ANGLICAN HIGH SCHOOL
SECONDARY FOUR
PRELIMINARY EXAMINATIONS 2020

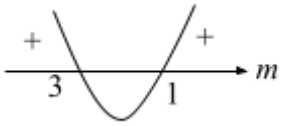
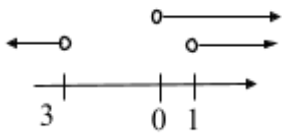
S4

ADDITIONAL MATHEMATICS **4047/02**

Paper 2 **28 August 2020 Friday**

2 hours 30 minutes

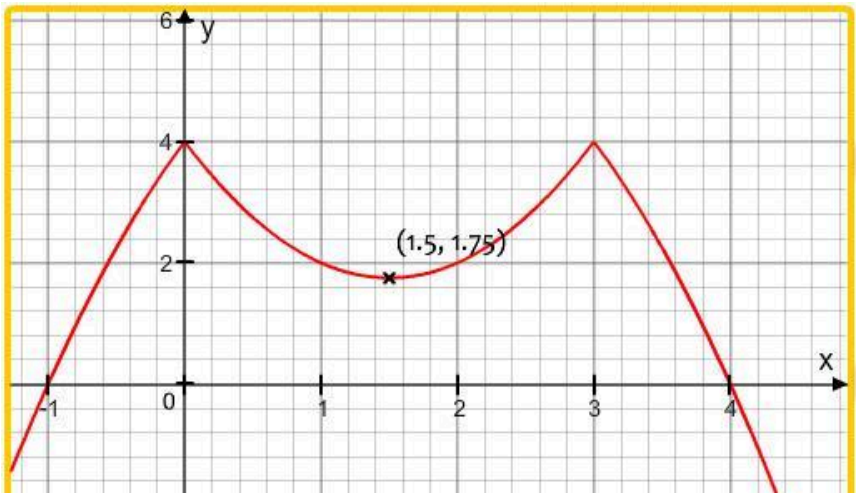
Marking Scheme

1a	The equation of a curve is $y = mx^2 + (m-3)x + m$, where m is a constant. Find the range of values of m for which the curve lies completely above the x-axis.	[5]
	Since the curve is completely above the x-axis, the curve must be a quadratic curve with minimum point , $a > 0$ $\therefore m > 0$	B1
	And there will be no point of intersection with the x-axis, $b^2 - 4ac < 0$ $(m-3)^2 - 4(m)(m) < 0$ $m^2 - 6m + 9 - 4m^2 < 0$ $-3m^2 - 6m + 9 < 0$ $m^2 + 2m - 3 > 0$ $(m+3)(m-1) > 0$ $m < -3$ or $m > 1$  	M1 M1 A1
	 Hence, $m > 1$	A1
1b	Given that $y = ax^2 - 4x + c$ is always negative, give an example of values of a and c which satisfy the condition.	[2]
	$b^2 - 4ac < 0$ $(-4)^2 - 4ac < 0$ $ac > 4, \quad a < 0$	M1

	$a = -1$ and $c = -5$ or $a = -2$ and $c = -2$ or $a = -5$ and $c = -1$ or any pairs of values that fulfill the above criteria	A1
2a	Given that $2x^4 + 3x^3 + ax^2 - 9x + 9 = \left(x^2 - 1\right)(x-2)\mathrm{Q}(x) - 3x^2 + bx + c$ is an identity, state, with reason, the degree of $\mathrm{Q}(x)$.	[1]
	Since degree of dividend = degree of quotient + degree of divisor, degree of $\mathrm{Q}(x) = 1$ ALTERNATIVE: Must multiply with x3 to give degree 4 in the polynomial on the left.	B1 for reason and answer
2b	Find the value of a, of b and of c.	[5]
	$2x^4 + 3x^3 + ax^2 - 9x + 9 = \left(x^2 - 1\right)(x-2)\mathrm{Q}(x) - 3x^2 + bx + c$ When $x = 1$, $2 + 3 + a - 9 + 9 = -3 + b + c$ $a = -8 + b + c \quad -------(1)$ When $x = -1$, $2 - 3 + a + 9 + 9 = -3 - b + c$ $a = -20 - b + c \quad --------(2)$ (1) = (2): $-20 - b + c = -8 + b + c$ $-12 = 2b$ $b = -6$ When $x = 2$, $32 + 24 + 4a - 18 + 9 = -12 - 12 + c$ $c = 47 + 4a + 24$ $c = 4a + 71 \quad --------(3)$ Sub (3) and b = −6 into (2) $a = -20 + 6 + 4a + 71$ $a = 4a + 57$ $a = -19$ Sub a = −19 into (2) $c = 4(-19) + 5$ $c = -5$	M1 M1 M1 M1 for solving A1 for 3 answers

	$a=-19,\quad b=-6,\quad c=-5$	
2c	Hence, find the remainder when $2x^4+3x^3+ax^2-9x+9$ is divided by $(3x-1)$.	[1]
	Let $\mathrm{f}(x)=2x^4+3x^3-19x^2-9x+9$ $\mathrm{f}\left(\frac{1}{3}\right)=2\left(\frac{1}{3}\right)^4+3\left(\frac{1}{3}\right)^3-19\left(\frac{1}{3}\right)^2-9\left(\frac{1}{3}\right)+9$ $\mathrm{f}\left(\frac{1}{3}\right)=\frac{326}{81}$ Therefore the reminder is $\frac{326}{81}$.	B1
3a	Given that $p=3^x$ and $q=3^y$, express $\log_3\frac{p^7q}{243}$ in terms of x and y.	[4]
	$p=3^x \Rightarrow \quad x=\log_3 p \qquad q=3^y \Rightarrow \quad y=\log_3 q$ $\log_3\frac{p^7q}{243}=7\log_3 p+\log_3 q-\log_3 3^5$ $\log_3\frac{p^7q}{243}=7x+y-5$	[M1 for change to log] M2 for using rules A1
3b	Given that $\log_2 x-\log_x x^2=\frac{1}{3}-\log_8 2x$, find the value of x by leaving your answer in index form.	[4]
	$\log_2 x-\log_x x^2=\frac{1}{3}-\log_8 2x$ $\log_2 x-2\log_x x=\frac{1}{3}-(\log_8 2+\log_8 x)$ $\log_2 x-2(1)=\frac{1}{3}-\frac{1}{3}\log_8 8-\frac{\log_2 x}{\log_2 8}$ $\log_2 x-2=\frac{1}{3}-\frac{1}{3}-\frac{\log_2 x}{3}$ $3\log_2 x-6=-\log_2 x$ $4\log_2 x=6$ $\log_2 x=\frac{3}{2}$	M1 for power rule and product rule M1 for change of base rule M1 A1

	$x = 2^{\frac{3}{2}}$ or $x = 8^{\frac{1}{2}}$ or $x = 64^{\frac{1}{4}}$	
4a	Without using a calculator, express $\frac{\sqrt{6}-\sqrt{5}}{\sqrt{15}+\sqrt{2}}$ in the form of $a\sqrt{10}+b\sqrt{3}$.	[4]
	$\frac{\sqrt{6}-\sqrt{5}}{\sqrt{15}+\sqrt{2}} = \frac{\left(\sqrt{6}-\sqrt{5}\right)\left(\sqrt{15}-\sqrt{2}\right)}{15-2}$ $\frac{\sqrt{6}-\sqrt{5}}{\sqrt{15}+\sqrt{2}} = \frac{\sqrt{90}-\sqrt{12}-\sqrt{75}+\sqrt{10}}{13}$ $\frac{\sqrt{6}-\sqrt{5}}{\sqrt{15}+\sqrt{2}} = \frac{3\sqrt{10}-2\sqrt{3}-5\sqrt{3}+\sqrt{10}}{13}$ $\frac{\sqrt{6}-\sqrt{5}}{\sqrt{15}+\sqrt{2}} = \frac{4\sqrt{10}-7\sqrt{3}}{13}$ $\frac{\sqrt{6}-\sqrt{5}}{\sqrt{15}+\sqrt{2}} = \frac{4}{13}\sqrt{10}-\frac{7}{13}\sqrt{3}$	M1 for rationalization M1 for expansion M1 for simplifying A1
4b	Without the use of a calculator, solve the equation $\sqrt[3]{27^x} - 81^{x+1} = 0.$	[3]
	$\sqrt[3]{27^x} - 81^{x+1} = 0$ $27^{\frac{x}{3}} - 81^{x+1} = 0$ $\left(3^3\right)^{\frac{x}{3}} - \left(3^4\right)^{x+1} = 0$ $3^x - 3^{4x+4} = 0$ $3^x = 3^{4x+4}$ By comparing powers, $x = 4x + 4$ $x = -\frac{4}{3}$	M1 for converting all terms to base 3 M1 for using if $a^m = a^n$ then $m = n$ [A1]
5ai	Given the curve $y = -\left\|3x - x^2\right\| + 4$, find the *x*-coordinates of the points where the curve meets the *x*-axis.	[2]
	When $y = 0$, $0 = -\left\|3x - x^2\right\| + 4$ $3x - x^2 = 4$ or $3x - x^2 = -4$	M1 for considering

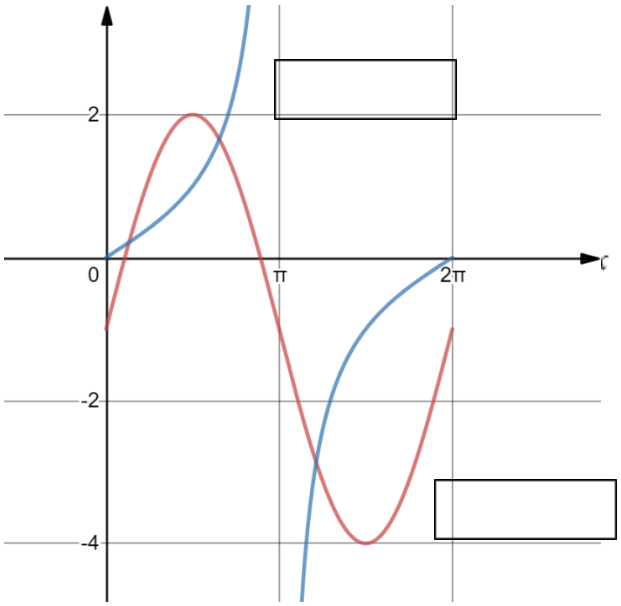
	$3x-x^2-4=0$ or $3x-x^2+4=0$ $x^2-3x+4=0$ or $x^2-3x-4=0$ $b^2-4ac=(-3)^2-4(1)(4)$ or $(x-4)(x+1)=0$ $b^2-4ac=-7$ or $x=4$ or $x=-1$	positive and negative A1 for answers
5aii	Sketch the curve $y=-\lvert 3x-x^2\rvert+4$, giving the coordinates of the maximum point and of the points where the curve meets the axes..	[3]
		B1 for shape B1 for coordinates of max pt and *x*-intercepts. B1 for labelling.
5b	Explain why there are only two solutions to the equation $-\lvert 3x-x^2\rvert=k-4$ for k < 1.75.	[2]
	$-\lvert 3x-x^2\rvert=k-4$ $-\lvert 3x-x^2\rvert+4=k$ $y=k$ $y=k$ is a **horizontal line** and when k is lesser than 2.25, it will be **below the turning point** and so it will only **intersect the curve at the two outer arms thereby giving two solutions only.**	B1 B1
5c	Determine the maximum value of m for which the line $y=mx+1$ intersects the graph of $y=-\lvert 3x-x^2\rvert+4$ in three points.	[1]
	$m=\frac{4-1}{3-0}$ $m=1$	B1
6(i)	Express $\frac{7x+11}{(x-1)(x+2)^2}$ in partial fractions.	[4]

	Let $\frac{A}{x-1}+\frac{B}{x+2}+\frac{C}{(x+2)^2}=\frac{7x+11}{(x-1)(x+2)^2}$ $A(x+2)^2+B(x-1)(x+2)+C(x-1)=7x+11$ when $x=1$, $9A=7+11$ $A=2$ when $x=-2$ $-3C=-14+11$ $C=1$ when $x=0$ $4A-2B-C=11$ $4(2)-2B-1=11$ $-2B=4$ $B=-2$ $\therefore \frac{7x+11}{(x-1)(x+2)^2}=\frac{2}{x-1}-\frac{2}{x+2}+\frac{1}{(x+2)^2}$	[M1 combining fractions together and equating the numerator] [M1 for substitution or any other method] [A2 for fully correct partial fractions. A1 only if at least two terms are correct]
6(ii)	Hence, find $\int \frac{7x+11}{2(x-1)(x+2)^2}dx$.	[3]
	$\int \frac{7x+11}{2(x-1)(x+2)^2}dx=\frac{1}{2}\int \frac{2}{x-1}-\frac{2}{x+2}+\frac{1}{(x+2)^2}dx$ $=\ln(x-1)-\ln(x+2)-\frac{1}{2(x+2)}+c$	M1 for taking out coefficient A2 for all correct, A1 if at least two terms are correct
7	A piece of wire which has a fixed length of k cm long is bent to form a rectangle. Show that the area of the rectangle is a maximum when it is a square.	[5]
	Let the length of the rectangle be x. Breadth of rectangle = $\frac{k-2x}{2}$ Let area of the rectangle be A.	[M1]

	$A = x\left(\frac{k-2x}{2}\right)$ $A = \frac{xk-2x^2}{2}$ $A = \frac{1}{2}xk - x^2$ $\frac{dA}{dx} = \frac{1}{2}k - 2x$ At the stationary value, $\frac{dA}{dx} = 0$ $\frac{1}{2}k - 2x = 0$ $2x = \frac{k}{2}$ $x = \frac{k}{4}$ $\frac{d^2A}{dx^2} = -2 < 0$ Therefore, since the stationary value occurs when the sides of the rectangle are $\frac{k}{4}$ cm, and it is a maximum value, the maximum area of the rectangle occurs when it is a square.	[M1] [M1] [M1] [A1]
8	Given a circle with the equation $(2x+5)(x+2)+(2y+1)(y-5)=0$,	
8(i)	Express the equation of the circle in standard form.	[5]
	$(2x+5)(x+2)+(2y+1)(y-5)=0$ $2x^2+9x+10+2y^2-9y-5=0$ $2x^2+2y^2+9x-9y+5=0$ $x^2+y^2+\frac{9}{2}x-\frac{9}{2}y+\frac{5}{2}=0$	M1 for expansion and simplification M1 – for getting the centre and radius

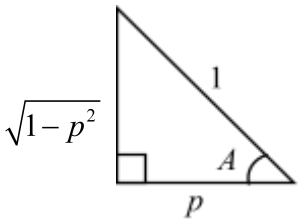
	$x^2+\frac{9}{2}x+\left(\frac{9}{4}\right)^2+y^2-\frac{9}{2}x+\left(\frac{9}{4}\right)^2=-\frac{5}{2}+\left(\frac{9}{4}\right)^2+\left(\frac{9}{4}\right)^2$	[A1]
	$\left(x+\frac{9}{4}\right)^2+\left(y-\frac{9}{4}\right)^2=\frac{61}{8}$	[A1]
	Coordinates of centre = $\left(-\frac{9}{4},\frac{9}{4}\right)$	[A1]
	Radius of circle = $\sqrt{\frac{61}{8}}$ units	
	Equation of circle, $\left(x+\frac{9}{4}\right)^2+\left(y-\frac{9}{4}\right)^2=\frac{61}{8}$	
8(ii)	Find the length of the chord when the line $y=-2x$ cuts the circle.	[5]
	$y=-2x$ -(1)	
	$\left(x+\frac{9}{4}\right)^2+\left(y-\frac{9}{4}\right)^2=\frac{61}{8}$ -(2)	
	Sub (1) into (2)	[M1 for substitution]
	$\left(x+\frac{9}{4}\right)^2+\left(-2x-\frac{9}{4}\right)^2=\frac{61}{8}$	
	$5x^2+\frac{27}{2}x+\frac{5}{2}=0$	[M1 for solving]
	$10x^2+27x+5=0$	
	$(5x+1)(2x+5)=0$	
	$x=-\frac{1}{5}$ *or* $x=-\frac{5}{2}$	
	when $x=-\frac{1}{5}, y=\frac{2}{5}$	[A1 for correct coordinates]
	when $x=-\frac{5}{2}, y=5$	
	Thus the coordinates of the end points of the chord are $\left(-\frac{1}{5},\frac{2}{5}\right)$ and $\left(-\frac{5}{2},5\right)$	[M1]

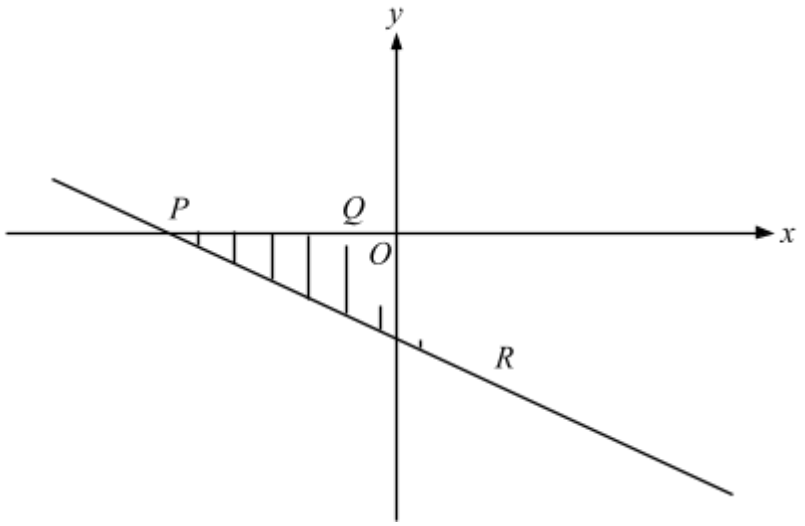
	Length of chord = $\sqrt{\left(2\frac{3}{10}\right)^2+\left(-4\frac{3}{5}\right)^2}$ = 5.14 units (to 3 s.f.)	[A1]
9ai	Prove the identity $\sin x\cos x+\cot x\cos^2 x=\cot x$.	[4]
	LHS = $\sin x\cos x+\cot x\cos^2 x$	
	$=\cos x\left(\sin x+\cot x\cos x\right)$	[M1]
	$=\cos x\left(\sin x+\frac{\cos x}{\sin x}\cos x\right)$	[M1]
	$=\cos x\left(\sin x+\frac{\cos^2 x}{\sin x}\right)$	
	$=\cos x\left(\frac{\sin^2 x+\cos^2 x}{\sin x}\right)$	[M1]
	$=\cos x\left(\frac{1}{\sin x}\right)$	[A1]
	$=\cot x=$ RHS	
	Or	
	LHS = $\sin x\cos x+\cot x\cos^2 x$	[M1]
	$=\sin x\cos x+\frac{\cos x}{\sin x}\cos^2 x$	
	$=\frac{\sin^2 x\cos x}{\sin x}+\frac{\cos^3 x}{\sin x}$	
	$=\frac{\sin^2 x\cos x+\cos^3 x}{\sin x}$	[M1]
	$=\frac{\cos x\left(\sin^2 x+\cos^2 x\right)}{\sin x}$	[M1]
	$=\frac{\cos x(1)}{\sin x}$	[A1]
	$=\cot x=$ RH	

9aii	Hence, solve $\sin 3x\cos 3x+\cot 3x\cos^2 3x=1$ for $0\le x\le \pi$.	[3]
	Since $\sin x\cos x+\cot x\cos^2 x=\cot x$ Therefore, $\sin 3x\cos 3x+\cot 3x\cos^2 3x=\cot 3x$ and $\sin 3x\cos 3x+\cot 3x\cos^2 3x=1 \quad\Rightarrow\quad \cot 3x=1$ $0\le x\le \pi$ $0\le 3x\le 3\pi$ Let the basic angle be α $\tan\alpha=1$ $\alpha=\frac{\pi}{4}$ $3x=\frac{\pi}{4},\pi+\frac{\pi}{4},2\pi+\frac{\pi}{4}$ $3x=\frac{\pi}{4},\frac{5\pi}{4},\frac{9\pi}{4}$ $x=\frac{\pi}{12},\frac{5\pi}{12},\frac{3\pi}{4}$ rad	[M1] [M1] [A1]
9bi	On the same axes, sketch the graphs of $y=3\sin x-1$ and $y=\tan\frac{x}{2}$ for $0\le x\le 2\pi$.	[5]
		For tan curve, [B1] for shape, [B1] for asymptote For Sine curve, [B1] for shape [B1 for correct max and minimum values

		[B1] for correct period for both graphs [subtract 1mark for wrong or no labels for axes or functions]
9bii	Hence, state the number of solutions of $3\sin x - 1 = \tan\frac{x}{2}$ for $0 \le x \le 2\pi$.	[1]
	From the sketch, the two functions intersect at three points. Hence there are three solutions for the equation for $0 \le x \le 2\pi$	[B1]
10a	Two particles, *A* and *B*, leaves a point *O* at the same time and travel in the same direction along the same straight line. Particle *A* starts with a velocity of 9 m/s and moves with a constant acceleration of 2 m/s^2. Particle *B* starts from rest and moves with an acceleration of a m/s^2, where $a = 1 + \frac{t}{3}$ and t seconds is the time since leaving *O*. Find an expression for the velocity of each particle in terms of t,	[3]
	For Particle *A*, $v_A = \int 2\, dt$ $v_A = 2t + c$ When $t = 0$, $v_A = 9$ $c = 9$ $v_A = 2t + 9$ For Particle *B*, $v_B = \int \left(1 + \frac{t}{3}\right) dt$ $v_B = t + \frac{1}{6}t^2 + c$ When $t = 0$, $v_B = 0$, $c = 0$ $v_B = t + \frac{1}{6}t^2$	B1 M1 A1
10b	an expression for the displacement of each particle in terms of t,	[3]
	For Particle *A*,	

	$s_A = \int (2t+9)\,dt$ $s_A = t^2 + 9t + c$ When $t = 0$, $s_A = 0$ $c = 0$ $s_A = t^2 + 9t$ For Particle B, $s_B = \int \left(t + \frac{1}{6}t^2 \right) dt$ $s_B = \frac{1}{2}t^2 + \frac{1}{18}t^3 + c$ When $t = 0$, $s_B = 0$, $c = 0$ $s_B = \frac{1}{2}t^2 + \frac{1}{18}t^3$	B1 M1 A1
10c	the distance from O at which particle B collides with A,	[3]
	When particle B collides with particle A, $s_A = s_B$ $t^2 + 9t = \frac{1}{2}t^2 + \frac{1}{18}t^3$ $18t^2 + 162t = 9t^2 + t^3$ $t^3 - 9t^2 - 162t = 0$ $t\left(t^2 - 9t - 162\right) = 0$ $t(t-18)(t+9) = 0$ $t = 0\,(\text{N.A.})$ or $t = 18$ or $t = -9\,(\text{N.A.})$ Distance from $O = (18)^2 + 9(18) = 486$ m	M1 M1 A1
10d	the speed of each particle at the point of collision.	[2]
	Speed of particle $A = 2(18) + 9 = 45$ m/s Speed of particle $B = (18) + \frac{1}{6}(18)^2 = 72$ m/s	B1 B1

11	Given that $\cos A = p$ and that A is acute, express the following in terms of p.	
i	$\sin 2A$	[3]
	 $\sin 2A = 2\sin A\cos A$ $= 2p\sqrt{1-p^2}$ Or $\cos^2 A + \sin^2 A = 1$ $\sin^2 A = 1 - \cos^2 A$ $\sin^2 A = 1 - p^2$ $\sin A = \sqrt{1-p^2}$ (reject negative as A is acute) $\sin 2A = 2\sin A\cos A$ $= 2p\sqrt{1-p^2}$	M1 for getting the length of opposite side M1 A1 M1 M1 A1
ii	$\cos\frac{A}{2}$	[3]
	$\cos A = 2\cos^2\frac{A}{2} - 1$ $\cos\frac{A}{2} = \pm\sqrt{\frac{\cos A + 1}{2}}$ $\cos\frac{A}{2} = \pm\sqrt{\frac{p+1}{2}}$ $\cos\frac{A}{2} = -\sqrt{\frac{p+1}{2}}$ (rejected) or $\cos\frac{A}{2} = \sqrt{\frac{p+1}{2}}$	M1 M1 for rejection A1
12	The diagram shows the curve, $y = 4e^{-2x} - 10$. The curve crosses the $x-$ axis at Q. The line PR is a tangent to the curve at R and intersects the $x-$ axis at P. The $x-$ coordinate of R is $\ln 2$.	[11]

	Find the area of the shaded region, PQR , which is the region enclosed by curve, the $x-$ axis and the line $x-$ axis and the line PR correct to 3 significant figures.  	
	At R , $x = \ln 2$ $y = 4e^{-2\ln 2} - 10 = -9$ $R(\ln 2, -9)$	B1
	$y = 4e^{-2x} - 10$ $\frac{dy}{dx} = 4(-2)e^{-2x} = -8e^{-2x}$ Gradient of line $PR = -8e^{-2\ln 2} = -2$ Equation of line PR $y-(-9) = -2(x - \ln 2)$ $y = -2x + 2\ln 2 - 9$ $y = -2x - 7.6137$	M1 A1
	At P , $y = 0$ $0 = -2x + 2\ln 2 - 9$ $x = \ln 2 - \frac{9}{2}$ $= -3.8069$ $P\left(\ln 2 - \frac{9}{2}, 0\right)$	B1
	At Q , $y = 0$	

	$0 = 4e^{-2x} - 10$ $4e^{-2x} = 10$ $e^{-2x} = \frac{5}{2}$ $-2x = \ln\frac{5}{2}$ $x = -\frac{1}{2}\ln\frac{5}{2} = -0.45815$ $Q\left(-\frac{1}{2}\ln\frac{5}{2}, 0\right)$	M1 – change index form to log form A1
	Area of the shaded region, PQR = Area of the triangle – Area between the curve and the $x-$ axis $\text{Area} = \frac{1}{2}\times\left(\left(\ln 2\right)-\left(\ln 2-\frac{9}{2}\right)\right)\times(9)-\left\lvert\int_{-\frac{1}{2}\ln\frac{5}{2}}^{\ln 2}\left(4e^{-2x}-10\right)dx\right\rvert$ $= \frac{81}{4}-\left\lvert\left[-2e^{-2x}-10x\right]_{-\frac{1}{2}\ln\frac{5}{2}}^{\ln 2}\right\rvert$ $= \frac{81}{4}-\left\lvert\left(\left[-2e^{-2\ln 2}-10\ln 2\right]-\left[-2e^{-2\left(-\frac{1}{2}\ln\frac{5}{2}\right)}-10\left(-\frac{1}{2}\ln\frac{5}{2}\right)\right]\right)\right\rvert$ $= \frac{81}{4}-\lvert -7.01295\rvert$ $= 13.23705 = 13.2 \text{ units}^2$	M1 M1 M1 M1 A1