Lesson 15:
Coordinate Systems

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Standards

This lesson makes a tactile model of 3D Cartesian and cylindrical coordinate systems. We see this as a model that supports many other learning objectives at a wide range of levels. Here are examples from 5th and 8th grade and high school CCSS standards.

- CCSS.Math.Content.5.G.A.1
    - "Graph points on the coordinate plane to solve real-world and mathematical problems. Use a pair of perpendicular number lines, called axes, to define a coordinate system, with the intersection of the lines (the origin) arranged to coincide with the 0 on each line and a given point in the plane located by using an ordered pair of numbers, called its coordinates. Understand that the first number indicates how far to travel from the origin in the direction of one axis, and the second number indicates how far to travel in the direction of the second axis, with the convention that the names of the two axes and the coordinates correspond (e.g., *x*-axis and *x*-coordinate, *y*-axis and *y*-coordinate)."
- CCSS.Math.Content.8.G.A.1
    - "Verify experimentally the properties of rotations, reflections, and translations"
- CCSS.Math.Content.HSG.CO.B.6
    - "Understand congruence in terms of rigid motions. Use geometric descriptions of rigid motions to transform figures and to predict the effect of a given rigid motion on a given figure; given two figures, use the definition of congruence in terms of rigid motions to decide if they are congruent."

Objectives

- Understand the elements of the Cartesian coordinate system (axes, origin, finding points in space).
- Be able to discuss and demonstrate rotation about an axis.
- Understand the equivalent elements of the polar (2D) and cylindrical (3D) coordinate systems.

Models Used

You can find the model for this Lesson at https://github.com/whosawhatsis/Geometry. See the Teacher's Guide for how to use Github.

- axes.scad
  - This model creates various coordinate planes, and a set of coordinate axes.

Coordinate Systems

Coordinate systems are the link that ties together algebra and geometry. Legend has it that René Descartes came up with the idea of coordinates in the early 1600s when he saw a fly wandering around on his window. The window was made up of small panes, and thus he made the leap to a grid. Be that as it may, Cartesian coordinates came into being around that time.

This lesson can be thought of as a supplemental lesson to all the others. If students are having trouble thinking about 3D coordinate systems, or visualizing what it means to rotate about an axis, this model can perhaps fill in those gaps. We cover Cartesian (2D and 3D), polar, and cylindrical coordinates.

The Model

This model prints three axes. Two of the axes end in an arrow. We will call the axes ending in these arrows the x and y axes. We will assume in our descriptions that the x and y axes are in the plane of a table. The x axis arrow points to the right, and the y-axis arrow points away from you. The third axis, which does not end in an arrow so that we can slip parts on and off it, is the z-axis. It is oriented toward the ceiling.

Model Parameters

The model has these parameters (lengths in mm, angles in degrees):

- radius = 100;
  - radius of the base. This also controls how far the grids extend in the x, y and z directions.
- base = 2;
  - thickness of the base part
- axes = 90;
  - length of the xy axes
- xythick = 4;

  - width of the xy axes
- zthick = 5;
  - width of the z axis
- zlength = 20;
  - length of the z axis
- angle = 90;
  - degrees of the wedge on the protractor
- clearance = .3;
  - clearance between connecting parts
- tube = 20;
  - length of the z axis connector
- tubewall = 1.2;
  - radial wall thickness of the connector tube
- line = 2;
  - thickness of lines on grids

Model Pieces

This model makes five different pieces, one at a time. You'll need to uncomment (remove the "//") from one line to make each of the elements. Run the OpenSCAD model five times, uncommenting a different line each time, to make all the pieces. Only remove the comments from ONE of these lines at a time. Otherwise you may get overlapping prints. Also, to make a set, leave all the other parameters we've just discussed the same from model to model. Otherwise, the pieces may not line up. Here are the lines to uncomment for each respective piece:

- Uncomment this line to get the x-y axes.
  - translate(-(xythick * 2.5 + zthick) * [1, 1, 0]) xyaxes();
- Uncomment this line to get a vertical grid.
  - translate(-(xythick * 2.5 + zthick) * [1, 1, 0]) rotate(angle) zgrid();
- Uncomment this line to get the protractor-style wedge, used to learn about rotating around an axis.
  - protractor();
- Uncomment this line to get the x-y plane grid.
  - xygrid();
- Uncomment this line to get the polar wedge grid.
  - polargrid();

The discussion that follows notes which model pieces to print for each activity.

Cartesian Coordinates

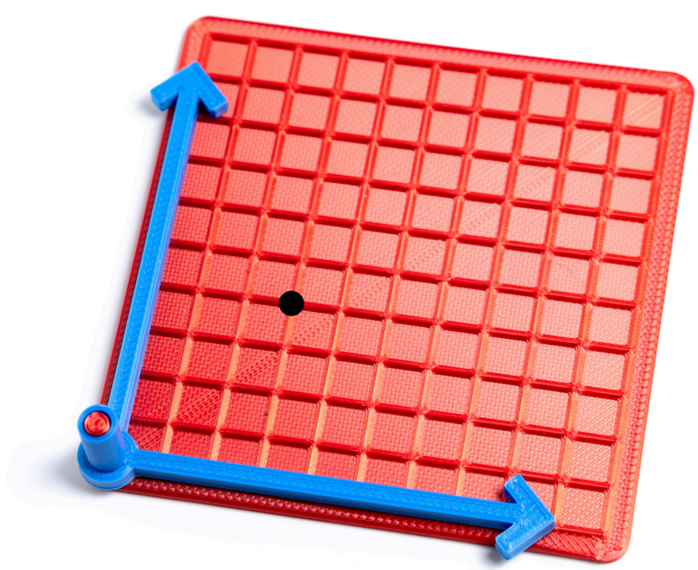
Cartesian coordinates use three axes that are at right angles to each other, called x, y and z. First we will use our models to find locations in 2D and 3D space, and then explore rotations around the z axis.

2D: Finding a point in x, y space

Print two models: the x-y plane and the x-y axes.

First, let's find a point in 2D space. First, take the x-y plane model piece (which has a grid and a piece sticking out of it). Let the point where the rod is sticking out be the (0,0) point -- the origin. Slide on the piece with the two axes.

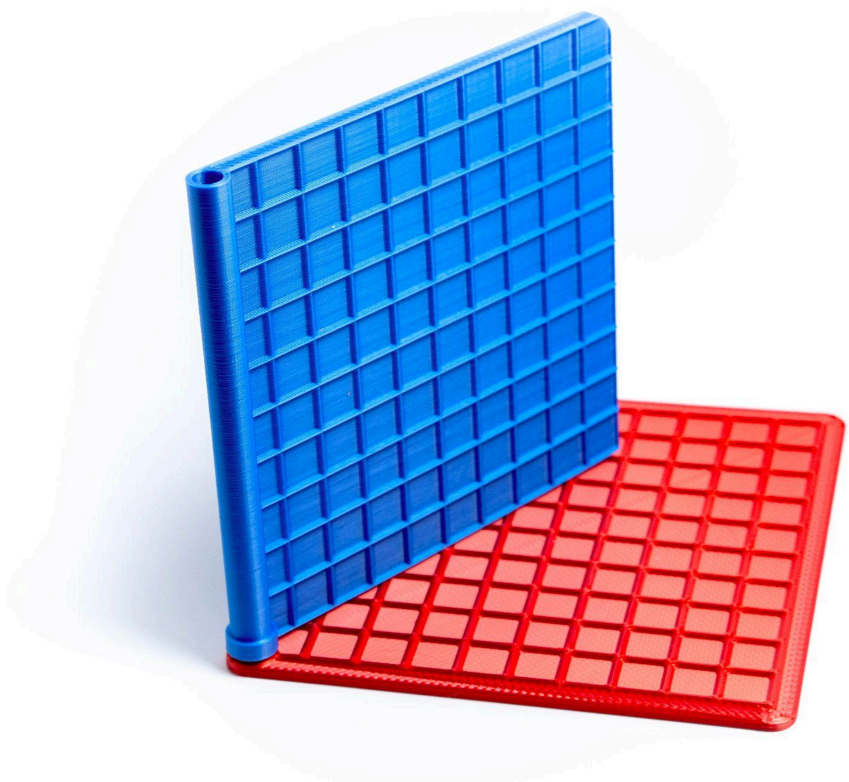
The x-axis should now be pointing to the right, and the y-axis away from you. To find the point (3, 4) count three points right of the origin and four points up. (This is shown by a dot in the following figure.)



3D: Finding a point in x, y, z space

Print two models: the x-y plane and the vertical plane

Leave the x-y plane where it was, but remove the x-y axes. Attach the second (vertical) plane by sliding the hollow part it has on one side over the rod at the origin of the x-y plane. As in the previous section, find the point x = 3, y = 4. Now swing the vertical plane so that it crosses over that point.



Let’s say we wanted to find the point (3, 4, 5). We would then count up 5 marks vertically.

Assessments

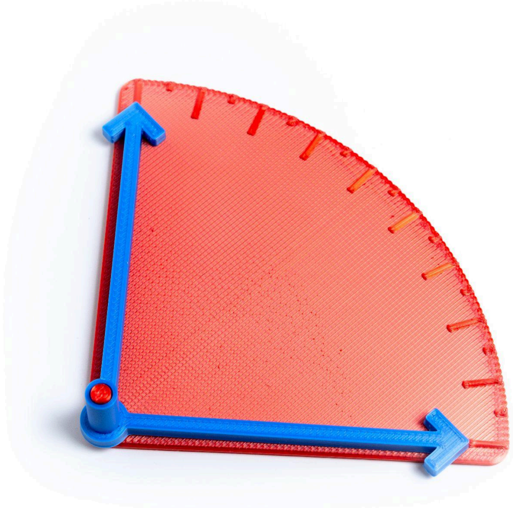
- Have students identify the origin and explain that it is the point (0, 0, 0).
- Have students find an arbitrary point in space using the method described. (Note that having integer x and y coordinates won't necessarily line up with integer lines on the z grid. Consider having students use the Pythagorean theorem to think about why.)
- Be able to point to the region of space where various combinations of x, y and/or z are negative.
- Bonus: the distance to that point from the origin is $\sqrt{x^2 + y^2 + z^2}$, by the Pythagorean theorem. First, you would find the distance in the x-y plane by finding the distance $\sqrt{x^2 + y^2}$. Then using that distance as one side of a right triangle and the vertical distance as the other, find the distance from the origin. Calculate what the distance should be, then use a piece of string to measure from the origin to the relevant point in space. Measure the length of the string and see how close you are. (You won't be able to measure the distance to the origin perfectly, since the pivot is occupying that space.)

Rotation

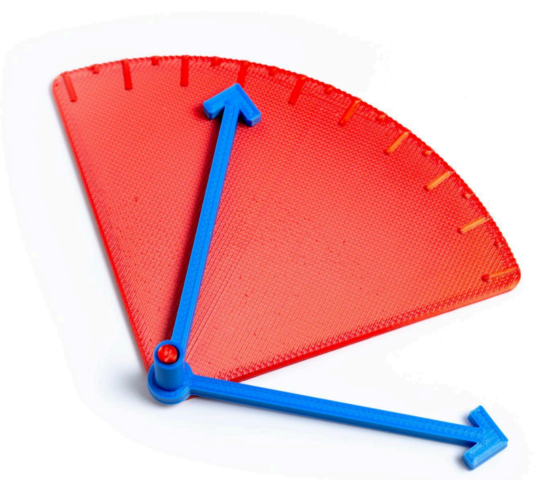
Next, let's explore what it means to rotate around an axis. You can rotate around any axis, of course, but this model allows you to rotate about the z axis.

Print two models: the x-y axes, and the protractor.

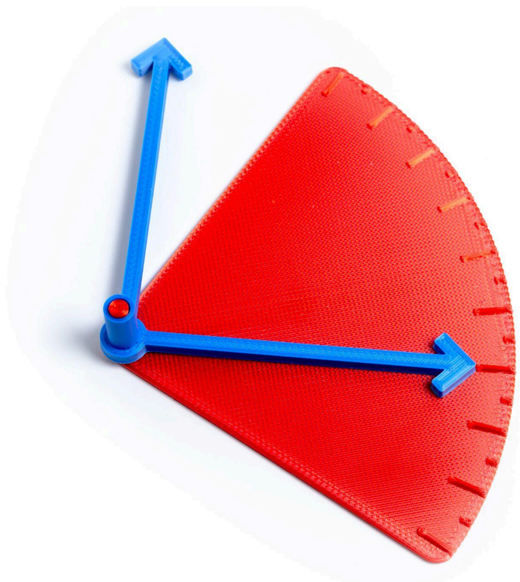
Start by lining up the 0 and 90 degree lines on the wedge with the x and y axes. We will call the line parallel to the x axis the 0 degree mark here.



Now, hold the axes still, and turn the wedge counterclockwise until the y axis is on the 60-degree mark. The angle between the x and y axis is 30 degrees. The wedge has been rotated positive 30 degrees.



Now turn the wedge the other way, so that the x axis is on the 30-degree mark. Now the wedge has turned -30 degrees from its starting point.



Assessments

- Have students turn the wedge positive or negative specified amounts. Note that the longer lines on the flat surface are in 10-degree increments and the shorter ones are in 5-degree increments. The little notches around the outside rim are 1-degree notches.
- Have students indicate, by reorienting part of the model, how they would show rotating about the x or y axis as opposed to the z-axis.
- Have students demonstrate that a LEGO brick or similar object does not change shape as it is translated or rotated.

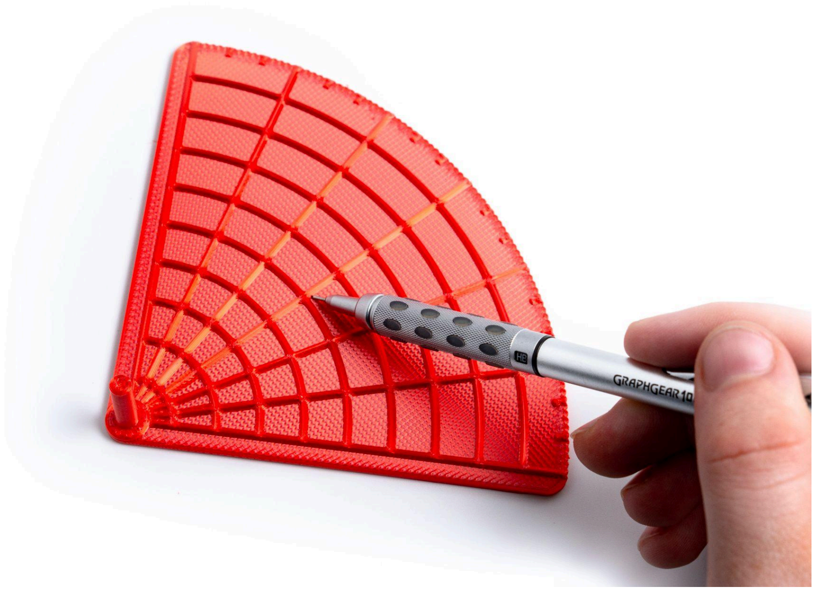
Polar and Cylindrical coordinates

Print out the alternative wedge shape, which has concentric circles and radial lines on it. The concentric circles measure the radial distance from the origin.

2D polar coordinates

Print one model: the polargrid piece.

The polar grid piece has a grid of radial and concentric lines. Align it so that one straight side is closest to you and one is on the left. In polar coordinates, we count out the number of radial marks from the origin (the rod sticking up) to find the value of the r-coordinate. We count radial lines starting at the zero line closest to you to find the angle, usually called theta. The long radial lines on this model are in 15-degree increments, and again there are shorter 5-degree lines and 1-degree notches along the edge. If we count four concentric lines from the center and three radial lines from the flat side nearest you, that is radius = 5 and theta = 15 * 3 = 45 degrees.



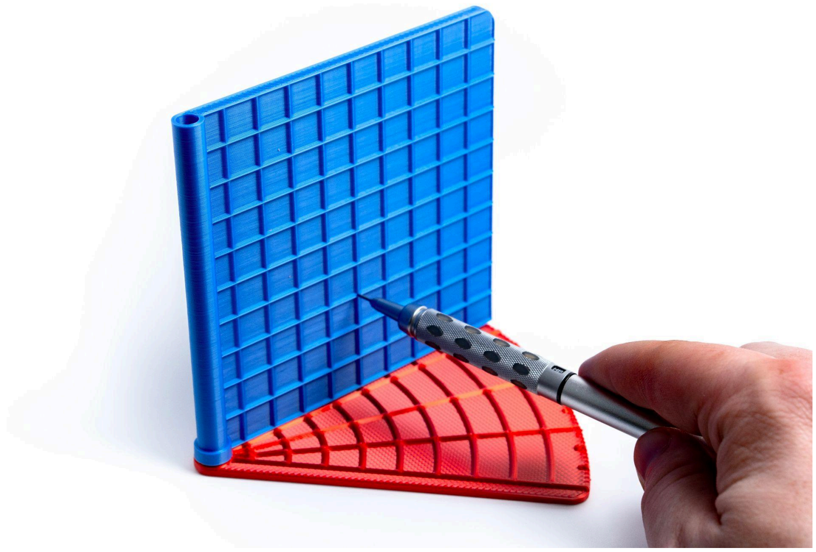
3D: Cylindrical coordinates

Print two models: the polar grid piece and the vertical grid piece.

What about 3D in this system? If we have polar coordinates in the plane of our table, we can rise straight above it with a z axis, like we did in Cartesian coordinates. This is then called cylindrical coordinates, since lines of constant r, theta and z will collectively draw a cylinder.

You can try the same exercise as we did for Cartesian coordinates to mark a place on the r, theta plane and then find a height z above it. Take the wedge-shaped piece you used for the 2D polar coordinates, plus the vertical piece you used in 3D Cartesian coordinates.

Use the same vertical grid as before to now find a point in terms of (r, theta,z). For example, to find the point (r = 5, theta = 45 degrees, z = 3) we would count 4 circles out from the origin, three radial lines up from the line closest to you, and three marks up the vertical grid.



Note that rotation about the z axis is handled by the theta coordinate. However rotation about axes other than z is tricky to write down.

Assessments

- Have students identify the origin and explain that it is the point (0, 0, 0).
- Have students find an arbitrary point in 2D and 3D space in terms of (r, theta) in 2D and (r, theta, z) in 3D.
- Bonus: the distance to that point from the origin is $\sqrt{r^2 + z^2}$, by the Pythagorean theorem (applied in the x-z or x-y plane). Calculate what the distance should be, then use a piece of string to measure from the origin to the relevant point in space. Measure the length of the string and see how close you are.

If You Don't Have a 3D Printer

As always, a public library or makerspace might be able to create this model for you if you have no access to a printer yourself. Failing that, you could use three straws stuck into some modeling clay for the axes, and use a protractor instead of the wedge.

Where to read more

- For Cartesian coordinates:*Cartesian Coordinates* (Wikipedia)
- For polar coordinates: *Polar coordinates* (Wikipedia)
- For cylindrical coordinates: *Cylindrical coordinates* (Wikipedia)
- For the big picture on geometry: *Geometry* (Wikipedia)