Linear Algebra
Lesson 21
Eigenvalues and Characteristic Polynomials

Be sure to put your notes and homework in a document:
MAT313F21-lesson21-lastname-firstname
If you have a question email me with QUESTION in the subject line.

Watch the [Playlist 313F20-Lesson21] pausing to do classwork and homework.

Lesson 21

I Eigenvalues and

II the Characteristic Polynomial

<u>Defn</u> Given a $n \times n$ matrix A we say a number λ is an eigenvalue for A if $\exists \vec{v} \in \mathbb{R}^n$ with $\vec{v} \neq 0$ s.t $A\vec{v} = \lambda \vec{v}$

We say $\vec{v}$ is the eigenvector for eigenvalue λ

20% Linear Algebra 2

Example $A = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$

Notice

$$\begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2\cdot 1 + 0\cdot 0 \\ 0\cdot 1 + 4\cdot 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

So 2 is an eigenvalue with eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2\cdot 0 + 0\cdot 1 \\ 0\cdot 0 + 4\cdot 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} = 4\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

So 4 is an eigenvalue with eigenvector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

1:43 AM Fri Oct 30 19%

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Example ESMS4 Eigenvalues, symmetric matrix of size 4
Consider the matrix

$$C = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

HW1

Check which of the following are eigenvectors

$$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$$

Do this before continuing!

Example ESMS4 Eigenvalues, symmetric matrix of size 4
Consider the matrix

$$C = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

Check which of the following are eigenvectors and find their eigenvalues:

$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$ $\begin{pmatrix} 1&0&1&1 \\ 0&1&1&1 \\ 1&1&1&0 \\ 1&1&0&1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1+0+3+4 \\ 0+2+3+4 \\ 1+2+3+0 \\ 1+2+0+4 \end{pmatrix} = \begin{pmatrix} 8 \\ 9 \\ 6 \\ 7 \end{pmatrix} \neq \lambda \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$

$8 = \lambda 1 \Rightarrow \lambda = 8$

$9 = 8 \cdot 2$ ⊗

Not an eigenvector

$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ $\begin{pmatrix} 1&0&1&1 \\ 0&1&1&1 \\ 1&1&1&0 \\ 1&1&0&1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1+0+1+1 \\ 0+1+1+1 \\ 1+1+1+0 \\ 1+1+0+1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 3 \\ 3 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ Yes is an evector with evalue $\lambda = 3$

$\begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \end{pmatrix}$ $\begin{pmatrix} 1&0&1&1 \\ 0&1&1&1 \\ 1&1&1&0 \\ 1&1&0&1 \end{pmatrix}\begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2+0+2+2 \\ 0+2+2+2 \\ 2+2+2+0 \\ 2+2+0+2 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \\ 6 \\ 6 \end{pmatrix} = 3\begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \end{pmatrix}$ Also an evector with evalue $\lambda = 3$

$\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}$

$\begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$

Thm If $\vec{v}$ is an evector for A with evalue λ $A\vec{v} = \lambda\vec{v}$ then any multiple of $\vec{v}$, say $\vec{w} = k\vec{v}$ for $k \neq 0$ is also an evector with evalue λ.

Proof $A\vec{w} = A(k\vec{v}) = kA\vec{v} = k\lambda\vec{v} = \lambda k\vec{v} = \lambda\vec{w}$ □

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Example ESMS4 Eigenvalues, symmetric matrix of size 4

Consider the matrix

$$C = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

Check which of the following are eigenvectors and find their eigenvalues:

$$\begin{pmatrix}1\\2\\3\\4\end{pmatrix} \quad \begin{pmatrix}1&0&1&1\\0&1&1&1\\1&1&1&0\\1&1&0&1\end{pmatrix}\begin{pmatrix}1\\2\\3\\4\end{pmatrix} = \begin{pmatrix}1+0+3+4\\0+2+3+4\\1+2+3+0\\1+2+0+4\end{pmatrix} = \begin{pmatrix}8\\9\\6\\7\end{pmatrix} \neq \lambda\begin{pmatrix}1\\2\\3\\4\end{pmatrix}$$

$8 = \lambda 1 \Rightarrow \lambda = 8$

$9 = 8 \cdot 2$ ⊗

Not an eigenvector

$$\begin{pmatrix}1\\1\\1\\1\end{pmatrix} \quad \begin{pmatrix}1&0&1&1\\0&1&1&1\\1&1&1&0\\1&1&0&1\end{pmatrix}\begin{pmatrix}1\\1\\1\\1\end{pmatrix} = \begin{pmatrix}1+0+1+1\\0+1+1+1\\1+1+1+0\\1+1+0+1\end{pmatrix} = \begin{pmatrix}3\\3\\3\\3\end{pmatrix} = 3\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$$

Yes is an evector with evalue $\lambda = 3$

$$\begin{pmatrix}2\\2\\2\\2\end{pmatrix} \quad \begin{pmatrix}1&0&1&1\\0&1&1&1\\1&1&1&0\\1&1&0&1\end{pmatrix}\begin{pmatrix}2\\2\\2\\2\end{pmatrix} = \begin{pmatrix}2+0+2+2\\0+2+2+2\\2+2+2+0\\2+2+0+2\end{pmatrix} = \begin{pmatrix}6\\6\\6\\6\end{pmatrix} = 3\begin{pmatrix}2\\2\\2\\2\end{pmatrix}$$

Also an evector with evalue $\lambda = 3$

$$\begin{pmatrix}-1\\1\\0\\0\end{pmatrix} \quad \begin{pmatrix}1&0&1&1\\0&1&1&1\\1&1&1&0\\1&1&0&1\end{pmatrix}\begin{pmatrix}-1\\1\\0\\0\end{pmatrix} = \begin{pmatrix}-1+0+0+0\\0+1+0+0\\-1+1+0+0\\-1+1+0+0\end{pmatrix} = \begin{pmatrix}-1\\1\\0\\0\end{pmatrix} = 1\begin{pmatrix}-1\\1\\0\\0\end{pmatrix}$$

So this is an evector with evalue $\lambda = 1$

$$\begin{pmatrix}0\\0\\-1\\1\end{pmatrix} \quad \begin{pmatrix}1&0&1&1\\0&1&1&1\\1&1&1&0\\1&1&0&1\end{pmatrix}\begin{pmatrix}0\\0\\-1\\1\end{pmatrix} = \begin{pmatrix}0+0-1+1\\0+0-1+1\\0+0-1+0\\0+0+0+1\end{pmatrix} = \begin{pmatrix}0\\0\\-1\\1\end{pmatrix} = 1\begin{pmatrix}0\\0\\-1\\1\end{pmatrix}$$

So also an evector with $\lambda = 1$

$$\begin{pmatrix}-1\\-1\\1\\1\end{pmatrix} \quad \begin{pmatrix}1&0&1&1\\0&1&1&1\\1&1&1&0\\1&1&0&1\end{pmatrix}\begin{pmatrix}-1\\-1\\1\\1\end{pmatrix} = \begin{pmatrix}-1+0+1+1\\0-1+1+1\\-1-1+1+0\\-1-1+0+1\end{pmatrix} = \begin{pmatrix}1\\1\\-1\\-1\end{pmatrix} = -1\begin{pmatrix}-1\\-1\\1\\1\end{pmatrix}$$

So yes an evector with evalue $\lambda = -1$

2:32 AM Fri Oct 30

Linear Algebra 2

How to find eigenvectors and eigenvalues?

$$A\vec{v} = \lambda\vec{v}$$ solve for $\vec{v} \neq 0$

$$A\vec{v} = \lambda I \vec{v}$$

$$A\vec{v} - \lambda I \vec{v} = \vec{0}$$

$$(A - \lambda I)\vec{v} = \vec{0}$$

Solve this homogeneous system for $\vec{v} \neq 0$

Need $A - \lambda I$ to be singular

$$\det(A - \lambda I) = 0$$

Linear Algebra 2

2×2 case

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\det\begin{pmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{pmatrix} = 0$$

$$(a_{11}-\lambda)(a_{22}-\lambda) - a_{12}a_{21} = 0$$

How to find eigenvectors and eigenvalues?

$$A\vec{v} = \lambda \vec{v}$$

solve for $\vec{v} \neq 0$

$$A\vec{v} = \lambda I \vec{v}$$

$$A\vec{v} - \lambda I \vec{v} = \vec{0}$$

$$(A-\lambda I)\vec{v} = \vec{0}$$

Solve this homogeneous system for $\vec{v} \neq 0$

Need $A-\lambda I$ to be singular

$$\det(A-\lambda I) = 0$$

Linear Algebra 2

$$\underbrace{\det(A-\lambda I)}_{\text{characteristic polynomial for the matrix } A} = 0$$

Solve for its roots.

$\lambda = \underline{\qquad\qquad}$ a list of roots

these answers are the eigenvalues.

For each eigenvalue solve the homogeneous system to find its eigenvectors.

2:42 AM Fri Oct 30

Linear Algebra 2

$$\underbrace{\det(A-\lambda I)}_{\text{characteristic polynomial for the matrix } A}=0$$

Solve for its roots.

$\lambda =$ ______ a list of roots

these answers are the eigenvalues.

For each eigenvalue solve the homogeneous system to find its eigenvectors.

Linear Algebra 2

Example $\begin{pmatrix} 2 & -3 \\ 1 & 6 \end{pmatrix}$

$$\det\begin{pmatrix} 2-\lambda & -3 \\ 1 & 6-\lambda \end{pmatrix} = (2-\lambda)(6-\lambda)-(-3)(1)=0$$

$$= 12-6\lambda-2\lambda+\lambda^2+3$$

$$= \lambda^2-8\lambda+15$$

$$A-\lambda I = \underbrace{\begin{pmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{pmatrix}}_{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}}$$

Example $\begin{pmatrix}2 & -3\\ 1 & 6\end{pmatrix}$

$$\det\begin{pmatrix}2-\lambda & -3\\ 1 & 6-\lambda\end{pmatrix} = (2-\lambda)(6-\lambda)-(-3)(1)=0$$
$$= 12-6\lambda-2\lambda+\lambda^2+3$$
$$= \lambda^2-8\lambda+15=0$$
$$(\lambda-3)(\lambda-5)=0$$

These are our eigenvalues → $\lambda=3 \quad \lambda=5$

$\lambda=3$ Solve the homog system

$$\left(\begin{array}{cc|c}2-3 & -3 & 0\\ 1 & 6-3 & 0\end{array}\right)=\left(\begin{array}{cc|c}-1 & -3 & 0\\ 1 & 3 & 0\end{array}\right)$$

$$\xrightarrow{\rho_1\to-\rho_1}\left(\begin{array}{cc|c}1 & 3 & 0\\ 1 & 3 & 0\end{array}\right)\xrightarrow{\rho_2\to\rho_2-\rho_1}\left(\begin{array}{cc|c}1 & 3 & 0\\ 0 & 0 & 0\end{array}\right)$$

$1x_1+3x_2=0$
$x_1=-3x_2$
$x_2=x_2$ (free)

$$\left\{\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\begin{pmatrix}-3x_2\\ x_2\end{pmatrix}=x_2\begin{pmatrix}-3\\ 1\end{pmatrix}\ \middle|\ x_2\in\mathbb{R}\right\}$$

$\begin{pmatrix}-3\\ 1\end{pmatrix}$ is the evector

Check our evector $\begin{pmatrix}-3\\ 1\end{pmatrix}$

$$\begin{pmatrix}2 & -3\\ 1 & 6\end{pmatrix}\begin{pmatrix}-3\\ 1\end{pmatrix}=\begin{pmatrix}-6-3\\ -3+6\end{pmatrix}=\begin{pmatrix}-9\\ 3\end{pmatrix}$$

$\lambda\begin{pmatrix}-3\\ 1\end{pmatrix}=3\begin{pmatrix}-3\\ 1\end{pmatrix}=\begin{pmatrix}-9\\ 3\end{pmatrix}$ ← Match!

$$\begin{pmatrix}2 & -3\\ 1 & 6\end{pmatrix}\begin{pmatrix}-3\\ 1\end{pmatrix}=3\begin{pmatrix}-3\\ 1\end{pmatrix}$$

any multiple of $\begin{pmatrix}-3\\ 1\end{pmatrix}$ is also an evector with evalue 3.

Try classwork

$\lambda=5$

pause

2:52 AM Fri Oct 30 — Linear Algebra 2

Example $\begin{pmatrix} 2 & -3 \\ 1 & 6 \end{pmatrix}$

$\det\begin{pmatrix} 2-\lambda & -3 \\ 1 & 6-\lambda \end{pmatrix} = (2-\lambda)(6-\lambda)-(-3)(1)=0$

$= 12-6\lambda-2\lambda+\lambda^2+3$

$= \lambda^2-8\lambda+15=0$

$(\lambda-3)(\lambda-5)=0$

These are our evalues

$\lambda=3 \quad \lambda=5$

$\lambda=3$ Solve the homog system

$\left(\begin{array}{cc|c} 2-3 & -3 & 0 \\ 1 & 6-3 & 0 \end{array}\right) = \left(\begin{array}{cc|c} -1 & -3 & 0 \\ 1 & 3 & 0 \end{array}\right)$

$\xrightarrow{\rho_1\to-\rho_1} \left(\begin{array}{cc|c} 1 & 3 & 0 \\ 1 & 3 & 0 \end{array}\right) \xrightarrow{\rho_2\to\rho_2-\rho_1} \left(\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$1x_1+3x_2=0$

$x_1=-3x_2$

$x_2=x_2$ (free)

$\left\{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -3x_2 \\ x_2 \end{pmatrix} = x_2\begin{pmatrix} -3 \\ 1 \end{pmatrix} \;\middle|\; x_2\in\mathbb{R}\right\}$

$\begin{pmatrix} -3 \\ 1 \end{pmatrix}$ is the evector

Check $\begin{pmatrix} 2 & -3 \\ 1 & 6 \end{pmatrix}\begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{pmatrix} -9 \\ 3 \end{pmatrix}$ ✓ $3\begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{pmatrix} -9 \\ 3 \end{pmatrix}$ ✓

2:52 AM Fri Oct 30 — Linear Algebra 2

$\lambda=5$ Solve homog system

$\left(\begin{array}{cc|c} 2-5 & -3 & 0 \\ 1 & 6-5 & 0 \end{array}\right) = \left(\begin{array}{cc|c} -3 & -3 & 0 \\ 1 & 1 & 0 \end{array}\right)$

$\xrightarrow{\rho_1\to\frac{-1}{3}\rho_1} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right) \xrightarrow{\rho_2\to\rho_2-\rho_1} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$1x_1+1x_2=0 \quad x_1=-x_2$

$x_2=x_2$ (free)

$\left\{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_2 \end{pmatrix} = x_2\begin{pmatrix} -1 \\ 1 \end{pmatrix} \;\middle|\; x_2\in\mathbb{R}\right\}$

evector is $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Check

$\begin{pmatrix} 2 & -3 \\ 1 & 6 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2-3 \\ -1+6 \end{pmatrix} = \begin{pmatrix} -5 \\ 5 \end{pmatrix}$

$5\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -5 \\ 5 \end{pmatrix}$ ← Match!

Example ESMS4 Eigenvalues, symmetric matrix of size 4
Consider the matrix

$$C = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

Example ESMS4 Eigenvalues, symmetric matrix of size 4
Consider the matrix

$$C = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

$$\det(C - \lambda I) =$$

$$\begin{pmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{pmatrix}$$

$$= \det \begin{pmatrix} 1-\lambda & 0 & 1 & 1 \\ 0 & 1-\lambda & 1 & 1 \\ 1 & 1 & 1-\lambda & 0 \\ 1 & 1 & 0 & 1-\lambda \end{pmatrix}$$

$$= (1-\lambda)\det\begin{pmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{pmatrix} - 0 + 1\det\begin{pmatrix} 0 & 1 & 1 \\ 1-\lambda & 1 & 1 \\ 1 & 0 & 1-\lambda \end{pmatrix} - 1\det\begin{pmatrix} 0 & 1 & 1 \\ 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 0 \end{pmatrix}$$

$$= (1-\lambda)\left((1-\lambda)\underline{\det\begin{pmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{pmatrix}} - 1\,\underline{\det\begin{pmatrix} 1 & 1 \\ 0 & 1-\lambda \end{pmatrix}} + 1\,\underline{\det\begin{pmatrix} 1 & 1 \\ 1-\lambda & 0 \end{pmatrix}}\right)$$

$$- 0 + 1\left(0 - (1-\lambda)\det\begin{pmatrix} 1 & 1 \\ 0 & 1-\lambda \end{pmatrix} + 1\,\underline{\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}}\right) - 1\left(0 - (1-\lambda)\det\begin{pmatrix} 1 & 1 \\ 1-\lambda & 0 \end{pmatrix} + 1\det\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right)$$

$$= (1-\lambda)\left(\underline{(1-\lambda)(1-\lambda)^2} - \left(1(1-\lambda) - 1\cdot 0\right) + 1\left(\underline{1\cdot 0 - 1(1-\lambda)}\right)\right)$$

$$+ 1\left(-(1-\lambda)\left(1(1-\lambda)\right) + 1\,\overset{0}{\cancel{(1\cdot 1 - 1\cdot 1)}}\right) - 1\left(-(1-\lambda)\left(1\cdot 0 - 1(1-\lambda)\right) + \overset{0}{\cancel{(1\cdot 1 - 1\cdot 1)}}\right)$$

$$= (1-\lambda)\left((1-\lambda)^3 - (1-\lambda) - (1-\lambda)\right) - (1-\lambda)^2 - (1-\lambda)^2 = (1-\lambda)^4 - 4(1-\lambda)^2$$

Do not multiply out.

Example ESMS4 Eigenvalues, symmetric matrix of size 4
Consider the matrix

$$C = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

$\det(C - \lambda I) =$

$$= \det\begin{pmatrix} 1-\lambda & 0 & 1 & 1 \\ 0 & 1-\lambda & 1 & 1 \\ 1 & 1 & 1-\lambda & 0 \\ 1 & 1 & 0 & 1-\lambda \end{pmatrix} = (1-\lambda)^4 - 4(1-\lambda)^2 = (1-\lambda)^2\left((1-\lambda)^2 - 4\right)$$

$$= (1-\lambda)^2(1-2\lambda+\lambda^2-4) = (1-\lambda)^2(\lambda^2-2\lambda-3) = (1-\lambda)^2(\lambda-3)(\lambda+1)$$

Our eigenvalues are 1, -1, 3.

Example ESMS4 Eigenvalues, symmetric matrix of size 4
Consider the matrix

$$C = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

$\det(C - \lambda I) =$

$$= \det\begin{pmatrix} 1-\lambda & 0 & 1 & 1 \\ 0 & 1-\lambda & 1 & 1 \\ 1 & 1 & 1-\lambda & 0 \\ 1 & 1 & 0 & 1-\lambda \end{pmatrix} = (1-\lambda)^4 - 4(1-\lambda)^2 = (1-\lambda)^2\left((1-\lambda)^2 - 4\right)$$

$$= (1-\lambda)^2(1-2\lambda+\lambda^2-4) = (1-\lambda)^2(\lambda^2-2\lambda-3) = (1-\lambda)^2(\lambda-3)(\lambda+1)$$

Our eogenvalues are 1, −1, 3.

Find the eigenvector for $\lambda = 1$:

$$\left(\begin{array}{cccc|c} 1-1 & 0 & 1 & 1 & 0 \\ 0 & 1-1 & 1 & 1 & 0 \\ 1 & 1 & 1-1 & 0 & 0 \\ 1 & 1 & 0 & 1-1 & 0 \end{array}\right) = \left(\begin{array}{cccc|c} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{array}\right) \xrightarrow{\rho_1 \leftrightarrow \rho_3} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{array}\right) \xrightarrow{\rho_4 \to \rho_4 - \rho_1} \left(\begin{array}{cccc|c} \boxed{1} & 1 & 0 & 0 & 0 \\ 0 & 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

subtract λ on diagonal

$$\xrightarrow{\rho_3 \to \rho_3 - \rho_2} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \qquad \begin{aligned} x_1 + x_2 &= 0 \\ x_3 + x_4 &= 0 \end{aligned} \qquad \begin{aligned} x_1 &= -x_2 \\ x_2 &= x_2 \text{ (free)} \\ x_3 &= -x_4 \\ x_4 &= x_4 \text{ (free)} \end{aligned}$$

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} \;\middle|\; x_2, x_4 \in \mathbb{R} \right\}$$

both eigenvectors
Can check them.

Example ESMS4 Eigenvalues, symmetric matrix of size 4
Consider the matrix

$$C = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

$\det(C - \lambda I) =$

$$= \det\begin{pmatrix} 1-\lambda & 0 & 1 & 1 \\ 0 & 1-\lambda & 1 & 1 \\ 1 & 1 & 1-\lambda & 0 \\ 1 & 1 & 0 & 1-\lambda \end{pmatrix} = (1-\lambda)^4 - 4(1-\lambda)^2 = (1-\lambda)^2\left((1-\lambda)^2 - 4\right)$$

$$= (1-\lambda)^2(1-2\lambda+\lambda^2-4) = (1-\lambda)^2(\lambda^2-2\lambda-3) = (1-\lambda)^2(\lambda-3)(\lambda+1)$$

Our eogenvalues are 1, −1, 3.

$\boxed{\text{HW2}}$

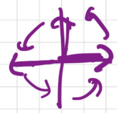
Find the eigenvector for $\lambda = -1$

Hint:
$$\left(\begin{array}{cccc|c} 1-(-1) & 0 & 1 & 1 & 0 \\ 0 & 1-(-1) & 1 & 1 & 0 \\ 1 & 1 & 1-(-1) & 0 & 0 \\ 1 & 1 & 0 & 1-(-1) & 0 \end{array}\right) = \left(\begin{array}{cccc|c} 2 & 0 & 1 & 1 & 0 \\ 0 & 2 & 1 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 2 & 0 \end{array}\right)$$

$\boxed{\text{HW3}}$ Find the evector for $\lambda = 3$.

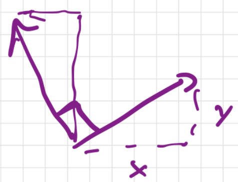
Show all work when taking the det as I do in the classwork to complete your HW and check your answers as taught in HW1.

Example



$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$$

Rotation by 90°



$$\underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}}_{\begin{pmatrix} -y \\ x \end{pmatrix}} \text{ is } \perp \text{ to } \begin{pmatrix} x \\ y \end{pmatrix}$$

because the dot product is 0

$$\begin{pmatrix} -y \\ x \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = -yx + xy = 0$$

Linear Algebra 2

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\vec{v} = \lambda \vec{v} \ ?$$

Impossible!

$$\det\begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} = 0$$

$$(-\lambda)(-\lambda) - (-1)(1) = 0$$

$$\lambda^2 + 1 = 0$$

$$\lambda^2 = -1$$

has no real solutions!

$$i = \sqrt{-1} \qquad \lambda = \pm i$$

$$-i = -\sqrt{-1}$$

3:31 AM Fri Oct 30

Linear Algebra 2

Use $\mathbb{C}$ complex numbers

$a+bi$ where $i^2=-1$
imaginary

$(a+bi)(c+di)=$
$= ac+bic+adi+dbi^2$
$= ac+bci+adi-bd$
$=(ac-bd)+(ad+bc)i$

$(a+bi)+(c+di)$
$=(a+c)+(b+d)i$

Linear Algebra 2

$\frac{1}{a+bi}=$

$=\left(\frac{1}{a+bi}\right)\left(\frac{a-bi}{a-bi}\right)=$

$=\frac{a-bi}{a^2-b^2\cancel{i^2}_{-1}}=\frac{a-bi}{a^2+b^2}$

$=\frac{a}{a^2+b^2}-\frac{b}{a^2+b^2}i$

Use $\mathbb{C}$ complex numbers

$a+bi$ where $i^2=-1$
imaginary

$(a+bi)(c+di) =$
$= ac + bic + adi + dbi^2$
$= ac + bci + adi - bd$
$= (ac-bd) + (ad+bc)i$

$(a+bi)+(c+di)$
$= (a+c) + (b+d)i$

Linear Algebra 2

$\frac{1}{a+bi} =$

$= \left(\frac{1}{a+bi}\right)\left(\frac{a-bi}{a-bi}\right) =$

$= \frac{a-bi}{a^2-b^2 \cancel{i^2}_{-1}} = \frac{a-bi}{a^2+b^2}$

$= \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2} i$

Can do row reduction with $\mathbb{C}$ numbers and find $\mathbb{C}$ evectors!

3:36 AM Fri Oct 30 | Linear Algebra 2 | Linear Algebra 2 | 8%

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \vec{v} = \lambda \vec{v} \; ?$$

Impossible!

$$\det \begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} = 0$$

$$(-\lambda)(-\lambda) - (-1)(1) = 0$$

$$\lambda^2 + 1 = 0$$

$$\lambda^2 = -1$$

has no real solutions!

$i = \sqrt{-1}$ $\qquad \lambda = \pm i$

$-i = -\sqrt{-1}$

$\lambda = +i$ Find its evector

Solve $\qquad -i(i) = -i^2 = -(-1) = +1$

$$\left(\begin{array}{cc|c} \boxed{-i} & -1 & 0 \\ 1 & -i & 0 \end{array}\right) \xrightarrow{\rho_1 \to i\rho_1} \left(\begin{array}{cc|c} 1 & -i & 0 \\ 1 & -i & 0 \end{array}\right)$$

$$\xrightarrow{\rho_2 \to \rho_2 - \rho_1} \left(\begin{array}{cc|c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$x_1 - i x_2 = 0 \qquad x_1 = i x_2$

$\qquad\qquad\qquad x_2 = x_2$ (free)

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} i x_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} i \\ 1 \end{pmatrix} \;\middle|\; x_2 \in \mathbb{C} \right\}$$

free in $\mathbb{C}$

evector is $\begin{pmatrix} i \\ 1 \end{pmatrix}$

Check

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} i \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \cdot i - 1 \cdot 1 \\ 1 \cdot i + 0 \cdot 1 \end{pmatrix} = \begin{pmatrix} -1 \\ i \end{pmatrix} \checkmark$$

$$i \begin{pmatrix} i \\ 1 \end{pmatrix} = \begin{pmatrix} i^2 \\ i \end{pmatrix} = \begin{pmatrix} -1 \\ i \end{pmatrix}$$

3:39 AM Fri Oct 30 — Linear Algebra 2

$\begin{pmatrix}0&-1\\1&0\end{pmatrix}\vec{v} = \lambda \vec{v}$?

Impossible!

$\det\begin{pmatrix}-\lambda & -1\\ 1 & -\lambda\end{pmatrix} = 0$

$(-\lambda)(-\lambda)-(-1)(1)=0$

$\lambda^2+1=0$

$\lambda^2=-1$

has no real solutions!

$i=\sqrt{-1}$ $\quad \lambda=\pm i$

$-i=-\sqrt{-1}$

$\lambda=+i$ Find its evector

Solve $\quad -i(i)=-i^2=-(-1)=+1$

$\left(\begin{array}{cc|c}\boxed{-i}&-1&0\\1&-i&0\end{array}\right) \xrightarrow{\rho_1\to i\rho_1} \left(\begin{array}{cc|c}1&-i&0\\1&-i&0\end{array}\right)$

$\xrightarrow{\rho_2\to\rho_2-\rho_1}\left(\begin{array}{cc|c}1&-i&0\\0&0&0\end{array}\right)$

$x_1 - i x_2 = 0 \qquad x_1 = i x_2$

$x_2=x_2$ (free)

$\left\{\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}ix_2\\x_2\end{pmatrix}=x_2\begin{pmatrix}i\\1\end{pmatrix} \;\middle|\; x_2\in\mathbb{C}\right\}$

So the evector is $\begin{pmatrix}i\\1\end{pmatrix}$ free in $\mathbb{C}$

Check

$\begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}i\\1\end{pmatrix}=\begin{pmatrix}0i-1\cdot1\\1\cdot i+0\cdot1\end{pmatrix}=\begin{pmatrix}-1\\i\end{pmatrix}$ ✓

$i\begin{pmatrix}i\\1\end{pmatrix}=\begin{pmatrix}i^2\\i\end{pmatrix}=\begin{pmatrix}-1\\i\end{pmatrix}$

$\boxed{\text{HW4}}$ Find evector for $\lambda=-i$.

Read Beezer Preliminaries on Complex numbers and then practice at IXL if you did not do this before or need a review.

See a nice youtube video on this topic: https://youtu.be/i8FukKfMKCI

$\boxed{HW5}$ Find the eigenvalues and eigenvectors for

$$B = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$$

$\boxed{HW6}$ Find the evalues and evectors for

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}.$$

$\boxed{HW7}$ Find the evalues and evectors for

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Hint: the characteristic polynomial is $(4-x)(1-x)^2$

Before you submit your work: Please check all your eigenvectors are not the 0 vector and multiply them by their matrix as in HW1 to check that they match their eigenvalue. If anything is not working email me with QUESTION in the subject line.

If your eigenvector is the zero vector, then this means either an error in row reduction or that you found the wrong eigenvalue. A common mistake when finding eigenvalues is forgetting parentheses:

page 4

HW6 Find the evalues & evectors for:

$$\begin{bmatrix} -1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} = \det \begin{bmatrix} 1-\lambda & -1 & 1 \\ -1 & 1-\lambda & -1 \\ 1 & -1 & 1-\lambda \end{bmatrix} \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$= (1-\lambda)\det\begin{pmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{pmatrix} - (-1)\det\begin{pmatrix} -1 & 1 \\ -1 & 1-\lambda \end{pmatrix} + (1)\det\begin{pmatrix} -1 & 1 \\ 1-\lambda & -1 \end{pmatrix} = 0$$

$$= (1-\lambda)((1-\lambda)(1-\lambda)-(-1)(-1)) + (1)((-1)(1-\lambda)-(-1)(-1)) + (1)((-1)(-1)-(1-\lambda)(1)) = 0$$

$$= (-\lambda+1)((-\lambda+1)(-\lambda+1)-1\cdot 1) + 2(-1\cdot(-\lambda+1)+1\cdot 1)$$

I do not know where this is from.

Maybe it would help if you used colors? It helps me.

The determinant parentheses in blue

$$= (1-\lambda)((1-\lambda)(1-\lambda)-(-1)(-1)) + (1)((-1)(1-\lambda)-(1)(-1)) + 1((-1)(-1)-1(1-\lambda))$$

Then work out what is inside the blue first

$$= (1-\lambda)(1-\lambda-\lambda+\lambda^2-1) + 1(-1+\lambda+1) + 1(1-1+\lambda)$$

$$= (1-\lambda)(\lambda^2-2\lambda) + 1(\lambda) + 1(\lambda)$$

Now we start the distribution of the terms outside blue parentheses

$$= 1\cdot\lambda^2 - \lambda\cdot\lambda^2 + 1(-2\lambda) - \lambda(-2\lambda) + \lambda + \lambda$$

$$= \lambda^2 - \lambda^3 - 2\lambda + 2\lambda^2 + \lambda + \lambda$$

Now combine like terms:

$$= -\lambda^3 + 3\lambda^2$$

Now factor

$$= \lambda^2(-\lambda+3)$$

So $\lambda = 0$ and $\lambda = 3$

After you submit, continue to the next lesson. Do not await feedback! You have checked yourself.

Note that I expect to see these checks completed.