory B is exhausted and the requirement of Warehouse G is reduced to 190. Hence, we cross out Row 1 and proceed to cell (C, G).

Thus, the allocations given using North – West corner rule are as shown in the following matrix along with the cost per unit of transportation:

Warehouse Factory	D	E	F	G	Capacity
A	42 80	48 80	38	37	160
B	40	49 10	52 11	51 30	150
C	39	38	40	43 190	190
Requirement	80	90	110	220	500

Thus, the total transportation cost for these allocations

$= 42 \times 80 + 48 \times 80 + 49 \times 10 + 52 \times 110 + 51 \times 30 + 43 \times 190$

= 3360 + 3840 + 490 + 5720 + 1530 + 8170 =23110

Exercise. Find the basic feasible solution of the following problem using North-West Corner Rule:

Origin/ Distribution Centre	1	2	3	4	5	6	Availability
1	4	6	9	2	7	8	10
2	3	5	4	8	10	0	12
3	2	6	9	8	4	13	4
4	4	4	5	9	3	6	18
5	9	8	7	3	2	14	20
Requirements	8	8	16	3	8	21	

Answer. Using North - West corner rule, the allocations are to be made as under:

8 units to cells (1,1), 2 units to cell (1,2), 6 units to cell (2,2), 6 units to cell (2,3), 4 units to cell (3 ,3), 6 units to cell (4,3), 3 units to cell (4,4), 8 units to cell (4,5), 1 unit to cell (4,6) and 20 units to cell (5,6) and the transportation cost is equals to 501.

Least Cost Method

This method is also known as the Matrix Minimum method or Inspection method. It starts by making the first allocation to the cell for which the transportation cost per unit is lowest. The row or column for which the capacity is exhausted or requirement is satisfied is removed from the transportation table. We follow the procedure with the reduced matrix until all the requirements are satisfied. If there is a tie for the lowest cost cell while making any allocation, the choice may be made for a row or a column by which maximum requirement is exhausted. If there is a tie in making this allocation as well, then we can arbitrarily choose a cell for allocation.

The method can be easily explained with the help of the following example.

Example. Find the basic feasible solution of the transportation problem of Example 4.3.1.1. by using the Least Cost method.

Solution. Here, the least cost is 37 in the cell (A, G). The requirement of the Warehouse G is 220 and the capacity of Factory A is 160. Hence, the maximum number of units that can be allocated to this cell is 160. Thus, Factory A is exhausted. The requirement of Warehouse G is reduced by 160.

Now, the least cost is 38, which is in the cell (C, E). The requirement of the Warehouse E is 90 and the capacity of Factory C is 190. Hence, the maximum number of units that can be allocated to this cell is 90. Moreover, we reduce the capacity of factory C by 90.

The least cost in the matrix is 39, which is in the cell (C, D). The requirement of Warehouse D is 80 and the capacity of Factory C is 100.Hence, the maximum number of units that can be allocated to this cell is
80. The requirement of Warehouse D is exhausted. The capacity of the Factory C is also reduced by 80.

The least cost in this matrix is 40 which is in the cell (C, F). The requirement of Warehouse F is 110 and the capacity of Factory C is 20. Hence, the maximum number of units that can be allocated to this cell is 20. Thus, Factory C is exhausted. The requirement of Warehouse F is reduced by 20. It is now 90 in the reduced matrix.

The least cost is 51 in the cell (B, G) and the requirement of warehouse G is 60 units. So, we allocate 60 units to cell (B, G) and the remaining 90 units to the cell (B, F). Thus, the allocations given using Least Cost method are as shown in the following matrix along with the cost per unit of transportation:

Warehouse / Factory	D	E	F	G	Capacity
A	42	48	38	37 [160]	160
B	40	49	52 [90]	51 [60]	150
C	39 [39]	38 [90]	40 [20]	43	190
Requirement	80	90	110	220	500

Thus, the total transportation cost = $37\times 160 + 52 \times 90 + 51 \times 80 + 38 \times 90 + 40 \times 20$

= 21000

Note: This method has reduced the total transportation cost in comparison to the NWC rule.

Exercise. Find the basic feasible solution of the following problem using the Least Cost method:

Origin/ Distribution Centre	1	2	3	4	5	6	Availability
1	4	6	9	2	7	8	10
2	3	5	4	8	10	0	12
3	2	6	9	8	4	13	4
4	4	4	5	9	3	6	18
5	9	8	7	3	2	14	20
Requirements	8	8	16	3	8	21	

Answer. Using Least Cost method, the allocations are to be made as under:

4 units to cell (1,1), 3 units to cell (1,4), 3 units to cell (1,6), 12 units to cell (2,6), 4 units to cell (3,1), 8 units to cell (4,2), 10 units to cell (4,3), 6 units to cell (5,3), 8 units to cell (5,5) and 6 units to cell (5,6). The transportation cost = 278.

Vogel's Approximation Method (VAM)

We describe the step by step procedure for finding the initial basic feasible solution by Vogel's Approximation method (Penalty method) in the following steps:

(i) In the transportation table calculate penalties for each row (column), by taking the difference between the least and second least costs in the same row (column). We display it to the right (below) of that row (column) in a new column (row) formed by extending the table on the right (bottom). The new column and row formed by extending the table at the right and bottom are labelled as penalty column and penalty row, respectively. The differences noted in the penalty row or penalty column indicates the penalty or extra cost. If two cells in a row (or column) contain the same least costs then the difference is taken as zero.

(ii) Select the row or column with the largest penalty (largest difference) and allocate the maximum possible units to the least cost cell in the selected column or row. If there is a tie in the values of penalties, the choice may be made for that row or column, which has the least cost. In case there is a tie in such least cost as well, choice may be made from that there is a tie in such least cost as well, choice may be made from that row or column by which maximum requirements are exhausted

.The cell so chosen is allocated the units and the corresponding exhausted row or column is removed or ignored from further consideration.

(iii) Now, we determine the column and row differences for the reduced transportation table and repeat the procedure until all column and row totals are exhausted.

This method is also known as the penalty method**.** Let us understand the procedure with the help of an example.

Example. Apply the Vogel's Approximation Method for finding the Basic Feasible Solution for the transportation problem of Example 4.3.1.1.

Solution. In the first row, the least and the second least costs are 37 and 38 and their difference is 1. We write 1 in a new column created on the right. It is labelled Penalty.Similarly, the differences between the least and the second least costs in the second and third row, respectively, are 49 – 40 = 9 and 39 – 38 = 1. So, we write the values (differences), i.e., 9 and 1 in the penalty column.

Next, we find the differences of the least and second least elements of each of the columns D, E, F and G. These are 40-39 = 1, 48-38 = 10, 40-38 = 2 and 43-37 = 6, respectively. We write them in a newly created penalty row at the bottom of the table.

We now select the largest of these differences in the penalty row and column, which are 10 in this case. This value (10) corresponds to the second column (Column E) and the least cost in the column is 38. Hence the allocation of 90 units (the maximum requirement of warehouse E) is to be made in the cell (C, E) from Factory C. Since the column corresponding to E is exhausted, it is removed for the next reduced matrix and the capacity of C is reduced by 90.

We now take the differences between the least and the Second least cost for each row and column of the reduced matrix. In the first row, the least and the second least costs are 37 and 38 and their differences is 1. We write it in the newly created penalty column. Similarly, we write the second difference element 51- 40 = 11 and third difference element 40-39 = 1 in the second and third row of this column. Likewise, the differences of the smallest and second smallest elements of each of the columns D, F, and G are 40–39 = 1, 4–38 = 2 and 43–37 = 6, respectively. We write these in a newly created penalty row at the bottom of the table.

Now, we select the largest of these differences in the penalty row and column, which is 11 in this case. This value (11) corresponds to Row B. Since the least cost in row is 40, we allocate 80 units (the maximum requirement of Warehouse D) to the cell (B, D). Thus, the requirement of Warehouse D is exhausted and we can remove it. We also reduce the capacity of Factory B by 80 in the next reduced matrix.

Again, in the first row, the least and the second least costs are 37 and 38 and their difference is 1. We write it to the right of this row in the newly created penalty column. Similarly, the second and third elements in the second and third rows of this column are 52–51=1 and

43–40 = 3, respectively. We write these in a newly created penalty row at the bottom of the table. Now, we select the largest of these differences, which are 6 in this case. It corresponds to Column G and the least cost in this column is 37. Hence, we allocate 160 units (the maximum capacity of Factory A) to the cell (A, G). Since Row A is exhausted, it is removed for the next reduced matrix. We also reduce the requirement of Warehouse G by 160 units.

Once again, the difference of the least costs in the first row is 52 – 51 = 1. We write it in the newly created penalty column. Similarly, for the second row, the difference is 43-40 = 3. Likewise, the differences of the least and second least elements of each of the columns F and G are 52 – 40 = 12 and 51 – 43 = 8, respectively. We write them in the newly created penalty row at the bottom of the table. The largest of these differences is 12 in this case. It corresponds to Column F and the least cost in this column is 40. Hence, we allocate 100 units from the row (the maximum capacity of Factory C) to the cell (C, F). Since Row C is exhausted, it is removed and the requirement of Warehouse F is reduced to 10 for the next reduced matrix.

At the end, of the 70 units available in Factory B, we allocate 60 units to the lower cost (51), i.e., to the cell (B, G) and the remaining 10 units to the cell (B, F).

The entire procedure of allocating units by Vogel's Approximation Method is given in the following table:

Warehouse / Factory	D	E	F	G	Capacity	$Diff_1$	$Diff_2$	$Diff_3$	$Diff_4$
A	42	48	38	**37** (160)	160	1	1	1	-
B	**40** (80)	49	**52** (10)	**51** (60)	150	9	11*	1	1
C	39	**38** (90)	**40** (100)	43	190	1	1	3	3
Requirement	80	90	110	220	500				
$Diff_1$	1	10*	2	6					
$Diff_2$	1	-	2	6					
$Diff_3$	-	-	2	6*					
$Diff_4$	-	-	12*	8					

Thus, the total transportation cost

$$=40 \times 80 + 38 \times 90 + 52 \times 10 + 40 \times 100 + 37 \times 160 + 51 \times 60$$

$$= 3200 + 3420 + 520 + 4000 + 5920 + 306 = 20120$$

Note: The total transportation cost obtained above is the lowest total transportation cost among the three methods. Clearly the solution obtained by VAM is nearest to the optimal solution.

Exercise. Find the basic feasible solution of the following problem using Vogel's Approximation method:

Origin /Distribution Centre	1	2	3	4	5	6	Availability
1	4	6	9	2	7	8	10
2	3	5	4	8	10	0	12
3	2	6	9	8	4	13	4
4	4	4	5	9	3	6	18
5	9	8	7	3	2	14	20
Requirement	8	8	16	3	8	21	

Answer. The total transportation cost= 242.

METHODS OF FINDING OPTIMAL SOLUTION

Once an initial basic feasible solution is determined, the next step is to check its optimality. For this, we performed an optimality test which tell us that whether the obtained feasible solution is optimal or not. This can be done by applying one of the following methods:

(a) Stepping Stone Method
(b) Modified Distribution (MODI) Methods

These methods not only tell us about optimality of the initial basic feasible solution, they also improve the solution until the optimal solution is obtained. Use of the Stepping Stone method is convenient to a problem of small dimension as its application to a problem of a large dimension is quite tedious and cumbersome.

The MODI method is usually preferred over the Stepping Stone method. Before studying the methods of performing optimality test and finding optimal solutions, we introduce the concept of independent allocations and state the conditions for performing optimality test.

Independent and Non-Independent Allocations

A set of allocations is said to be independent if they do not form a loop in the transportation table. Here, formation of closed loop means that it is possible to travel from any allocation, back to itself by a series of horizontal and vertical jumps from one occupied cell (i.e., the cell containing allocation) to another, without a direct reversal of route (see Fig.). If such a loop can be formed using some or all of the allocations under consideration, the allocations are known as non-independent.

	•		•
•	•	•	
			•
		•	

(a)

•	•		•
•	•		
	•		•

(b)

Clearly allocations in (a) are independent allocations and of (b) are dependent.

Note: Every loop has an even number of cells.

Test for Optimality

An optimality test can be applied to the feasible solution only if it satisfies the following conditions:

(i) It contains exactly m+n-1 allocations where m and n represent the number of rows and columns, respectively, of the transportation table.

(ii) These allocations are independent.

Stepping Stone Method

Steps involved in Stepping Stone method for obtaining the optimal solution of a transportation problem can be summarized as follows:

1. First find an initial basic feasible solution.
2. Check optimality conditions if conditions are satisfied then evaluate all unoccupied cells for the effect of transferring one unit from an occupied cell to the unoccupied cell as follows:
   a) Select an unoccupied cell to be evaluated.
   b) Starting from this cell, from a closed path (or loop) through at least three occupied cells. The direction of movement is immaterial because the result will be same in both directions. Note that expect for the evaluated cell, all cells at the corners of the loop have to be occupied.
   c) At each corner of the closed path, assign plus (+) and minus (-) sign alternatively; beginning with the plus sign for the unoccupied cell to be occupied.
   d) Compute the net change in cost with respect to the costs associated with each cell traced in the closed path.
   e) Repeat steps 2(a) to 2(d) until the net change in cost has been calculated for all occupied cells.
3. If the net changes are positive or zero, an optimal solution has been arrived at. Otherwise go to step 4.
4. If some net changes are negative, select the unoccupied cell having the most negative net change. If two negative values are equal, select the one that results in moving more units into the selected unoccupied cell with the minimum cost.
5. Assign as many units as possible to this unoccupied cell.
6. Go the Step 2 and repeat the procedure until all unoccupied cells are evaluated and the value of net change, i.e., net evaluation is positive or zero.

Let us now take up an example to illustrate the Stepping Stone method.

4.4.3.1. Example. A company is spending Rs1000 on transportation of its units from three plants to four distribution centres. The availability of unit per plant and requirement of units per distribution centre, with unit cost of transportation are given as follows:

D. Centres Plants	D_1	D_2	D_3	D_4	Availability
P_1	19	30	50	12	7
P_2	70	30	40	60	10
P_3	40	10	60	20	18
Requirement	5	8	7	15	

What is the maximum possible saving by optimum distribution? Use the Stepping Stone method to solve the problem.

Solution. First, we determine the initial feasible solution by applying VAM is given as follows:

D. Centers Plant	D_1	D_2	D_3	D_4	Av.	$Diff_1$	$Diff_2$	$Diff_3$	$Diff_4$
P_1	**19** (5)	30	50	**12** (2)	7	7	18	38	38
P_2	*70*	30	**40** (7)	**60** (3)	10	10	10	20	20
P_3	40	**10** (8)	6	**20** (10)	18	10	10	40*	-
Requirement	5*	8	7	15					
$Diff_1$	21*	20	10	8					
$Diff_2$	-	20*	10	8					
$Diff_3$	-	-	10	8					
$Diff_4$	-	-	10	48*					

In the above table, the encircled values are the allocations. The total transportation cost associated with this initial basic feasible solution is

$= 19 \times 5 + 12 \times 2 + 40 \times 7 + 60 \times 3 + 10 \times 8 + 20 \times 10$

$= 95 + 24 + 280 + 180 + 80 + 200 = 859$

Clearly, here we have 3+4-1= 6 occupied cells. These are independent as it is not possible to form any closed loop through these allocations (see Fig.).

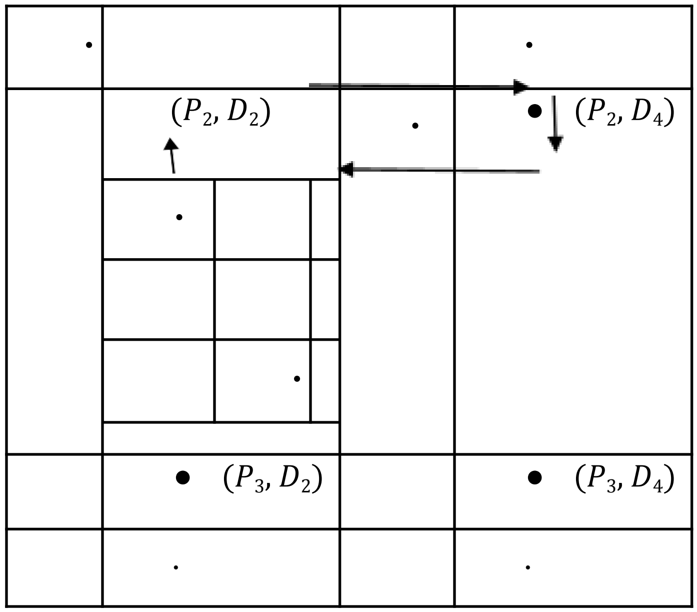
Hence, all the conditions of applying optimality test is satisfied so we apply Stepping Stone method to obtain optimal solution.

Optimality Test using the Stepping Stone Method

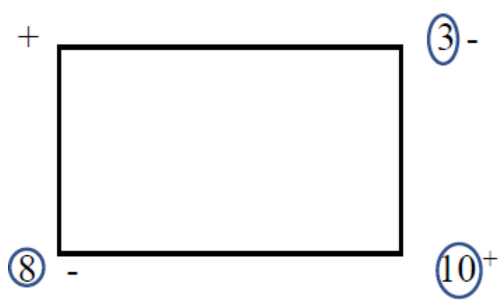
We evaluate the effect of allocating one unit to each of the unoccupied cells making closed paths. Note that the unoccupied cells are (P_1,D_2), (P_1,D_3),(P_2,D_1), (P_2,D_2), (P_3,D_1) and (P_3,D_3). We have to make closed paths so that each path contains at least three occupied cells. We also have to evaluate the net change in cost for each and every unoccupied cell. Then we have to select the one unoccupied cell, which has most negative opportunity cost and allocate as many units as possible to reduce the total transportation cost. The computations are shown as follows:

Unoccupied Cell	Closed Path	Net Change in Cost (Rs)
(P_1, D_2)	$(P_1, D_2) \to (P_1, D_4) \to (P_3, D_4) \to (P_3, D_2)$	30-12+20-10 = 28
(P_1, D_3)	$(P_1, D_3) \to (P_1, D_4) \to (P_2, D_4) \to (P_2, D_3)$	50-12+60-40 = 58
(P_2, D_1)	$(P_2, D_1) \to (P_1, D_1) \to (P_1, D_4) \to (P_2, D_4)$	70-19+12-60 = 3
(P_2, D_2)	$(P_2, D_2) \to (P_3, D_2) \to (P_3, D_4) \to (P_2, D_4)$	30-10+20-60 = -20
(P_3, D_1)	$(P_3, D_1) \to (P_1, D_1) \to (P_1, D_4) \to (P_3, D_4)$	40-19+12-20 = 13
(P_3, D_3)	$(P_3, D_3) \to (P_2, D_3) \to (P_2, D_4) \to (P_3, D_4)$	60-40+60-20 = 60

The cell (P_2, D_2) has the most negative opportunity cost (net change in cost). Therefore, transportation cost can be reduced by making allocation to this unoccupied cell. This means that if one unit is shifted to this unoccupied cell through the loop shown in the fourth row of the table, then Rs20 can be saved (see Fig.4.3a). hence, we shall shift as many units as possible to the cell (P_2, D_2)through this loop. The maximum number of units that can be allocated to (P_2, D_2) through this loop is 3.



(a)



(b)

This is because the shifting can be done only from the corners of the loop and more than 3 units cannot be shifted to (P_2, D_2)as explained below:

The allocations at the corners of the loops are 3,10 and 8. If we try to shift more than 3 units, say, 4 units from the corner (P_2, D_4), then 4 units will have to be subtracted from the corner (P_3, D_2) so that the total of the column D4 remains unchanged. But this will give 3-4 = -1 allocations to the cell (P_2, D_4), which is impossible as negative allocations cannot be made.

We obtain the maximum number of units that can be allocated to the cell (P_2, D_2) through the loop as follows:

1. First, we assign (+) sign to the unoccupied cell (P_2, D_2) to be evaluated and then (-) and (+) signs alternatively to other corners of the closed loop (moving in one direction) as shown in Fig.4.3b.
2. Then we take the minimum of the values at the corners that have been assigned the negative sign. In this case, the maximum number of units that can be allocated to the cell (P_2, D_2) through

the mentioned loop is the minimum of 3 and 8. It is 3. So, we write it as:

$$\min.\begin{cases} \text{the no. of units in}(P_2, D_4) = 3 \\ \text{the no. of units in } (P_2, D_2) = 8 \end{cases} = 3$$

The new table with these changes becomes:

	D_1	D_2	D_3	D_4
P_1	19 5	30	50	2 12
P_2	70	30 3	40 7	60
P_3	40	10 5	60	13 20

In the above table, note that we have also allocated 3 units from (P_3, D_2) to (P_3, D_4) so that column D_4 remains unchanged. This leaves 5 units in the cell (P_3, D_2) and there are13 units in (P_3, D_4). The total transportation cost associated with this solution is

Total cost =19 × 5 + 12 × 2 + 30 × 3 + 40 × 7 + 10 × 5 + 20 × 13

= 95 + 24 + 90 + 280 + 50 + 260 = 799

Now, we repeat the optimality test to see if further allocation can be made to reduce the total transportation cost. The computation for the unoccupied cells is as follows:

Unoccupied Cell	Closed Path	Net Change in Cost (Rs)
(P_1, D_2)	$(P_1, D_2) \to (P_1, D_4) \to (P_3, D_4) \to (P_3, D_2)$	30-12+20-10 = 28
(P_1, D_3)	$(P_1, D_3) \to (P_1, D_4) \to (P_3, D_4) \to (P_3, D_2)$ $\to (P_2, D_2) \to (P_2, D_3)$	50-12+20-10+30-40 = 38
(P_2, D_1)	$(P_2, D_1) \to (P_1, D_1) \to (P_1, D_4) \to (P_3, D_4)$ $\to (P_3, D_2) \to (P_2, D_2)$	70-19+12-20+10-30 = 23
(P_2, D_2)	$(P_2, D_4) \to (P_3, D_4) \to (P_3, D_2) \to (P_2, D_2)$	60-20+10-30 = 20
(P_3, D_1)	$(P_3, D_1) \to (P_1, D_1) \to (P_1, D_4) \to (P_3, D_4)$	40-19+12-20 = 13
(P_3, D_3)	$(P_3, D_3) \to (P_2, D_3) \to (P_2, D_2) \to (P_3, D_2)$	60-40+30-10 = 40

Since, all opportunity costs in the unoccupied cells are non-negative, the current solution is an optimal solution with total transportations cost 799. Hence the maximum saving by optimum distribution is (1000-799) = 201.

Note: The Stepping Stone method should be applied only to problems of small dimensions as it becomes quite tedious for large dimensions. To solve large dimensions problems, we use another method known as (MODI). This method may be conveniently applied to problems of small dimension as well.

Modified Distribution Method (MODI)

The difference between the two methods is that in the Stepping Stone method, closed loops are drawn for all unoccupied cells for determining their opportunity costs. However, in the MODI method, the opportunity costs of all the unoccupied cells are calculated and the cell with opportunity costs of all the unoccupied cells are calculated and the cell with the highest negative opportunity cost is identified without drawing any closed loop. Then only one loop is drawn for the highest negative opportunity cost. The procedure for finding out the optimal solution of a transportation problem with the help of the MODI method is summarized in the following steps:

(i) First find an initial feasible solution using a suitable method.

(ii) Check optimality conditions for the current basic feasible solution if it has exactly (m+n-1) independent allocations, write the cost matrix for only the allocated cells.

(iii) Now, by using this cost matrix, determine the set of m+n numbers u_i(i =1,2,…,m) and v_j (j=1,2,..,n) such that for each occupied cell (i, j), $C_{ij} = u_i + v_j$, taking one of u_i or v_j as zero.

(iv) Fill the vacant cells using $u_i + v_j$.

(v) Compute all net evaluations (unit cost differences) $\Delta_{ij}= C_{ij} - (u_i + v_j)$ by subtracting the values so obtained in Step 4 from the corresponding values of the original cost matrix.

(vi) Examine the sign of each Δ_{ij}

(a) If all $\Delta_{ij}> 0$, for all i, j; then the current basic feasible solution is an optimal solution.

(b) If all $\Delta_{ij}\geq 0$, for all i, j and at least one Δ_{ij} is zero then the current basic feasible solution is an optimal solution and alternate optimal solution exists.

(c) If some of $\Delta_{ij}< 0$ for some (i, j), the current solution is not optimal. Then select the cell having the most negative Δ_{ij} and tick it.

(vii) Construct a closed path/loop for the unoccupied cell ticked in Step 5(ii) using the already allocated cells. At each corner of the closed path, assign plus (+) and minus (-) signs alternatively, beginning with the plus sig for the unoccupied cell. Then θ units are to be allocated to this cell and the same numbers of units are to be added and subtracted alternatively at the corners assigned plus and minus signs, respectively.

The value of θ is the maximum number of units that can be allocated to this cell through the loop. It is obtained by equating the allocations at the negative sign corners to zero. In this way, allocations have been improved.

(viii) Write the cost matrix for only those cells which have improved allocations. Go to Step 3 and repeat the procedure until all $\Delta_{ij}\geq 0$. Calculate the associated total transportation cost.

4.4.4.1. Example. Apply the MODI method for the problem taken in example 4.4.2.1.

Solution. We find initial basic feasible solution of the problem by VAM as given in table and check optimality conditions

D. Centres / Plant	D_1	D_2	D_3	D_4	Av.
P_1	**19** (5)	30	50	**12** (2)	7
P_2	*70*	30	**40** (7)	**60** (3)	10
P_3	40	**10** (8)	60	**20** (10)	18
Requirement	5	8	7	15	35

Now we perform optimality test by applying the MODI method. First of all, we write the cost matrix for only allocated cells:

19			12	u_1
		40	60	u_2
	10		20	u_3
v_1	v_2	v_3	v_4	

Let us denote the row numbers by u_1, u_2, $u_3$and column numbers by v_1,v_2,$v_3$and $v_4$such that

$$u_1 + v_1 = 19, \quad u_1 + v_4 = 12, \quad u_2 + v_3 = 40,$$

$$u_2 + v_4 = 60, \quad u_3 + v_2 = 10, \quad u_3 + v_4 = 20.$$

Taking u_1=0, we have $v_1 = 19$ and $v_4 = 12$ from the first two equations.

Putting the value of v_4 in the fourth equation, we have $u_2 = 48$.

Similarly, we find $u_3 = 8$, $v_2 = 2$ *and* $v_3 = -8$

Using these values, we fill the vacant cells of the above table using $C_{ij} = u_i + v_j$ and put dots in the already filled cells so that these cells are not considered again.

	$v_1 = 19$	$v_2 = 2$	$v_3 = -8$	$v_4 = 12$
$u_1 = 0$	.	2	-8	.
$u_2 = 48$	67	50	.	.
$u_3 = 8$	27	.	0	.

Now, subtracting these values from the corresponding values of the original cost matrix, we have the net evaluations, i.e., $\Delta_{ij} = C_{ij} - (u_i + v_j)$

.	30-2=28	50-(-8) =58	.
70-67=3	30-50= -20	.	.
40-27=13	.	60-0=60	.

Since Note that the cell (P_2, D_2)has the most negative opportunity cost (net change in cost). Therefore, the transportation cost can be reduced by making allocation to this unoccupied cell. This means that if one unit is shifted to this unoccupied cell through the closed loop formed, beginning from this cell and using allocated cells, Rs20 can be saved. We form the loop beginning from this cell, i.e., the cell (P_2, D_2) (see Fig. 4.4). We shift θ units to this unoccupied cell through the loop and add and subtract θ from the cells at the other corners of the loop which are assigned '+' and '-' signs. So, we get

$8-\theta$ $10+\theta$

The maximum number of the units that can be allocated to the cell (P_2, D_2) through this loop is given by the minimum of the solution of the equation 3- $\theta = 0$ and 8- $\theta = 0$, i.e.,

$$\theta = \min.\begin{cases} the\ no.\ of\ units\ in (P_2, D_4) = 3 \\ the\ no.\ of\ units\ in\ (P_2, D_2) = 8 \end{cases} = 3$$

So, with the improved allocations, the table now becomes:

19 (5)	30	50	12 (2)
70	30 (3)	40 (7)	60
40	10 (5)	60	20 (13)

Thus, the total cost of transportation for this set

=19×5+12×2+30×3+40×7+10×5+20×13

= 95+24+90+280+50+260 =799

Now, let us apply the optimality test to the improved solution. Proceeding in the same way as in the first iteration, first of all, we write the cost matrix for only allocated cells:

19			12	u_1
	30	40		u_2
	10		20	$\boldsymbol{u_3}$
v_1	v_2	v_3	v_4	

Let us denote the row numbers by $u_1, u_2, u_3$and column numbers by $v_{1,}v_{2,}v_3$and $v_4$such that

$$u_1 + v_1 = 19, \quad u_1 + v_4 = 12, \quad u_2 + v_3 = 40,$$

$$u_2 + v_2 = 30, \quad u_3 + v_2 = 10, \quad u_3 + v_4 = 20.$$

Taking u_1=0, we have $v_1 = 19$ and $v_4 = 12$ from the first two equations.

Putting the value of v_4 in the fourth equation, we have $u_3 = 8$.

Similarly, $u_2 = 8, v_2 = 2\ and\ v_3 = 12$

Using these values, we fill all the vacant (unoccupied) cells of the table using $C_{ij} = u_i + v_j$for each unoccupied cell and put dots in the already filled cells so that these cells are not considered again.

	$v_1 = 19$	$v_2 = 2$	$v_3 = 12$	$v_4 = 12$
$u_1 = 0$	.	2	-8	.
$u_2 = 28$	47	.	.	40
$u_3 = 8$	27	.	20	.

Now, subtracting these values from the corresponding values of the original cost matrix, we have the net evaluations, i.e., $\Delta_{ij} = C_{ij} - (u_i + v_j)$

•	28	38	•
23	•	•	20
13	•	40	•

Since none of the net evaluation is negative, this solution is optimal. Thus, the total minimum transportation cost is Rs799 and the maximum saving = (1000-799) = 201.

UNBALANCED TRANSPORTATION PROBLEM

The transportation problem wherein the total capacity of all sources and total requirement (demand) of all destinations are not equal is called the unbalanced transportation problem. An unbalanced transportation problem may occur in two different forms (i) Excess of availabilities (ii) Shortage in availabilities. In case (i), we add a dummy destination in the transportation table with zero transportation cost. The requirement of this dummy destination is assumed to be equal to the difference of total availability of sources and total demand so that the problem becomes balanced. Similarly, in case (ii) we introduce a dummy source and take the corresponding steps. The following example explain the whole procedure.

Example. A company has factories at A, B and C which supply warehouses at D, E, F, and G. The monthly factory capacities are 160, 150 and 190 units, respectively. Monthly warehouse requirements are 80, 90, 110 and 160, respectively. Unit shopping costs (in rupees) are as follows:

From	To				
		D	E	F	G
	A	42	48	38	37
	B	40	49	52	51
	C	39	38	40	43

Determine the optimum distribution for this company to minimize shipping costs.

Solution. Here the total capacity of sources (factories) is 160 + 150 + 190 = 500. It is greater than the total requirement of all the destinations (warehouses) which is 80 + 90 + 110 + 160 = 440.

Therefore, we added dummy destination in the transportation table with zero transportation cost and take 60 as its requirement. Thus, the problem becomes balanced that is the total capacity and total requirement are equal the balanced., the problem is as follows:

From	To						Capacity
		D	E	F	G	H	
	A	42	48	38	37	0	160
	B	40	49	52	51	0	150
	C	39	38	40	43	0	190
Requirement		80	90	110	160	60	500

We can solve this problem using VAM to determine the basic feasible solution and then use MODI method to find the optimal solution. We have solved this problem in Example 3. You may like to try the remaining solution yourself.

The optimum allocation is:

160 to (A, F), 80 to (B, D), 10 to (B, E), 60 to (B, G), 80 to (C, E) and 110 to (C, F).

Note: If the number of allocations in the basic feasible solution is less than m + n – 1, we first go through the special case of degeneracy and then apply the test of optimality.

DEGENRACY

Degeneracy occurs in the transportation problem if the number of occupied cells is less than m + n – 1. To resolve degeneracy, we introduce an infinitesimally allocation e (say)in the least cost and independent cell so that number of occupied cells becomes m+n-1. Also notice that if 'e' is added to or subtracted from any quantity, the quantity remains unaltered.

Remark: We do not write 0 in place of e as it is one of the allocations and will not come into counting if it is written as 0 so we write e and not 0 in its place and count it as one of the allocations.

Degeneracy occurs in Example 4.5.1. To solve this problem, we proceed as follows:

Using Vogel Approximation method, we get the basic feasible solution as:

Warehouse / Factory	D	E	F	G	H	Av.	D_1	D_2	D_3	D_4
A	42	48	38	**37** (160)	0 (e)	160	37	1	1	1
B	**40** (80)	49	**52** (10)	**51** (60)	0 (60)	150	40*	9	11*	1
C	39	**38** (90)	**40** (100)	43	0	190	38	1	3	3
Requirement	80	90	110	160	60	500				
$Diff_1$	1	10	2	6	0					
$Diff_2$	1	10*	2	6	-					
$Diff_3$	1	-	2	6	-					
$Diff_4$	-	-	2	6*	-					

Here the number of allocations is 6 but the number of allocations should be 3 + 5 – 1 = 7 to perform optimality test. Hence it is the case of degeneracy to resolve such a problem, we introduce infinitesimally small allocation e in the least cost and independent cell, i.e., the cell (A,H). Another cell (C, H) also has same least cost as the cell (A, H). But this cell is not independent because a loop (C,H)→ (C,F) → (B,F) → (B,H) can be formed beginning with the cell as shown in figure 4.5. Now the number of allocations is 7, i.e., as many as required for performing the optimality test.

42	48	38	•	0
•	49	• ↑ (B, F)	51	• ↓ (B, H)
39	•	• (C, F)	43	0 (C, H)

Thus, applying optimality test by MODI method, the improved allocations after first iteration will be:

42	48	38 e	37 160	0
40 80	49	52 10	51	0 60
39	38 90	40 100	43	0

Again, applying the optimality test, the improved solution is

	D	E	F	G	H
A	42	48	38 e	37 160	0
B	40 80	49 10	52	51	0 60
C	39	38 80	40 110	43	0

Again, applying the optimality test, you will see that these allocations give the optimal solution and the total transportation cost = rupees 17050.

CHECK YOUR PROGRESS

1. Explain transportation problem. Is this a linear programming problem?
2. Explain method for obtaining initial basic feasible solution of Transportation problem:
   (i) The North-West Corner Rule,
   (ii) Vogel's Approximation Method (VAM),
   (iii) The least Cost Method.
3. Describe the computational procedure of the following optimality tests for solving transportation problem:
   (i) Stepping stone method,
   (ii) MODI method.
4. Write a short note on unbalanced transportation problem.
5. What is degeneracy in a transportation problem?

SUMMARY

In this chapter, we have introduced transportation problems. As one can express these problems in terms of LPPs so transportation problems is also considered as one of the sub – classes of Linear Programming Problems and the objective of the problem is to determine the optimal amount to ship/ transport from each origin to destination. Here, we have explained how to formulate mathematically a transportation problem, finding initial basic feasible solution, performing optimality test and moving towards optimal solution.

In this chapter, we have discussed three methods for finding initial basic feasible solution namely North – West Corner Rule, Least Cost Method and Vogel's Approximation Method**.** Vogel's approximations method is preferred over other methods of finding the initial basic feasible solutions as it generally gives the solution closest to the optimum solution. After finding the initial basic feasible solution, optimality test is performed by applying the Stepping Stone Method or Modified Distribution Method (MODI)to find whether the obtained feasible solution is optimal or not. In the last we have discussed the case of unbalanced transportation problems and degeneracy.

5

ASSIGNMENT PROBLEMS

Structure

Introduction

Assignment Problems

Hungarian Method

Unbalanced Assignment Problem

Case of Maximization of an Assignment Problem

Travelling Salesman Problem

Check Your Progress

Summary

INTRODUCTION

Assignment problem is a special type/case of transportation problem and hence is of linear programming problem which deals with the allocation of the various resources to the various activities on one to one basis. It does it in such a manner that the cost or time involved in the process is minimum and profit or sale is maximum. In chapter 4, we have already discussed methods of finding optimal solution for the given problem but here we discuss another method namely Hungarian method for solving an assignment problem. Hungarian method is shorter and easier than stepping stone and MODI methods which we have discussed in previous chapter. In this chapter, we shall explain the assignment problems including travelling salesman problem and apply Hungarian method for solving these problems.

Objectives

After studying this chapter, reader should be able to explain the following concepts like:

- Assignment Problems
- Hungarian Method.
- Unbalanced Assignment Problem
- Case of Maximization of an Assignment Problem
- Travelling Salesman Problem

ASSIGNMENT PROBLEMS

An assignment problem may be considered as a special type of transportation problem in which there are as many jobs/sources as the number of machines/destinations so that the jobs can be assigned to machines in a one-to-one way only. The capacity of each source as well as the requirement of each destination is taken as 1. The main difference between an assignment problem and transportation problem is that in the case of assignment problem, the given matrix must necessarily be a square matrix which is not the condition for a transportation problem.

Let there be n persons and n jobs and let C_{ij} represents the amount of time taken by i^{th} person to complete the j^{th} job then our objective is assignment of jobs on one-to-one basis in such a way that the total cost is minimum. The assignment problem can be stated in the form of an n×n matrix of real numbers called the cost matrix as given in the following table:

Person	Job 1	2 …	j …	n
1	C_{11}	C_{12} …	C_{1j} …	C_{1n}
2	C_{21}	C_{22} …	C_{2j} …	C_{2n}
.	.			
.	.			
.	.			
i	C_{i1}	C_{i2} …	C_{ij} …	C_{in}
.	.			
.	.			
.	.			
n	C_{n1}	C_{n2} …	C_{nj} …	C_{nn}

Let x_{ij} denote the j^{th} job assigned to the i^{th} person. Then, mathematically, the assignment problem can be stated as follows:

Minimize $Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}x_{ij}$, where i= 1, 2, …, n and j=1, 2, …, n.

subject to

$$x_{ij} = \begin{cases} 1 & \text{if the } i^{th} \text{ person is assigned the } j^{th} \text{ job} \\ 0 & \text{if the } i^{th} \text{ person is not assigned the } j^{th} \text{ job} \end{cases}$$

Since one job assigned to one person only, we have

$x_{i1} + x_{i2} + \ldots + x_{in} = 1$, i= 1, 2, …, n

$x_{1j} + x_{2j} + \ldots + x_{nj} = 1$, j= 1, 2, …, n

Note: The constant c_{ij} in the above problem represents time, in many situations it may be cost or some other parameter which is to be minimized in the assignment problem under consideration.

Remark: An assignment problem can be looked upon as a special type of transportation problem in which the jobs stands for sources, the machines for destinations and all the availabilities and requirements are equal to one.

HUNGARIAN METHOD

Hungarian method is also known as Reduced Matrix Method, it is an efficient method for solving assignment problems. Hungarian method is developed by Hungarian mathematician D. Konig. The step by step procedure for obtaining an optimal solution of an assignment problem are as follows:

1. Develop the cost table from the given problem then check whether the given matrix is square i.e. is number of sources/machines is equal to the number of destinations/jobs. If not, make it square by adding a suitable number of dummy row (or column) with 0 cost/time element.
2. Locate the smallest cost element in each row of the given cost matrix and then subtract the smallest element of each column from every element of that column.
3. In the resulting cost matrix, locate the smallest element in each column and subtract the smallest element of each column from every element of that column.
4. In the modified matrix, search for an optimal assignment as follows:

a) Examine the row successively until a row with exactly one zero is found. Draw a rectangle around this zero like 0 and cross out all other zeroes in the corresponding column. Proceed in this way until all the row have been considered. If there is more than one zero in any row, don't touch that row, pass on to the next row.

b) Repeat step (a) above for the columns of the resulting cost matrix.

c) If a row or column of the reduced matrix contains more than one zeroes, arbitrarily choose a row or column having the minimum number of zeroes. Arbitrarily select any zero in the row or column so chosen. Draw a rectangle around it and cross out all the zeroes in the corresponding row and column. Repeat steps (a), (b), and (c) until all the zeroes have either been assigned (by drawing a rectangle around them) or crossed.

d) If each row or column of the resulting matrix has one and only one assigned zero, i.e. number of assigned zeroes are equal to the number of rows/columns, then the optimum assignment is made in the cells corresponding to 0 and the optimum solution of the problem is attained and we can stop here.

Otherwise, go to the next step.

5. Draw the minimum number of horizontal and/or vertical lines through all the zeroes as follows:
   i. Mark (√) the rows in which assignment has not been made.
   ii. Mark (√) column, that have zeroes in the marked rows.

iii. Mark (√) rows(not already marked) which have assignments in marked columns. Then mark (√) columns, which have zeroes in newly marked rows, if any. Mark (√) rows (not already marked), which have assignments in these newly marked columns.

6. Revise the cost matrix as follows:
   i. Find the smallest elements not covered by any of the lines.
   ii. Subtract this from all the uncovered elements and add it to the elements at the intersection of the two lines.
   iii. Other elements covered by the lines remain unchanged.
7. Repeat the procedure until an optimal solution attained.

Note: By drawing lines through all the unmarked rows and marked columns, we will get the required minimum number of lines.

The following example illustrate the method.

Example. A computer centre has four expert programmers & needs to develop four application programmes. The head of the computer Centre, estimates the computer time (in minutes) required by the respective experts to develop the application programmes as follows:

PROGRAMMES / PROGRAMMERS	A	B	C	D
1	120	100	80	90
2	80	90	110	70
3	110	140	120	100
4	90	90	80	90

Find the assignment pattern that minimize the time required to develop the application programmes.

Solution. Let us subtract the minimum element of each row from every element of that row. Note that the minimum element in the first row is 80. So, 80 is subtracted from every element of the first row. Similarly, we obtain the elements of the other rows of the resulting matrix. Thus, the modified matrix is:

	A	B	C	D
1	40	20	0	10
2	10	20	40	0
3	10	40	20	0
4	10	10	0	10

Let us now subtract the minimum element of each column from every element of that column in the resulting matrix. The minimum element in the first column is 10. So, 10 is to be subtracted from every element of the first column. Similarly, we obtain the elements of the other columns of the resulting matrix. Thus, the resulting matrix is:

	A	B	C	D
1	30	10	0	10
2	0	10	40	0
3	0	30	20	0
4	0	0	0	10

Now starting from the first row onward, we draw a rectangle around the zero in each row having a single zero and cross all other zeros in the corresponding column. Here, in the very first row we find a single zero. So, we draw a rectangle around it and cross all the other zeroes in the corresponding column. We get

	A	B	C	D
1	30	10	[0]	10
2	0	10	40	0
3	0	30	20	0
4	0	0	0×	10

In the second, third and fourth row, there is no single zero. Hence, we move column – wise. In the second column, we have a single zero. Hence, we draw a rectangle around it and cross all other zeroes in the corresponding row. We get

	A	B	C	D
1	30	10	[0]	10
2	0	10	40	0
3	0	30	20	0
4	0×	[0]	0×	10

In the matrix above, there is no row or column, which has a single zero. Therefore, we first move row – wise to locate the row having more than one zero. The second row has two zeroes. So, we draw a rectangle arbitrarily around one of these zeroes and cross the other one. Let us draw a rectangle around the zero in the cell (2, A) and cross the zero in cell (2, D). We cross out the other zeroes in the first column. Note that we could just as well have selected zero in the cell (2, D), drawn a rectangle around it and crossed all other zeroes. This would have led to an alternative solution.

In this way, we are left with only one zero in every row and column around which a rectangle has been drawn. This means that we have assigned only one operation to one operator. Thus, we get the optimum solution as follows:

	A	B	C	D
1	30	10	[0]	10
2	[0]	10	40	0×
3	0×	30	20	[0]
4	0×	[0]	0×	10

Note that the assignment of jobs should be made on the basis of the cells corresponding to the zeroes around which rectangles have been drawn. Therefore, the optimum solution for this problem is:

$1 \rightarrow C, 2 \rightarrow A, 3 \rightarrow D, 4 \rightarrow B$

This means that programmer 1 is assigned programme C, programmer 2 is assigned programme A, and so on. The minimum time taken in developing the programmes is

$= 80 + 80 + 100 + 90 = 350$ min.

Example. A company is producing a single product and selling it through five agencies situated in the different cities. All of a sudden, there is a demand for the product in five more cities that do not have any agency of the company. The company is faced with the problem of deciding on how to assign the existing agencies to dispatch the product to the additional cities in such a way that the travelling distance is minimized. The distances (in km) between the surplus and the deficit cities are given in the following distance matrix:

Deficit city / Surplus city	I	II	III	IV	V
A	160	130	175	190	200
B	135	120	130	160	175
C	140	110	155	170	185
D	50	50	80	80	110
E	55	35	70	80	105

Determine the optimum assignment schedule.

Solution: Subtracting the minimum element of each row from every element of that row and then subtracting the minimum element of each column from every element of that column, we have

	I	**II**	**III**	**IV**	**V**
A	30	0	35	30	15
B	15	0	0	10	0
C	30	0	35	30	20
D	0	0	20	0	5
E	20	0	25	15	15

We now assign zeroes by drawing rectangles around them as explained in above example. Thus, we get

	I	**II**	**III**	**IV**	**V**

A	30	[0]	35	30	15
B	15	0×	[0]	10	0×
C	30	0×	35	30	20
D	[0]	0×	20	0×	5
E	20	0	25	15	15

Since the number of assignments is less than number of rows (or columns), we proceed from step 5 onwards of the Hungarian method described as follows:

i. We mark (√) the rows in which the assignment has not been made . These are the 3rd& 5th row.

ii. We mark (√) the columns which have zeroes in the marked rows . This is the 2nd column.

iii. We mark (√) the rows which have assignments in marked columns . This is the 1st row.

iv. Again we mark (√) the columns which have zeroes in the newly marked row . This is the 2nd column (which has been already marked).

There is no other such column. So, we have

	I	II	III	IV	V	
A	30	[0]	35	30	15	√
B	15	0×	[0]	10	0×	
C	30	0×	35	30	20	√
D	[0]	0×	20	0×	5	
E	20	0×	25	15	15	√
		√				

We draw straight lines through unmarked rows and marked columns as follows:

	I	II	III	IV	V	
A	30	[0]	35	30	15	√
B	15	0×	[0]	10	×0	
C	30	0×	35	30	20	√
D	[0]	×0	20	×0	5	
E	20	0×	25	15	15	√
		√				

We proceed as follows, as explained in the step 6 of the Hungarian method:

1) We find the smallest element in the matrix not covered by any of the lines. It is 15 in this case.

2) We subtract the number '15' from all uncovered elements and add it to the elements at the intersection of the two lines.

3) Other elements covered by the lines remain unchanged. Thus, we have

	I	II	III	IV	V
A	15	0	20	15	0
B	15	15	0	10	0
C	15	0	20	15	5
D	0	15	20	0	5
E	5	0	10	0	0

We repeat steps 1 to 4 of the Hungarian method and obtain the following matrix:

	I	II	III	IV	V
A	15	0✕	20	15	[0]
B	15	15	[0]	10	0✕
C	15	[0]	20	15	5
D	[0]	15	20	0✕	5
E	5	0✕	10	[0]	0✕

Since each row and each column of this matrix has one and only one assigned 0, we obtain the optimum assignment schedule as follows:

A → V, B → III, C → II, D → I, E→ IV

Thus, the minimum distance is 200+130+110+50+80 = 570 km.

Exercise. A solicitor's firm employs typists on an hourly piece – rate basis for their daily work. There are five typists for service and their charges and speeds are different. According to the contract, only one job is given to one typist. Find the least cost allocation for the following data:

	P	Q	R	S	T
A	85	75	65	85	75
B	90	180	66	90	78
C	75	66	57	75	69
D	80	72	60	80	72
E	76	64	56	72	68

Answer. Applying the Hungarian method of solving an assignment problem, we finally get

	P	Q	R	S	T
A	2	4	2	4	0
B	4	106	0	6	0✕
C	0✕	0	2	2	2
D	0	4	0✕	2	0✕
E	2	2	2	0	2

Thus, the least cost allocation is given by:

A->T, B->R, C->Q, D->P, E->S

and the total minimum cost is Rs. (75+66+66+80+72)

=Rs.359.

UNBALANCED ASSIGNMENT PROBLEM

Some assignment problems may be unbalanced, i.e. the number of machines may be different from the number of jobs. In this case, in the obtained matrix the number of rows is not equal to the number of columns and the problem said to be an unbalanced Assignment problem. Such a problem is handled by introducing dummy row(s) if the number of rows is less than the number of columns and dummy column(s) if the number of columns is less than the number of rows. All the elements of such a dummy row/column are taken as zero. After creating dummy rows or columns, we get a balanced assignment problem and now we solved it by Hungarian method.

The following example will make the procedure clear.

Example. To stimulate interest and provide an atmosphere for intellectual discussion, the faculty of mathematical sciences in an institute decides to hold special seminars four contemporary topics - Statistics, Operations Research, Discrete Mathematics, Matrices. Each such seminar is to be held once a week. However, scheduling these seminars (one for each topic and not more than one seminar per a day) has to be done carefully so that the numbers of students unable to attend is kept to a minimum. A careful study indicates that the number of students who cannot attend a particular seminar on a specific day is as follows:

	Statistics	**Operations Research**	**Discrete mathematics**	**Matrices**
Monday	50	40	60	20
Tuesday	40	30	40	30
Wednesday	60	20	30	20
Thursday	30	30	20	30
Friday	10	20	10	30

Find an optimal schedule for the seminars. Also fine the number of students who will be missing at least one seminar.

Solution. Here the number of rows is 5 and the number of columns is 4. Therefore, the given assignment problem is unbalanced. As the number of columns is one less than the number of rows, we introduce one dummy column to convert the given assignment problem into a balanced problem. The number of students in each cell of this column is taken as zero. Thus, the problem takes the following form:

	Statistics	Operations Research	Discrete mathematics	Matrices	Dummy
Monday	50	40	60	20	0
Tuesday	40	30	40	30	0
Wednesday	60	20	30	20	0
Thursday	30	30	20	30	0
Friday	10	20	10	30	0

Now, on applying the Hungarian method (Steps 1 to 4), we get

	Statistics	Operations Research	Discrete mathematics	Matrices	Dummy
Monday	40	20	50	[0]	0✕
Tuesday	30	10	30	10	[0]
Wednesday	50	[0]	20	0✕	0✕
Thursday	20	10	10	10	0✕
Friday	[0]	0✕	0✕	10	0✕

Since the number of assigned zeroes < number of rows, we apply Step 5 of the Hungarian method and draw the minimum number of horizontal/ vertical lines that cover all the zeros as shown in the following table:

	Statistics	Operations Research	Discrete mathematics	Matrices	Dummy
Monday	40	20	50	[0]	✕0
Tuesday	30	10	30	10	[0]
Wednesday	50	[0]	20	✕0	✕0
Thursday	20	10	10	10	✕0
Friday	[0]	✕0	✕0	10	✕0

We select the minimum element from amongst the uncovered elements, which in 10 in this case. We subtract this element, i.e., 10 from each uncovered element and add it to the elements which lie at the intersection of the horizontal/vertical lines. Other covered elements will remain unaltered. Thus, the resulting matrix is:

40	20	50	0	10
20	0	20	0	0
50	0	20	0	10
10	0	0	0	0
0	0	0	10	10

Now on applying the Hungarian method, we have

40	20	50	0	10
20	0✕	20	0✕	[0]
50	[0]	20	0✕	10
10	0✕	[0]	0✕	0✕
[0]	0✕	0✕	10	10

Since each row and each column of the matrix has one and only one assigned 0, optimum assignment is made in the cells containing those zeroes around which rectangles have been drawn as Monday → Matrices, Wednesday → Operations Research, Thursday → Discrete Mathematics, Friday → Statistics The total number of students who will be missing at least one seminar = 20 + 20 + 20 + 10 = 70

CASE OF MAXIMIZATION OF AN ASSIGNMENT PROBLEM

All problems dealt with so far were all cost-minimizing problems but assignment problem exists with profit maximization problem also. For example, profits (or anything else like revenues), which need maximization may be given in the cells instead of costs/ times. The method of solving such problems is a simple modification of the method of solving cost-minimizing assignment problems. To solve such a problem, we find the opportunity loss matrix by subtracting the value of each cell from the largest value chosen from amongst all the given cells. When the value of a cell is subtracted from the highest value, it gives the loss of amount caused by not getting the opportunity which would have given the highest value. The matrix so obtained is handled in the same way as the minimization problem. The following example illustrate the method.

Example. Five salesmen are to be assigned to five districts. Estimates of sales revenue (in thousands) for each salesman are given as following:

	A	**B**	**C**	**D**	**E**
1	32	38	40	28	40
2	40	24	28	21	36
3	41	27	33	30	27
4	22	38	41	36	36
5	29	33	40	35	39

Find the assignment pattern that maximizes the sales revenue.

Solution. Since we are to maximize the sales revenue, we need to convert it into minimization form before applying the Hungarian method. For this, we obtain the opportunity loss matrix by subtracting every element in the given table from the largest element is 41. Thus, we obtain the following opportunity loss matrix:

9	3	1	13	1
1	17	13	20	5
0	14	8	11	4
19	3	0	5	5
12	8	1	6	2

Now, we apply the Hungarian method (Steps 1 to 4) and finally obtain the following result matrix:

	A	B	C	D	E
1	8	[0]	0✕	7	0✕
2	[0]	14	12	14	4
3	0✕	12	8	6	4
4	19	1	[0]	0✕	5
5	11	5	0✕	[0]	1

Since the number of assigned zero is less than the number of rows, we apply Step 5 of the Hungarian method and draw the minimum number of horizontal/vertical lines that cover all the zeroes as shown in the following table:

	A	B	C	D	E	
1	8	[0]	0✕	7	0✕	
2	[0]	14	12	14	4	√
3	0✕	12	8	6	4	√
4	19	1	[0]	0✕	5	
5	11	5	0✕	[0]	1	
	√					

Let us now, select the minimum element from amongst the uncovered elements, which is 4 in the case. We subtract the element 4 from each of the uncovered elements and add it to the elements which lie at the intersection of the horizontal and vertical lines. Other covered elements remain unaltered. Then applying the Hungarian method to the resulting matrix. We get

	A	B	C	D	E
1	12	[0]	0✕	7	0✕
2	[0]	10	8	10	0✕
3	0✕	8	4	2	[0]
4	23	1	[0]	0✕	5
5	15	5	0✕	[0]	1

Since the number of assigned zeroes is equal to the number of rows, the optimum assignment has been attained and is given as

1 → B, 2 → A, 3 → E, 4 → C, 5 → D

Thus, the maximum sales revenue = 38+ 40 +37 +41 +35 thousand rupees

= 191 thousand rupees.

TRAVELLING SALESMAN PROBLEM

Consider a travelling salesman who has to visit a certain number of cities allotted to him. He wishes to visit each city once and only once and arrive back in the city from where he started. He knows the distances (or cost or time) of journey between every pair of cities, and he wishes to determine the tour schedule that represents the least distance/cost/time. Such type of problems can be solved by the assignment algorithm.

The difference between a travelling salesman and assignment problem is that in an assignment problem, different destinations are assigned to different sources but in a travelling salesman problem, a destination is assigned to a source. Then this destination becomes another source to which we assign another destination, which in turn becomes another source, and so on. Let us explain this point further with the help of an example.

Example: A travelling salesman has to visit five cities. He wishes to start from a particular city, visit each city once and then return to his starting point. The travelling times for each city from a particular city is given below

To / **From**	A	B	C	D	E
A	∞	5	8	4	5
B	5	∞	7	4	5
C	8	7	∞	8	6
D	4	4	8	∞	8
E	5	5	6	8	∞

What is the sequence of visits of the salesman so that the total travelling time is minimized?

Solution: Appling the Hungarian method to this problem, we get

	A	B	C	D	E
A	∞	0×	2	[0]	0×
B	[0]	∞	1	0×	0×
C	2	1	∞	3	[0]
D	0×	[0]	3	∞	4
E	0	0	[0]	4	∞

As per the above assignment, the salesman should travel from A to D, D to B , B to A i.e., A→D→B→A.

The above solution is not a complete solution of the travelling salesman problem as the salesman returns to A without travelling through all the cities.so we proceed as follows.

Since the assignment of zeroes has not given the solution of the travelling salesman problem, we bring the next minimum non-zero element into the solution. Thus, we obtain the next best solution by bringing

into the solution. had there not been 1 in any cell, we would have taken the minimum, but greater than 1 value from amongst all the values of the tables. here we have 1and it appears in two places in this problem.one of these is chosen arbitrarily. let us choose the cell (B, C) and from a rectangle around the values in this cell and cross out the zeroes in its row and column. now we apply the Hungarian method for the assignment of zeroes. Thus, we have

	A	B	C	D	E
A	∞	0✗	2	[0]	0✗
B	0✗	∞	[1]	0✗	0✗
C	2	1	∞	3	[0]
D	[0]	0✗	3	∞	4
E	0	[0]	0	4	∞

Alternatively, we get

	A	B	C	D	E
A	∞	0✗	2	0✗	[0]
B	0✗	∞	1	[0]	0✗
C	2	[1]	∞	3	0✗
D	[0]	0✗	3	∞	4
E	0	0	[0]	4	∞

In the case of the first alternative, the optimum assignment is A→D→A, but this is not the solution of the travelling salesman problem.

In the case of the second alternative, the optimum assignment is

A→E→C→B→D→A

This is the complete solution for the problem as starting from as the salesman returns to A visiting all the other cities. The minimum time taken by him to travel to all the cities is 5+6+7+4+4=26hrs.

Exercise. Solve the following travelling salesman problem so as to minimize the cost per cycle:

To From	A	B	C	D	E
A	∞	3	6	2	3
B	3	∞	5	2	3
C	6	5	∞	6	4
D	2	2	6	∞	6
E	3	3	4	6	∞

What is the sequence of visits of the salesman so that travelling time is minimized?

Answer. Applying the Hungarian method of solving an assignment problem, we reduce the cost matrix and make assignments in rows and columns having single zeroes.

	A	B	C	D	E	
A	∞	1✕	3	0	1	√
B	1	∞	2	0✕	1	√
C	2	1	∞	2	0	
D	0✕	0	3	∞	4	
E	0✕	0✕	0	3	∞	
				√		

Now, we draw the minimum number of lines to cover all the zeroes. Then we subtract the lowest element from all the elements not covered by any of lines and add the same at the intersection of two lines. We have

	A	B	C	D	E	
A	∞	✕0	2	0	✕0	√
B	0	∞	1	✕0	✕0	√
C	2	1	∞	3	0	
D	✕0	0	3	∞	4	
E	✕0	✕0	0	4	∞	
				√		

As per the above assignment, the salesman should travel from A to D, D to B, B to A,

i.e., A→D → B → A.

The above solution is not a complete solution of the travelling salesman problem as the salesman returns to A without travelling through all the cities. So, we proceed as follows:

Since the assignment of zeroes has not given the solution of the travelling salesman problem, we bring the next minimum non-zero element in the solution.

Thus, we obtain the next best solution by bringing 1 into the solution. Had there not been 1 in any cell, we could have taken the minimum, but greater than 1 value from amongst all the values of the table. Here, we have 1 and it appears in two places in this problem. one of these is chosen arbitrarily. Let us choose the cell (B, C) and form a rectangle around the value in this cell and cross out the zeroes in its row and column. Now we apply the Hungarian method for the assignment of zeroes. Thus, we have

	A	B	C	D	E
A	∞	0✕	2	0	0✕
B	0✕	∞	1	0✕	0✕
C	2	1	∞	3	0
D	0	✕0	3	∞	4
E	✕0	0	✕0	4	∞

Alternatively, we get

	A	B	C	D	E
A	∞	0×	2	0×	0
B	0×	∞	1	0	0×
C	2	1	∞	3	0×
D	0	0×	3	∞	4
E	0×	0×	0	4	∞

In this case of the first alternative, the optimum assignment is A→D→A, but this is not the solution of the travelling salesman problem.

In this case of the second alternative, the optimum assignment is

A→E→C→B→D→A, i.e., 3 + 4 + 5 + 2 + 2 = 16 hr

CHECK YOUR PROGRESS

1. Define an assignment problem.
2. Explain the difference between a transportation problem and assignment problems.
3. Explain step by step procedure of Hungarian method to solve the assignment problem.
4. What is an unbalanced assignment problem?
5. Explain briefly travelling salesman problem.

SUMMARY

In this chapter, we have observed that an assignment problem may be considered as a special type of transportation problem in which the capacity of each of the sources as well as the requirements of each of the destination is taken as 1.In an assignment problem, the given matrix must necessarily be a square matrix , which is not the condition for a transportation problem . Such problem is solved as Hungarian method, which is shorter and easy compared to any other method of finding the optimal solution. We have discussed the concept of unbalanced transportation problem; such problem is handled by introducing dummy rows/columns. In last, we have introduced travelling salesman problem.

9

GAME THEORY

Structure

Introduction

Some Basic Definitions

Two-Person Zero-Sum Game

Pure Strategies: Game with Saddle Points

The Rule of Dominance

Mixed Strategies: Game without Saddle Points

- Algebraic Method
- Graphical Method
- Linear Programming Method

Check Your Progress

Summary

INTRODUCTION

In real-life, we can see a great variety of competitive situations. Game theory provides tools for analysing situations in which parties, called players, make decisions that are interdependent. This interdependence causes each player to consider the other player's possible decisions, or strategies, in formulating strategy. A solution to a game describes the optimal decisions of the players, who may have similar, opposed, or mixed interests, and the outcomes that may result from these decisions. So, one can say that it is a type of decision theory. Game theory was originally developed by John von Neumann (called the father of game theory) and his colleague Oskar Morgenstern to solve problems in economics.

In this chapter, first we define some basic terms used in game theory then we shall discuss two-person zero-sum-games (also known as rectangular games), games with saddle point in which we study minimax and maximin criterion. Also, we shall explain rules of dominance which are used to reduce the size of the payoff matrix and discuss solution methods for game without saddle point namely algebraic method, graphical method and linear programming method.

Objectives. The objective of these contents is to get familiar reader with game theory. After studying this chapter, reader should be able to describe the following concepts like:

- Minimax and Maximin Principle
- Pure Strategies: Game with Saddle Points
- The Rule of Dominance
- Mixed Strategies: Game without Saddle Points

SOME BASIC DEFINITIONS

Game: A competitive situation is called a game if it has the following properties

a) There are finite numbers of participants called players.
b) Each player has finite number of strategies available to him.
c) Every game result in an outcome.

Player: Each participant of a game is called a player.

Number of players: If a game involves any two payers, it is called a two-person game. However, if the number of players is more than two, the game is known as *n*-person game.

Payoff: A quantitative measure of satisfaction, a person gets at the end of each play, is called a payoff.

Play: A play is said to occur when each player chooses one of his activities.

Strategy: The strategy for a payer is the list of al possible actions or moves available to him. Generally, two types of strategies are employed by players in a game.

(i) **Pure strategy**: It is a decision rule which is always used by the player to select any one particular course of action. The objective of the payer is to maximize gains or minimize losses.

(ii) **Mixed strategy**: When the players use a combination of strategies and each player always keep guessing as to which course of action is to be selected by the other on a particular occasion, then this is known as mixed strategy. Thus, the mixed strategy is a selection among pure strategies with fixed probabilities.

Zero-sum game. A game in which the algebraic sum of the outcomes for all the participants equals zero for very possible combination of strategies, is called a zero-sum game.

A game which is not zero-sum is called a non-zero-sum game.

Optimal strategy. A course of action or play which puts the player in the most preferred position, irrespective of the strategy of his competitors, is called optimal strategy.

Value of the game. The expected payoff when the players follow their optimal strategy is called the value of the game.

TWO-PERSON ZERO-SUM GAME

A game with only two-persons is said to be two-person zero-sum game if the gain of one player is equal to the loss of the other so that total sum is zero.

Payoff Matrix: In a two-person game, the payoffs in terms of gains or losses, when players select their particular strategies can be represented in the form of a matrix, called the payoff matrix of the player. If the game is zero-sum, the gain of one player is equal to the loss of the other and vice-versa. So, one player's payoff table would contain the same amounts I payoff table of the other payer with the sign changed. If the player A has strategies $A_1, A_2, \ldots, A_n$ and the player B has strategies $B_1, B_2, \ldots, B_n$ and if a_{ij} represent the payoffs that the player A gains from player B when player A chooses strategy i, and player B chooses strategy j then payoff matrix for player A is given by

Player B's strategies

$$\text{Player } A\text{'s strategies} \quad \begin{array}{c} \\ A_1 \\ A_2 \\ \\ A_m \end{array} \begin{array}{c} \begin{array}{cccc} B_1 & B_1 & \ldots\ldots & B_n \end{array} \\ \begin{bmatrix} a_{11} & a_1 & \ldots\ldots & a_{1n} \\ a_{21} & a_{21} & \ldots\ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots\ldots & a_{mn} \end{bmatrix} \end{array}$$

Basic Assumptions of the Game:

Rules of the game are given as follows:

- Each player has available to him a finite number of possible courses of action. The list may not be the same for each player.
- Players act rationally and intelligently.
- The decisions of both the payers are made individually, prior to the play, with no communication between them.
- One player attempts to maximize gains and the other attempts to minimize losses.
- The players simultaneously select their respective courses of action.
- The payoff is fixed and determined in advance.
- List of strategies of each player and the amount of gain or loss on an individual's choice of strategy is known to each player in advance.

PURE STRATEGIES: GAMES WITH SADDLE POINT

Consider the payoff matrix of a game which represents payoff of player *A*. Now, the objective of the study is to know how these players must select their respective strategies so that they may optimize their payoff. Such a decision-making criterion is referred to as the **minimax-maximin principle**.

For payer *A*, the minimum value in each row represents the least gain to him if he chooses his particular strategy. He will then select the strategy that gives the largest gain among the row minimum values. This choice of player *A* is called the **maximin principle** and the corresponding gain is called the maximin value of the game denoted by $\underline{v}$.

For player *B*, who is assumed to be loser, the maximum value in each column represents the maximum loss to value in each column represents the maximum loss to him if he chooses his particular strategy. He will then select the strategy that gives minimum loss among the column maximum values. This choice of player *B* is called the **minimax principle** and the corresponding loss is called the minimax value of the game, denoted by v .

Saddle point. A saddle point of a payoff matrix is that position in the payoff matrix where maximum of row minima coincides with the minimum of the column maxima. The saddle point need not be unique.

Value of the game. The amount of payoff at the saddle point is called the value of the game, denoted by $\underline{v}$.

Fair game. A game is said to be fair if $\underline{v} = 0 = v$.

Strictly determinable game. A game is said to be strictly determinable if $\underline{v} = v = v$.

Procedure to Determine Saddle Point

a) Select the minimum element in each row and enclose it in a rectangle box.
b) Select the maximum element in each column and enclose it in a circle.
c) Find the element which is enclosed by the rectangle as well as the circle such element is the value of the game and that position is a saddle point.

Example. For the game with payoff matrix:

		Player *B*		
		B_1	B_2	B_3
Player *A*	A_1	–1	2	–2
	A_2	6	4	–6

Determine the optimal strategies for players *A* and *B*. Also determine the value of game.

Is this game (i) fair? (ii) strictly determinable?

Solution. Select the row minimum and enclose it in a rectangle. Then select the column maximum and enclose it in a circle.

	B_1	B_2	B_3
A_1	–1	2	–2
A_2	6	4	–6

Saddle point is (A_1, B_3).

Value of game = –2

Optimal strategy for A is A_1 and for B is B_3.

The game is strictly determinable. Since value of game is not zero, the game is not fair.

Example. Determine which of the following two-person zero-sum games are strictly determinable and fair. Give optimum strategies for each player in case of strictly determinable games:

(a) Player B

$$\text{Player } A \begin{bmatrix} 0 & 2 \\ -1 & 4 \end{bmatrix}$$

(b) Player B

$$\text{Player } A \begin{bmatrix} -3 & 1 \\ 3 & -1 \end{bmatrix}$$

Solution. (a) Payoff matrix for player A is:

Player A \ Player B	B_1	B_2	Row minima
A_1	0	2	0
A_2	–1	4	–1
Column maxima	0	4	

$\underline{v}$ (maximin value) = 0

v (minmax value) = 0

Since $\underline{v} = -1$, $v = 1$, game is not strictly determinable.

Example. Solve the game hide payoff matrix is given by

Player B

$$\text{Player } A \quad \begin{matrix} & B_1 & B_2 & B_3 \\ A_1 & 1 & 3 & 1 \\ A_2 & 0 & -4 & -3 \\ A_3 & 1 & 5 & -1 \end{matrix}$$

Solution. Select the row minimum and enclose it in a rectangle select the column maximum and enclose it in a circle.

$$
\begin{array}{cc}
 & \text{Player } B \\
 & \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\
\text{Player } A \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} & \begin{bmatrix} \textcircled{\boxed{1}} & 3 & \textcircled{\boxed{1}} \\ 0 & \boxed{-4} & -3 \\ 1 & \textcircled{5} & \boxed{-1} \end{bmatrix}
\end{array}
$$

We observe that there exist two saddle points at positions (1, 1) and (1, 3). Thus, the solution of the game is given by

(i) the optimum strategy for player A is A_1.

(ii) the optimum strategies for player B are B_1 and B_3.

(iii) the value of game is 1 for A and B.

Since $v \neq 0$, the game is not fair.

Example. Consider the game G with the following payoff matrix:

$$
\begin{array}{cc}
 & \text{Player } B \\
\text{Player } A & \begin{bmatrix} 2 & 6 \\ -2 & \lambda \end{bmatrix}
\end{array}
$$

(a) Show that G is strictly determinable, whatever λ may be

(b) Determine the value of G.

Solution. First, ignoring the value of λ , we determine the maximin and minimax values of the payoff

$$
\begin{array}{ccc}
 & \text{Player } B & \\
 & \begin{array}{cc} B_1 & B_2 \end{array} & \text{Row minima} \\
\text{Player } A \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} 2 & 6 \\ -2 & \lambda \end{bmatrix} & \begin{array}{c} 2 \\ -2 \end{array} \\
\text{Column maxima} & \begin{array}{cc} 2 & 6 \end{array} &
\end{array}
$$

matrix, as shown below:

Since maximin value = 2 = minimax value, the game G is strictly determinable, whatever λ may be

value of game G is 2

Example. For what value of λ , the game with following payoff matrix is strictly determinable?

	B_1	B_2	B_3
A_1	λ	6	2
A_2	–1	λ	–7
A_3	–2	4	λ

Solution. Ignoring the value of λ , we determine the maximin and minimax values of the payoff matrix,

		B_1	B_2	B_3	Row minimum
	A_1	–5	6	2	2 ← Maximin
Player A	A_2	–1	6	–7	–7
	A_3	–2	4	1	–2
Column maximum		–1	6	2	
		↑ Minimax			

as shown below

Here

maxim

in

value

= 2,

minim

ax

value

= –1.

The

value

of

game

lies

betwee

n –1

and 2.

For strictly determinable game, since maximin value equals minimax value, we must have

$$-1 \le \lambda \le 2 .$$

Exercises. Solve the games whose payoff matrices are given below.

1. Player A (rows), Player B (columns)

$$\begin{bmatrix} 1 & \textcircled{0} \\ \boxed{4} & \textcircled{-3} \end{bmatrix} \begin{matrix} 1 \\ -3 \end{matrix}$$

$$\begin{matrix} 4 & 1 \end{matrix}$$

Answer. $v = 1$

2. Player A (rows), Player B (columns)

$$\text{Player } A \begin{bmatrix} \textcircled{15} & \boxed{2} & \boxed{3} \\ \boxed{6} & 5 & \textcircled{7} \\ -7 & 4 & 0 \\ 15 & 5 & 3 \end{bmatrix} \begin{matrix} 2 \\ 5 \\ 0 \\ \end{matrix}$$

Answer. (A_2, B_2), $v = 5$

3. Player A (rows), Player B (columns)

$$\text{Player } A \begin{matrix} \\ A_1 \\ A_2 \\ A_3 \end{matrix} \begin{matrix} B_1 & B_2 & B_3 \\ \textcircled{\boxed{1}} & 2 & \textcircled{\boxed{1}} \\ \textcircled{0} & \boxed{-4} & -1 \\ \textcircled{1} & \textcircled{3} & \boxed{-2} \end{matrix} \begin{matrix} \\ 1 \\ -4 \\ -2 \end{matrix}$$

Answer. (A_1, B_1), (A_1, B_3), $v = 1$ for A.

4.

$$\begin{matrix} \\ A_1 \\ A_2 \end{matrix} \begin{matrix} B_1 & B_2 & B_3 \\ 6 & 8 & 6 \\ 4 & 12 & 2 \end{matrix} \begin{matrix} \\ 1 \\ -4 \end{matrix}$$

Answer. (A_1, B_1), (A_1, B_2), $v = 6$

5.

$$\text{Player } A \begin{matrix} & \text{Player } B \\ & \begin{matrix} \text{I} & \text{II} & \text{III} & \text{IV} & \text{V} \end{matrix} \\ \begin{matrix} \text{I} \\ \text{II} \\ \text{III} \\ \text{IV} \end{matrix} & \begin{bmatrix} 9 & 3 & 1 & 8 & 0 \\ 6 & 5 & 4 & 6 & 7 \\ 2 & 4 & 4 & 6 & 7 \\ 5 & 6 & 2 & 2 & 1 \end{bmatrix} \end{matrix}$$

Answer. (II, III), $v = 4$

6 Determine which of the following two-person zero-sum games are strictly determinable and fair? Give the optimum strategies for each player in case of strictly determinable games.

(a)

$$\text{Player } A \begin{matrix} & \text{Player } B \\ & \begin{matrix} B_1 & B_2 \end{matrix} \\ \begin{matrix} A_1 \\ A_2 \end{matrix} & \begin{bmatrix} -5 & 2 \\ -7 & -4 \end{bmatrix} \end{matrix}$$

(b)

$$\text{Player } A \begin{matrix} & \text{Player } B \\ & \begin{matrix} B_1 & B_2 \end{matrix} \\ \begin{matrix} A_1 \\ A_2 \end{matrix} & \begin{bmatrix} 10 & 6 \\ 8 & 2 \end{bmatrix} \end{matrix}$$

Answer. (a) (A_1, B_1), $v = -5$, not fair(b) (A_1, B_2) , value = 6, not fair.

PRINCIPLE OF DOMINANCE

The principle of dominance is used to reduce the size of a games payoff matrix by eliminating a course of action which is so inferior to another as never to be used. Such a course of action is said to be dominated by the other. It is applicable to both pure and mixed strategy problems. However, this rule is especially useful for the evaluation of two-person zero-sum games where a saddle point does not exist.

In general, the following rules of dominance are used to reduce the size of payoff matrix.

Rule 1. If all the elements in a column are greater than or equal to the corresponding elements in another column, then that column is dominated and can be deleted from the matrix.

Rule 2. If all the elements in a row are less than or equal to the corresponding elements in another row, then that row is dominated and can be deleted from the matrix.

Rule 3. If all the elements in a column are greater than or equal to the average of the corresponding elements of two or more other columns, then that column can be deleted.

Rule 4. If all the elements in a row are less than or equal to the average of the corresponding elements of two or more other rows, then it can be deleted.

Example. Reduce the following game to 2 × 2 game using principle of dominance.

		Player *B*					
		I	II	III	IV	V	VI
	I	4	2	0	2	1	1
	II	4	3	1	3	2	2
Player *A*	III	4	3	7	–5	1	2
	IV	4	3	4	–1	2	2
	V	4	3	3	–2	2	2

Solution. Column I, II and IV are dominated by column V, so columns I, II and VI are deleted. The reduced matrix is

Player *B*

Player *A*		III	IV	V
	I	0	2	1
	II	1	3	2
	III	7	–5	1
	IV	4	–1	2
	V	3	–2	2

Now row I is dominated by row 2 and row 5 is dominated by row 4. Hence deleting rows I and V, we have

Player *B*

Player *A*		III	IV	V
	II	1	3	2
	III	7	–5	1
	IV	4	–1	2

Now none of single row (or column) dominates another row (or column. However, column V is dominated by the average of columns III and IV. Hence deleting column V, we have

Player *B*

Player *A*		III	IV
	II	1	3
	III	7	–5
	IV	4	–1

Now average of row II and row III gives the row (4, –1) which dominates the row IV. Hence deleting row IV, we have

Player *B*

Player *A*		III	IV
	II	1	3
	III	7	–5

Example. Reduce the following game into 2 × 2 game using the rules of dominance.

Player *B*

Player *A*		B_1	B_2	B_3
	A_1	1	7	2
	A_2	6	2	7
	A_3	5	1	6

Solution. First, we delete the column 3 as all the elements of this column are greater than that of first column after that we delete 3rd row as all the elements of row 3 are less than the corresponding elements of row 2. Hence the reduced matrix is

		Player B	
		B_1	B_2
Player A	A_1	1	7
	A_2	6	2

MIXED STRATEGIES: GAMES WITHOUT SADDLE POINT

Pure strategies are available as optimal strategies only for those games which have a saddle point. For games which do not have a saddle point can be solved by applying the concept of mixed strategies. Her, we study algebraic, graphical and linear programming method to solve mixed strategies games.

Algebraic Method

Consider the two-person zero-sum game with the following payoff matrix:

	Strategy	Player B: B_1	B_2	Probability
Player A	A_1	a_{11}	a_{12}	p
	A_2	a_{21}	a_{22}	$1-p$
	Probability	q	$1-q$	

If this game is to have no saddle point, the two largest elements of the matrix must constitute one of the diagonals. We have assumed this and therefore both players use mixed strategies. Our task is to determine the probabilities with which both players choose their course of action.

In this game, let player A play the strategies A_1 and A_2 with respective probabilities p and $1-p$

and let player B play his strategies B_1 and B_2 with respective probabilities q and $1-q$. The expected payoffs to player A when B plays any one of his strategies B_1 or B_2 throughout the game, are given by

B's Strategy	A's Strategy
B_1	$a_{11}p + a_{21}(1-p)$
B_2	$a_{12}p + a_{22}(1-p)$

Now in order that player A is unaffected with whatever choice of strategies B makes, we must have

$$a_{11}p + a_{21}(1-p) = a_{12}p + a_{22}(1-p)$$

$$\Rightarrow (a_{11}-a_{12})p + (a_{22}-a_{21})p = a_{22}-a_{21}$$

$$\Rightarrow p = \quad a \quad 2$$

2 −

$$\frac{a_{21}}{(a_{11} - a_{12}) + (a_{22} - a_{21})}$$

and

$$1-p=\frac{a_{11}-a_{12}}{\left(a_{11}-a_{12}\right)+\left(a_{22}-a_{21}\right)}$$

similarly, by equating the expected payoffs of the player *B*, for whatever choice of strategies player *A* makes, we have

$$a_{11}q+a_{12}\left(1-q\right)=a_{21}q+a_{22}\left(1-q\right)$$

This implies

$$\left[\left(a_{11} - a_{22}\right) + \left(a_{22} - a_{12}\right)\right] q = a_{22} - a_{12}$$

So $q = \dfrac{a_{22} - a_{12}}{\left(a_{11} - a_{21}\right) + \left(a_{22} - a_{12}\right)}$ and $1 - q = \dfrac{\left(a_{11} - a_{21}\right)}{\left(a_{11} - a_{21}\right) + \left(a_{22} - a_{12}\right)}$

The value of game, v is found by substituting the value of p in one of the expressions for the expected gains of A so that

$$v = a_{11} p + a_{21}\left(1 - p\right)$$

$$= \frac{a_{11}\left(a_{22} - a_{21}\right)}{\left(a_{11} - a_{12}\right) + \left(a_{22} - a_{21}\right)} + \frac{a_{21}\left(a_{11} - a_{12}\right)}{\left(a_{11} - a_{12}\right) + \left(a_{22} - a_{21}\right)}$$

$$= \frac{a_{11} a_{22} - a_{12} a_{21}}{\left(a_{11} - a_{12}\right) + \left(a_{22} - a_{21}\right)}$$

Hence the solution of the game is

A plays $\left(p, 1 - p\right)$ where $p = \dfrac{a_{22} - a_{21}}{\left(a_{11} - a_{12}\right) + \left(a_{22} - a_{21}\right)}$

B plays $(q, 1-q)$ where $q = \dfrac{a_{22} - a_{12}}{(a_{11} - a_{12}) + (a_{22} - a_{21})}$

and value of game, $v = \dfrac{a_{11}a_{22} - a_{12}a_{21}}{(a_{11} - a_{21}) + (a_{22} - a_{12})}$.

Example. Solve the following game:

$$\begin{array}{cc} & \text{Player } B \\ & \begin{array}{cc} B_1 & B_2 \end{array} \\ \text{Player } A \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} 25 & 5 \\ 10 & 15 \end{bmatrix} \end{array}$$

Solution. Here maximin value = 10 and minimax value = 15.

So, game has no saddle point.

Let the player A play *h*is first strategy A_1 with probability p, then he would play his second strategy A_2 with probability $(1-p)$. Then expected gain of A is B selects B_1, is equal to $25p + 10(1-p)$ i.e. $10 + 15p$ and the expected gain of A if B selects strategy B_2, is equal to $5p + 15(1-p)$ i.e. $15 - 10p$.

Now in order that the player *A* may be unaffected with whatever choice *B* makes, the optimal plan for the player *A* should be such that the expected payoffs for each of *B*'s strategies should be equal is

$10 + 15p = 15 - 10p$

$\therefore p = \frac{5}{25} = \frac{1}{5}$ and $1 - p = 1 - \frac{1}{5} = \frac{4}{5}$

Hence, the player *A* would play his first strategy A_1 with probability $\frac{1}{5}$ and second strategy A_2 with probability $\frac{4}{5}$.

Similarly, if the player *B* selects strategies B_1 and B_2 with probabilities *q* and $1 - q$ respectively, then the expected loss to *B* when *A* adopts the strategy A_1, is $25q + 5(1 - q)$ and the expected loss to *B* when the player *A* adopts the strategy A_2, is $10q + 15(1 - q)$. By equating the expected losses of player *B*, for whatever choice of strategies player *A* makes, we have

$$25q + 5(1 - q) = 10q + 15(1 - q)$$

$$\Rightarrow 20q + 5 = 15 - 5q$$

$$\Rightarrow q = \frac{10}{25} = \frac{2}{5} \quad \text{and } 1 - q = \frac{3}{5}$$

Hence the player *B* would play his strategies B_1 and B_2 with probabilities $\frac{2}{5}$ and $\frac{3}{5}$ respectively.

Value of the game = expected payoff to player *A*

$$= 25p + 10(1 - p)$$

$$= 25\times\frac{1}{5}+10\times\frac{4}{5} = 13.$$

Example. For the following game:

		Firm B			
		B_1	B_2	B_3	B_4
Firm A	A_1	35	65	25	5
	A_2	30	20	15	0
	A_3	40	50	0	10
	A_4	55	60	10	15

Determine the optimal strategies for each firm and value of the game.

Solution. Since maximin value = 10 and minimax value = 15, there is no saddle point.

We apply rules of dominance to reduce the size of payoff matrix. Since each element of second row is less than the corresponding elements of first row, second row is dominated by first row. So, deleting the second row, the reduced matrix becomes

		Firm B			
		B_1	B_2	B_3	B_4
	A_1	35	65	25	5
Firm A	A_3	40	50	0	10
	A_4	55	60	10	15

In the reduced matrix, each element of second column is more than the corresponding elements in first column, so second column is dominated by first column. Thus after deleting the second column, the reduced matrix becomes

		Firm B		
		B_1	B_3	B_4
	A_1	35	25	5
Firm A	A_3	40	0	10
	A_4	55	10	15

Further second row is dominated by third row, so we delete second row to get reduced matrix as

		Firm B		
		B_1	B_3	B_4
Firm A	A_1	35	25	5
	A_4	55	10	15

Now column one is dominated by column two. So, we delete column one and reduced matrix becomes

		Firm B	
		B_3	B_4
Firm A	A_1	25	5
	A_4	10	15

Now we solve the game by algebraic method in the same manner as we did in example 9.6.1.1. Thus, firm A would select strategy A_1 with probability $\frac{1}{5}$ and strategy A_4 with probability $\frac{4}{5}$ and firm B would select strategy B_3 with probability $\frac{2}{5}$ and strategy B_4 with probability $\frac{3}{5}$ and Value of game =

$$25p + 10(1 - p) = 25\cdot\frac{1}{5} + 10\cdot\frac{4}{5} = 5 + 8 = 13.$$

Example. In a game of matching coins with two players, suppose one player wins Rs. 2 when there are two heads and wins nothing when there are two tails, and losses Re. 1 when there are one head and one tail. Determine the payoff matrix, the best strategies for each player and the value of the game.

Solution. Let the two players be A and B. Then the payoff matrix for player A is

Player A \ Player B	H	T
H	2	–1
T	–1	0

Here maximin value $(\underline{v}) = -1$; minimax value $(\overline{v}) = 2$.

Since $\underline{v} \neq \overline{v}$, given game has no saddle point. Let the player *A* plays H with probability *p* and T with probability $1 - p$. Then *A*'s expected gains when *B* plays H and T respectively, are $2p + (-1)(1-p)$ and $-p + 0(1-p)$.

For best strategy of *A*, we have

$$2p + (-1)(1-p) = -p.$$

so that $p = \frac{1}{4}$ and $1 - p = \frac{3}{4}$. Therefore, best strategy for player *A* is to play H and T with probabilities $\frac{1}{4}$ and $\frac{3}{4}$ respectively.

For player *B*, let the probability of the choice of H be *q* and that of T be $1 - q$. For best strategy of *B*, we have

$$2q + (-1)(1-q) = (-1)q + 0(1-q)$$

so that $q = \frac{1}{4}$ and $1 - q = \frac{3}{4}$.

Hence player *B* should play H and T with probabilities $\frac{1}{4}$ and $\frac{3}{4}$ respectively.

Value of game $= 2p + (-1)(1-p) = -\frac{1}{4}$

for player A.

9.6.1.4 Exercises. Solve the following games without saddle points.

1. $$\begin{array}{cc} & \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} 2 & 5 \\ 7 & 3 \end{bmatrix} \end{array}$$

Answer. $\left(4, 3\right)$, for player A, $\left(2, 5\right)$, for player B and Value of game = 0.

2. In a game of matching coins with two players, suppose A wins one unit of value, when there are two heads, wins nothing when there are two tails and loses $\frac{1}{2}$ unit of value when there are one head and one tail. Determine the payoff matrix, the best strategies for each player and the value of game to A.

Answer. Payoff matrix for *A* is

$$\begin{array}{cc} & \begin{array}{cc} H & T \end{array} \\ \begin{array}{c} H \\ T \end{array} & \begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{bmatrix} \end{array}$$

and value of the game is $-\frac{1}{8}$.

Optimal strategies for two players are $\left(\frac{1}{4}, \frac{3}{4}\right), \left(\frac{1}{4}, \frac{3}{4}\right)$

3. The firms are competing for businesses under the conditions so that one firm's gain is another firm's loss. Firm *A*'s payoff matrix is given below:

		Firm *B* No advertising	Medium advertising	Heavy advertising
	No Adv.	10	5	–2
Firm *A*	Medium Adv.	13	12	15
	Heavy Adv.	16	14	10

Suggest the optimum strategies for the firms.

Answer. $\left(0, \frac{4}{7}, \frac{3}{7}\right), \left(0, \frac{5}{7}, \frac{2}{7}\right), v = \frac{90}{7}$.

Graphical method

The graphical method is useful for solving two person–zero–sum–game. A Game having saddle point can be easily solved, so, we consider games without saddle point, where the payoff matrix is of size 2× n or m × 2.

Optimal strategies for both the players assign no–zero probabilities to the same number of pure games. Therefore, if one player has only two strategies, the other will also use the same number of strategies. Hence, this method is useful in finding out which of the two strategies can be use. Consider the following 2 × n payoff matrix of a game without saddle point.

Player A	B_1	B_2	B_3	Probability
$A1$	a_{11}	a_{12}	a_{13}	p_1
A_2	a_{21}	a_{22}	a_{23}	p_2
Probability	q_1	q_2	q_3	

To solve this game, we draw two vertical lines at unit distance, for representing $p_1=0$ and $p_2=0$ where $p=(p_1, p_2)$ is the strategy of A and $q=(q_1, q_2, \ldots, q_n)$ is the strategy of B.

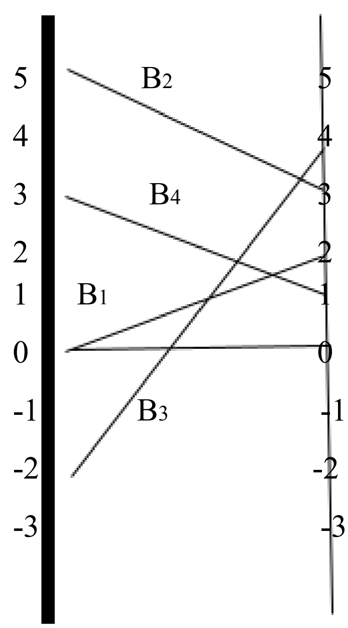
We now draw n line segments joining the points $(0, a_{2j})$ and $(1, a_{1j})$, j= 1, 2, …, n but excluding the end points. The lower envelope of these lines gives the minimum expected gain of A as a function of p_1. The highest point o of this lower boundary of these lines will give maximum of the minimum gain of A, i.e. maximin of A.

Now, the two strategies of player B corresponding to those lines which pass through the maximum point can be determined. It helps in reducing the size of the game to (2 × 2), which can be easily solved by any of the methods discussed earlier.

Remark: The (m×2) games are also treated in the same way except the upper boundary of the straight lines corresponding to B's expected payoff will give the maximum expected payoff to player B and the lower point on this boundary will then give the minimum expected payoff (minimax value) and the optimum value of probability q_1 and q_2.

9.6.2.1. Example. Use graphical method in solving the following game and find the value of the game Solve the following game:

Solution. Since maximin $a_{ij} = 3 < minimax\ a_{ij} = 4$, the game is to be solved by mixed strategies. We therefore use the graphical method to reduce this to a 2× 2 by game as follows:



We join the points 0, 5, -2 and 3 on the left line given by p_1= 0 to the points 2, 3, 4 and 1 on the right line given by p_2=0 respectively. Clearly the highest point of the lower envelop determines the strategies B_1 and B_4.

So, the reduced game is:

		Player B	
		B_1	B_4
Player A	A_1	0	3
	A_2	2	1

Solving this game, we get

p_1^* =1/4, p_2^* =3/4; q_1^* =1/2, q_4^*=1/2

Hence the required solution is

Note: An m x n game is solvable if it has a saddle point but if it has no saddle point, it cannot be solved by graphical method unless it is reducible to the form m x 2 or 2 x n game by the dominance principle.

Linear Programming Method

The two person – zero – sum - game can also be solved by linear programming. The major advantages of using the programming technique is to solve mixed-strategy games of larger dimension payoff matrix.

To illustrate the transformation of the game problem to a linear programming problem, consider a payoff matrix of size m × n. Let a_{ij} be the element in the ith row and jth column of game payoff matrix, and letting the probabilities of m strategies (i = 1, 2, 3, ... , m) for player A, for each of player B's strategies will be

$$\sum_{i=1}^{m} p_i . a_{ij}, \qquad j = 1, 2, ..., n$$

The aim of player A is to select a set of strategies with probability p_1, the value of the game to the played A for all strategies by the player B must be at least equal to V. Thus, to miximize the minimum expected gains, it is necessary that

$$a_{11}p_1 + a_{12}p_2 + \cdots + a_{m1}p_m \geq V$$
$$a_{21}p_1 + a_{22}p_2 + \cdots + a_{m2} \geq V$$
$$.$$
$$.$$
$$.$$
$$a_{1n}p_1 + a_{2n}p_2 + \cdots + a_{mn}p_m \geq V$$
$$p_1 + p_2 + \cdots + p_m = 1; p_i \geq 0 \text{ for all i}$$

Dividing both sides of the m inequalities of the and equation by V the division is valid as long as V > 0. In case V < 0, The direction of inequality constraints must be reversed. But if V = 0, division would be meaningless. In this case a constraint can be added to all entries of the matrix ensuring that the value of the game (V) for the revised matrix becomes more than zero. After optimal solution is obtained for the value of the game is obtained by subtracting the $= x_i$, (≥ 0). Then we get,

same constant value. Let p_i/V

≥ 1

≥ 1

P_1 p_2 p_m

.

.

.

$a_{1n} \frac{}{V} + p_{2n} \frac{}{V} + \cdots + a_{mn} \frac{}{V} \geq 1$

$\frac{p_1}{V} + \frac{p_2}{V} + \cdots + \frac{p_m}{V} = 1$

Since, the objective of player A is to maximize the value of the game, V which is equivalent to minimize $\frac{1}{V}$, the resulting linear programming problem can be stated as

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{m1}x_m \geq V$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{m2}x_m \geq V$$

.

.

.

$$a_{1n}x_1 + a_{2n}x_2 + \cdots + a_{mn}x_m \geq V$$

And x_1, x_2, ..., x_m ≥ 0, $x_i = \frac{p_i}{V}$; i = 1, 2, ..., m

Similarly, Player B has a similar problem with the inequalities of the constraints reversed, i.e., minimize the expected loss. Since minimizing of V is equivalent to maximizing $\frac{1}{V}$, therefore, the resulting programming problem can be stated as:

Maximize

$$Z_q \left(= \frac{1}{V}\right) = y_1 + y_2 + \cdots + y_n$$

Subjected to the constraints

$$a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n \leq 1$$

$$a_{21}y_1 + a_{22}y + \cdots + a_{2n}y_n \leq 1$$

.

. $a_{m1}y_1 + a_{m2}y_2 + \cdots + a_{mn}y_n \leq 1$

.

And $y_1, y_2, \ldots, y_n \geq 0$

Where, $y_j = \frac{q_i}{V} \geq 0$; i = 1, 2, ..., n

It may be noted that the LP problem for the player B is the dual of LP problem for plater A and vice-versa. Therefore, solution of the dual problem can be obtained from the primal simplex table. Since both the players $Z_p = Z_q$, the expected gain to player A in the game will be exactly equal to expected payoff to player B.

Remark: Linear programming technique require all variables to be non-negative and therefore to derive a non – negative value V of the game, the data to the problem, i.e., a_{ij} in the payoff table should be non – negative. If there are some negative elements in the payoff table, a constant to every elements of the payoff table must be added so as to make the smallest element zero; the solution to this new game give an optimal mixed strategy for the new game. The value of the original game then equals to the value of the new game minus the constant.

Example. For the following payoff matrix, transform the zero-sum game into an equivalent linear programming problem and solve it by using simplex method.

Player B

Player A	B1	B2	B3
A1	1	-1	3
A2	3	5	-3
A3	6	2	-2

Solution: The first step is to find out the saddle point (if any) in the payoff matrix as shown below

Player B

Player A	B1	B2	B3	Row minimum	
A1	1	-1		-1	← Maximin
A2	3	5	-3	-3	
A3	6	2	-2	-2	
Column maximum	6	5	3 ← Minimax		

The given game payoff matrix does not have a saddle point. Since, the maximin value is -1, therefore, it is possible that the value of game (V) may be negative or zero because -1< V < 1. Thus, a constant which is at least equal to the negative of maximin value, i.e., more than -1 is added to all elements of the payoff matrix. Thus, adding a constant number 4 to all the elements of the payoff matrix, the payoff matrix becomes:

Player B

Player A	B1	B2	B3	Probability
A1	5	3	7	p_1
A2	7	9	1	p_2
A3	10	6	2	p_3
Probability	q_1	q_2	q_3	

Let p_i $(i = 1,2,3)$ and q_j $(j = 1,2,3)$ be the probabilities of selecting strategies $A_i (i = 1,2,3)$ and B_j $(j = 1,2,3)$ by players A and B, respectively.

The expected gain for player A will be as follows:

$5p_1 + 7p_2 + 10p_3 \geq V$ (if B uses strategy B_1)

$3p_1 + 9p_2 + 6p_3 \geq V$ (if B uses strategy B_2)

$7p_1 + p_2 + 2p_3 \geq V$ (if B uses strategy B_3)

$$p_1 + p_2 + p_3 = 1$$

and $p_1, p_2, p_3 \geq 0$

Dividing each inequality and equality by V, we get,

$$5\frac{p_1}{V} + 7\frac{p_2}{V} + 10\frac{p_3}{V} \geq 1$$

$$3\frac{p_1}{V} + 9\frac{p_2}{V} + 6\frac{p_3}{V} \geq 1$$

$$7\frac{p_1}{V} + \frac{p_2}{V} + 2\frac{p_3}{V} \geq 1$$

$$\frac{p_1}{V} + \frac{p_2}{V} + \frac{p_3}{V} = \frac{1}{V}$$

In order to simplify, we define new variables:

$x_1 = p_1/V$, $x_2 = p_2/V$ $and\ x_3 = p_3/V$

The problem for player A, therefore becomes,

Minimize Z_p (=1/V) $= x_1 + x_2 + x_3$ subject to the constraints

$$5x_1 + 7x_2 + 10x_3 \geq 1$$

$$3x_1 + 9x_2 + 6x_3 \geq 1$$

$$7x_1 + x_2 + 2x_3 \geq 1$$

and $x_1, x_2, x_3 \geq 0$

player B's objective is to minimize his expected losses which can be reduced to minimizing the value of the game V. Hence, the problem of player B can be expressed as follows:

Minimize Z_q (=1/V) $= y_1 + y_2 + y_3$

subject to the constraints

$$5y_1 + 7y_2 + 10y_3 \leq 1$$

$$3y_1 + 9y_2 + 6y_3 \leq 1$$

$$7y_1 + y_2 + 2y_3 \leq 1$$

and $y_1, y_2, y_3 \geq 0$

where $y_1 = q_1/V$, $y_2 = q_2/V$ $and\ y_3 = q_3/V$.

It may be noted that problem of player A is the dual of the problem of player B. Therefore, solution of the dual problem can be obtained from the optimal simplex table of primal.

To solve the problem of player B, introduce slack variables to convert the three inequalities to equalities. The problem becomes

Minimize Z_q (=1/V) $= y_1 + y_2 + y_3 + 0s_1 + 0s_2 + 0s_3$

subject to the constraints

$$5y_1 + 7y_2 + 10y_3 + s_1 = 1$$

$$3y_1 + 9y_2 + 6y_3 + s_2 = 1$$

$7y_1 + \ y_2 + 2y_3 + s_3 = 1$ and $y_1, y_2, y_3, s_1, s_2, s_3 \geq 0$

The initial solution is shown in Table 12.7.

Table 12.7 Initial Solution

$c_j \rightarrow$			1	1	1	0	0	0	
Unit Cost $\boldsymbol{c_B}$	Variables in Basis **B**	Solution Values $\boldsymbol{y_B}$**(=b)**	y_1	y_2	y_3	s_1	s_2	s_3	Min. Ratio $\boldsymbol{y_B/y_1}$
0	s_1	1	5	3	7	1	0	0	1/5
0	s_2	1	7	9	1	0	1	0	1/7
0	s_3	1	(10)	6	2	0	0	1	1/10→
Z=0		z_j	0	0	0	0	0	0	
		$c_j - z_j$	1	1	1	0	0	0	

		Player B			
		B1	B2	B3	B4
Player A	A_1	0	5	-2	3
	A_2	2	3	4	1

$\frac{P_1}{V} + p_{21} \frac{p_2}{V} + a_{m1} \cdots + \frac{p_m}{V}$

$\frac{P_1}{V} + p_{22} \frac{p_2}{V} + a_{m2} \cdots + \frac{p_m}{V}$

Proceeding with usual simplex method, the optimal solution is shown in Table 12.8.

Table 12.8 Optimal Solution

$c_j \rightarrow$	1	1	1	0	0	0

Unit Cost c_B	Variables Solution in Basis Values B	y_B(=b)	y_1	y_2	y_3	s_1	s_2	s_3
1	y_3	1/10	2/ 5	0	1	3/20	-1/10	0
1	y_2	1/10	11/15	1	0	-1/60	7/60	0
0	s_3	1/5	24/5	0	0	-1/5	-3/5	1
Z=1/5		z_j	17/15	0	0	2/15	1/15	0
	$c_j - z_j$		-2/15	1	1	-2/15	-1/15	0

The optimal solution (mixed strategies) for B is: $y_1 = 0$; $y_2 = 1/10$ and $y_3 = 1/10$ and expected value of the game is: Z = 1/V – constraint (= 4) = 5-4 = 1.

These solution values are now converted back into the original variables; if 1/V = 1/5 then V=5

$y_1 = q_1/V$, then $q_1 = y_1 \times V = 0$

$y_2 = q_2/V$, then $q_2 = y_2 \times V = {}^1/_{10} \times 5 = 1/2$

$y_3 = q_3/V$, then $q_3 = y_3 \times V = {}^1/_{10} \times 5 = 1/2$

The optimal strategies for player A are obtained from the $c_j - z_j$ row of the Table 12.8.

$x_1 = {}^2/_{15}$, $x_2 = {}^1/_{15}$ and $x_3 = 0$

Then $p_1 = x_1 \times V = ({}^2/_{15}) \times 5 = {}^2/_3$; $p_2 = x_2 \times V = ({}^1/_{15}) \times 5 = {}^1/_3$

$$p_3 = x_3 \times V = 0$$

Hence, the probabilities of using strategies by both the players are:

Player A: (2/3, 1/3, 0), Player B: (0, 1/2, 1/2) and Value of the game is V = 1.

Exercises.

1. A soft drink company calculated the market share of two products against its major competitor having products and found out the impact of additional advertisement in any one of its products against the other

Company A		Company B		
		B_1	B_2	B_3
	A_1	6	7	15
	A_2	20	12	10

What is the best strategy for the company as well as competitor? What is the payoff obtained by the company and the competitor in the long run? Use graphical method to obtain the solution.

Answer. Company A: (2/3, 1/3, 0), Company B: (7/12, 5/12) and V = 1/3.

2. In a town there are only two discount stores ABC and XYZ. Both stores run annual pre – Diwali sales. Sales are advertised through local newspapers with the aid of an advertising firm. ABC stores constructed following payoff in units of Rs 1,00,000. Find the optimal strategies for both stores and the value of the game.

Answer. Add 2 (absolute value of the smallest negative payoff value) to each element of the payoff matrix. Then formulae an LP model for store XYZ. Optimal values of decision variables are: y_1 =
y_2 = y_3 = 1/6 and Z = 1/2 = 1/V or V = 2. Subtract 2 from V=2 to get V=0.

CHECK YOUR PROGRESS

1. What do you mean by a game in game theory? What are the assumptions made in game theory?
2. Explain maximin and minimax criterion used in game theory.
3. Define the following terms:
    i. Saddle point,
    ii. Two-person zero -sum game,
    iii. Strictly determinable game,
    iv. Value of the game.
4. Explain the rules of dominance.
5. Explain the algebraic method for solving rectangular games.

SUMMARY

In this chapter, we discussed about basic concepts/terminologies of game theory such as payoff matrix, pure strategy, mixed strategy etc. We explain basic assumptions of the game and discuss two-person zero- sum game i.e. rectangular games. In certain cases, we observed that there is no pure strategy solution for a game i.e. no saddle point exists. In all such cases, one can use methods which involve concept of mixed strategy. Here, we study three methods namely algebraic, graphical and linear programming methods for solving the problems having no saddle point. However, game theory is limited in scope as it has been capable of analysing simple competitive situations.