Solving and interpreting LP Models
Slack Variables

Given Problem

$$\begin{aligned} \text{Max} \quad & C_1X_1 \\ \text{s.t.} \quad & A_1X_1 \leq b \\ & X_1 \geq 0 \end{aligned}$$

Add Slacks

$$\begin{aligned} \text{Max} \quad & C_1X_1 + 0S \\ \text{s.t.} \quad & A_1X_1 + IS = b \\ & X_1 , S \geq 0 \end{aligned}$$

From now on

$$\begin{aligned} \text{Max} \quad & CX \\ \text{s.t.} \quad & AX = b \\ & X \geq 0 \end{aligned}$$

Where $C= (C_1\ 0)$, $X=(X_1\ S)$, $A=(A_1\ I)$

Solving and interpreting LP Models
Matrix Solution

Mathematicians have found the above can be solved by choosing M variables to be nonzero and setting the rest to zero then inverting. The set of variables chosen to be non-zero (the set of M) is called basic. The remaining variables are called non-basic. So, we partition the problem

$$\begin{aligned} \text{Max} \quad & C_B X_B + C_{NB} X_{NB} \\ \text{s.t.} \quad & B X_B + A_{NB} X_{NB} = b \\ & X_B, \quad X_{NB} \geq 0. \end{aligned}$$

where X_B are the basic variables,
X_{NB} are the non-basic variables
B is the basis matrix.

So

$$B X_B + A_{NB} X_{NB} = b$$

And setting X_{NB} to zero

$$B X_B + 0 = b$$

Or

$$X_B = B^{-1} b$$

Solving and interpreting LP Models
Matrix Solution

And if we consider non basic

$$X_B = B^{-1} b - B^{-1}A_{NB}X_{NB}$$

Now suppose we don't know if we have the right basis

Our objective function is

$Z = C_BX_B + C_{NB}X_{NB}$ Substituting in the X_B equation yields

$$Z = C_B(B^{-1}b – B^{-1}A_{NB}X_{NB}) + C_{NB}X_{NB}$$

or

$$Z = C_BB^{-1}b - C_BB^{-1}A_{NB}X_{NB} + C_{NB}X_{NB}$$

or

$$Z = C_BB^{-1}b - (C_BB^{-1}A_{NB} - C_{NB})X_{NB}$$

Solving and interpreting LP Models
Matrix Solution

Now we know

$Z = C_B B^{-1} b - (C_B B^{-1} A_{NB} - C_{NB}) X_{NB}$

Or

$$Z = C_B B^{-1} b - \sum_{j \in NB} \left(C_B B^{-1} a_j - c_j \right) X_j$$

The current incumbent value when X_j is set to zero is

$Z = C_B B^{-1} b$

But note

$$\frac{\partial Z}{\partial x_j} = -(C_B B^{-1} a_j - c_j) \quad j \in NB$$

So if we have a basis and want to increase the objective we would choose to add a variable with a (most) negative term

When $C_B B^{-1} A_j - C_j$ is all positive we are optimal

Solving and interpreting LP Models
Matrix Solution

So we want to increase non basic variable η when $C_B B^{-1} A_\eta - C_\eta < 0$. We know we need to maintain $X_B \geq 0$. So using our above formulae when we increase $X\eta$ which is a member of the non basic variables, but leaving other X_{NB}'s at zero. The altered values are

$$X_B = B^{-1} b - B^{-1} A_{NB} X_{NB}$$

When using just $X\eta$ and recalling the basic variables must remain non negative

$$X_{Bi} = (B^{-1} b)_i - (B^{-1} A\eta)_i X\eta \geq 0$$

An element of the basis reaches zero when

$$X_{Bi} = (B^{-1} b)_i - (B^{-1} A\eta)_i X\eta = 0 \text{ for some } i$$

Since all X_{Bi} must remain positive or zero

$$X\eta \leq (B^{-1} b)_i / (B^{-1} A\eta)_i \text{ for all } i \text{ when } (B^{-1} A\eta)_i > 0$$

and to find first one to zero or the biggest possible value of $X\eta$ we have the so called minimum ratio rule

$$X\eta = \min \left((B^{-1} b)_i / (B^{-1} A\eta)_i \right) \text{ over } i \text{ when } (B^{-1} A\eta)_i > 0$$

Solving and interpreting LP Models
Simplex Method

1) Select an initial feasible basis matrix (B).

2) Calculate the Basis inverse (B^{-1}).

3) Calculate $C_B B^{-1} a_j - c_j$ for the non-basic variables.
Identify the entering variable as one with the most negative value of that calculation;
if there are none, go to step 6.

4) Calculate the minimum ratio rule.
$X_{\boldsymbol{\eta}} = \min\left((B^{-1} b)_i / (B^{-1} A_{\boldsymbol{\eta}})_i \right)$ over i
when $(B^{-1} A_{\boldsymbol{\eta}})_i > \mathbf{0}$
Denote row with minimum ratio as row i^*;
If there are no rows with $(B^{-1} A_{\boldsymbol{\eta}})_i > \mathbf{0}$ then go to step 7.

5) Replace the basic variable in row i^* (minimum ratio) with variable $\boldsymbol{\eta}$ recalculate the basis inverse. Go to step 3.

6) The solution is optimal. Terminate.
Optimal basic variable values = $B^{-1}b$
Non-basic variables = 0
Reduced costs = $C_B B^{-1} a_j - c_j$
Objective function = $C_B B^{-1} b$.

7) The problem is unbounded. Terminate.

Set Up Simple LP Models

Consider a farmer named Blake located somewhere in the Midwestern United States who is trying to determine how much of three crops (corn, soybeans and wheat) to grow on available land. Blake owns 120 acres of cropland and has at most 320 hours of labor that can be allocated to crop production. Based upon prior yields, Blake's expectation of crop prices at harvest and the cost of producing each crop, calculated net income from producing corn is $40 per acre, $30 per acre of soybeans and $35 per acre of wheat. In addition, calculations indicate that it takes 4 hours of labor to grow one acre of corn, 2 hours of labor to grow one acre of soybeans, and 5 hours of labor to grow one acre of wheat. Blake's sole objective is to maximize the net returns from crop production. To achieve this objective, Blake wants to use modeling to determine the number of acres of corn, soybeans and/or wheat to plant.

$$
\begin{array}{llllll}
\text{Maximize } Z= & 40X_1 & +30X_2 & +35X_3 & \\
\text{s.t.} & X_1 & + \; X_2 & + \; X_3 & \leq 120 \\
 & 4\;X_1 & +2\;X_2 & +5\;X_3 & \leq 320 \\
 & X_{1,} & X_{2,} & X_3 & \geq 0
\end{array}
$$

Solving and interpreting LP Models
Simplex Example

Problem

Maximize Z=	$40X_1$	$+30X_2$	$+35X_3$	
s.t.	X_1	$+ \; X_2$	$+ \; X_3$	≤ 120
	$4 \; X_1$	$+2 \; X_2$	$+5 \; X_3$	≤ 320
	$X_{1,}$	$X_{2,}$	X_3	≥ 0

Add Slacks

Max Z=	$40X_1$	$+30X_2$	$+35X_3$	$+0S_1$	$+0S_2$	
s.t.	X_1	$+ \; X_2$	$+ \; X_3$	$+ \; S_1$		$= 120$
	$4 \; X_1$	$+2 \; X_2$	$+5 \; X_3$		$+ \; S_2$	$= 320$
	$X_{1,}$	$X_{2,}$	$X_{3,}$	$S_{1,}$	$+ \; S_2$	≥ 0

Matrix Setup

$C = [40 \;\; 30 \;\; 35 \;\; 0 \;\; 0]$

$A = [1\ 1\ 1\ 1\ 0\ 4\ 2\ 5\ 0\ 1\ ] \;\; b = (120\ 320\ )$

Initial Basis

$X_B = [S_1 \; S_2] \;\; X_{NB} = [X_1 \; X_2 \; X_3] = [0 \;\; 0 \;\; 0]$

$B = [1\ 0\ 0\ 1\ ] \;\; A_{NB} = [1\ 1\ 1\ 4\ 2\ 5\ ] \;\; C_B = [0 \;\; 0] \;\; C_{NB} = [40 \;\; 30 \;\; 35]$

$B^{-1} = [1\ 0\ 0\ 1\ ] \quad C_B B^{-1} A_{NB} - C_{NB} = [-\ 40 \quad -\ 30 \quad -\ 35]$

$X_B = B^{-1} b = [120\ 320\ ] \;\; C_B B^{-1} b = 0$

Entering Variable

X_1

Solving and interpreting LP Models
Simplex Example (continued)

Min ratio rule

$X_B = B^{-1}b - B^{-1}A_{X_1}X_1 = [120\ 320\ ] - [1\ 4\ ]X_1 \geq 0$

$X_1 \leq [120/1\ 320/4\ ] = [120\ 80\ ]$

So take out second basic variable (S_2) replace with X_1

Remove S_2 Insert X_1

New Basis

$B = [1\ 1\ 0\ 4\ ]$ $A_{NB} = [1\ 1\ 0\ 2\ 5\ 1\ ]$ $C_B = [0\ \ 40]$ $C_{NB} = [30\ \ 35\ \ \ 0]$

$X_B = [S_1\ X_1]$ $X_{NB} = [X_2\ \ X_3\ \ S_2\ ]$

$B^{-1} = [1\ \ -\ 1/4\ 0\ 1/4\ ]$ $C_B B^{-1} A_{NB} - C_{NB} = [-\ 10\ \ \ 15\ \ \ 0]$

$X_B = B^{-1}b = [40\ 80\ ]$ $C_B B^{-1}b = 3200$

Entering Variable

X_2

Solving and interpreting LP Models
Which Variable Leaves

Min ratio rule

$X_B = B^{-1}b - B^{-1}A_{X_2}X_2 = [40\ 80\ ] - [1\ \ -\ 1/4\ 0\ 1/4\ ][1\ 2\ ]X_2 \geq 0$

$X_2 \leq [40/0.5\ 80/0.5\ ] = [80\ 160\ ]$

So take out first basic variable (S_1) replace with X_2

Remove S_1 Insert X_2

New Basis

$B = [1\ 1\ 2\ 4\ ]\ \ A_{NB} = [1\ 1\ 0\ 5\ 0\ 1\ ]\ \ C_B = [30\ \ 40]\ \ \ C_{NB} = [35\ \ 0\ \ \ 0]$

$X_B = [X_2\ X_1]$ $\quad$ $X_{NB} = [X_3\ S_1\ S_2\ ]$

$B^{-1} = [2\ \ -\ 1/2\ \ -\ 1\ 1/2\ ]$ $\quad$ $C_B B^{-1} A_{NB} - C_{NB} = [10\ \ \ 20\ \ 5]$

$X_B = B^{-1}b = [80\ 40\ ]$ $\quad$ $C_B B^{-1} b = 4000$

Optimal Values

$Z = C_B X_B = 4000$ $\quad$ $U = C_B\ B^{-1} = [20\ 5]$

$X_1 = 40\ \ X_2 = 80\ \ X3 = 0$ $\quad$ $C_B B^{-1} A_{NB} - C_{NB} = [10\ \ \ 20\ \ 5]$

Solving and interpreting LP Models
Sensitivity Information

Two Equations

$$X_B = B^{-1}b - \sum_{j \in NB} B^{-1} a_j x_j$$

$$Z = C_B B^{-1} b - \sum_{j \in NB} \left(C_B B^{-1} a_j - c_j \right) x_j ,$$

Sensitivity Results

$$\frac{\partial X_B}{\partial b} = B^{-1}$$

$$\frac{\partial Z}{\partial b} = C_B B^{-1}$$

$$\frac{\partial Z}{\partial x_j} = -(C_B B^{-1} a_j - c_j)$$

$$\frac{\partial X_B}{\partial x_j} = -B^{-1} a_j \quad j \in NB$$

For example

$B^{-1} = [2 \ - 1/2 \ - 1\ 1/2\,]$ $\quad C_B B^{-1} A_{NB} - C_{NB} = [10 \ \ 20 \ \ 5]$

$C_B B^{-1} = [20 \ \ \ 5]$ $\quad B^{-1} A_{NB} = \left[-\frac{1}{2} \ - 2 \ - \frac{1}{2}\ \frac{3}{2} \ - 1\, \frac{1}{2} \right]$

Solving and interpreting LP Models
Right Hand Side Ranging

Two Fundamental Equations

$$X_B = B^{-1}b \qquad \geq 0 \; -\frac{\partial Z}{\partial X_j} = C_B B^{-1} a_j - c_j \geq 0$$

Now Suppose We Alter RHS

$$b_{new} = b_{old} + \theta r$$

Effect on Equations

$$X_B = B^{-1} b_{new} = B^{-1}\left(b_{old} + \theta r\right) = B^{-1} b_{old} + \theta B^{-1} r \geq 0$$

while $C_B B^{-1} a_j - c_j$ is unchanged.

The net effect is that the new solution levels are equal to the old solution levels plus $\theta B^{-1} r$. Objective function changes by this times C_B

Limits on θ

$$\theta \geq -\frac{\left(B^{-1} b_{old}\right)_i}{\left(B^{-1} r\right)_i}, \quad \text{where} \left(B^{-1} r\right)_i > 0$$

$$\theta \leq -\frac{\left(B^{-1} b_{old}\right)_i}{\left(B^{-1} r\right)_i}, \quad \text{where} \left(B^{-1} r\right)_i < 0$$

Solving and interpreting LP Models
Right Hand Side Ranging

Now Suppose We Alter RHS

$b_{new} = b_{old} + \theta r$
$r = [1\ 0]$

Effect on Equations

$$X_B = B^{-1} b_{new} = B^{-1}\left(b_{old} + \theta r\right) = B^{-1} b_{old} + \theta B^{-1} r \geq 0$$

$$[80\ 40] + \theta[2\ \ -1/2\ \ -1\ 1/2][1\ 0] \geq 0$$

$$[80\ 40] + \theta[2\ \ -1] \geq 0$$

Limits on θ

$$\theta \geq -\frac{\left(B^{-1} b_{old}\right)_i}{\left(B^{-1} r\right)_i}, \quad \text{where} \left(B^{-1} r\right)_i > 0$$

$$\theta \leq -\frac{\left(B^{-1} b_{old}\right)_i}{\left(B^{-1} r\right)_i}, \quad \text{where} \left(B^{-1} r\right)_i < 0$$

$$-80/2 \leq \theta \leq 40$$

$$80 \leq b_1 \leq 160$$

Solving and interpreting LP Models
Cost Ranging

Two Fundamental Equations

$$X_B = B^{-1}b \qquad \geq 0 \; -\frac{\partial Z}{\partial X_j} = C_B B^{-1} a_j - c_j \geq 0$$

Now Suppose We Alter Obj

$$C_{new} = C_{old} + \gamma T$$

Effect on Equations

$$\left(C_B B^{-1} a_j - c_j\right)_{new} = C_{B_{new}} B^{-1} a_j - c_{j_{new}} \geq 0$$

while $B^{-1}b$ is unchanged.

The net effect is that the new reduced costs are equal to the old ones plus a term

$$\left(C_B B^{-1} a_j - c_j\right)_{new} = \left(C_B B^{-1} a_j - c_j\right)_{old} + \gamma(T_B B^{-1} a_j - T_j) \geq 0$$

Recalling this must be non zero we can solve for γ

In turn, we discover the cost range for non-basic variables ($T_B = 0$)

$$\gamma \leq \left(C_B B^{-1} a_j - c_j\right)_{old}$$

Solving and interpreting LP Models
Cost Ranging

while for basic variables

$$\gamma \leq -\frac{\left(C_B B^{-1} a_j - c_j\right)_{old}}{\left(T_B B^{-1} a_j - T_j\right)},\ where \left(T_B B^{-1} a_j - T_j\right) < 0 \gamma \geq -\frac{\left(C_B B^{-1} a_j -\right.}{\left(T_B B^{-1} a_j\right.}$$

Example

Suppose in our example problem we want to alter the objective function on X_1 so it equals 40 + γ. The setup then is

$$C_{new} = [40\ \ 30\ \ 35\ \ 0\ \ 0] + \gamma[1\ \ 0\ \ 0\ \ 0\ \ 0]$$

and

$$T_B = [0\ \ 1]$$

$$C_B = [30\ \ 40]\quad C_{NB} = [35\ \ 0\ \ 0]$$

note the order that variables appear in the T_B vector reflects the order they appear in the basis so X_2 is first then X_1

Solving and interpreting LP Models
Cost Ranging

So for the non-basic variables' reduced cost equals

$$\left(C_B B^{-1} a_j - c_j\right)_{new} = \left(C_B B^{-1} a_j - c_j\right)_{old} + \gamma\left(T_B B^{-1} a_j - T_j\right) = ($$

which implies -20/3 $\leq \gamma \leq$ 20 or that the basis is optimal for any objective function value for X_1 between 33.5 and 50. This shows a range of prices for X_1 for which optimal level is constant.

Solving and interpreting LP Models
A_{ij} Ranging

Suppose we alter the matrix of the technical coefficients

$$\overline{A} = A + uv^T = A + M$$

where $\overline{A}$, A, and M are $m\times n$ matrices, u is $m\times 1$ vector and v is $n\times 1$ vector. The matrix M = uv^T indicates a set simultaneous of changes in columns or rows to be made in A.

When varying an element or columns in A_{NB}, only the reduced cost of that non-basic variable changes.
When varying an element or columns in B or a row in A, the model solution X, reduced costs, shadow prices and objective functions are all changed.
The expected change in the optimal objective function is

$$Z_{new} - Z_{old} = -\frac{U_{old} M X_{old}}{1+v^T B_{old}^{-1} u}$$

where $U_{old} = C_B B_{old}^{-1}$

The proof of the formula used Sherman–Morrison formula

$$\left(A + uv^T\right)^{-1} = A^{-1} + \frac{A^{-1} uv^T A^{-1}}{1+v^T A^{-1} u}$$

Solving and interpreting LP Models

A_{ij} Ranging

Example

Maximize Z=	$40X_1$	$+30X_2$	$+35X_3$	
s.t.	X_1	$+X_2$	$+X_3$	≤ 120
	$(4+\theta)X_1$	$+(2+\theta)X_2$	$+(5+\theta)X_3$	≤ 320
	$X_1,$	$X_2,$	X_3	≥ 0

$$M = [0\ 0\ \theta\ \theta\ ],\ u = [0\ \theta\ ],\ v^T = [1\ \ 1\ ]$$

Thus, the change in the value of the objective function is given by

$$Z_{new} - Z_{old} = -\frac{[20\ 50\ ][0\ 0\ \theta\ \theta\ ][80\ 40\ ]}{1+[1\ 1\ ]\left[2\ -\frac{1}{2}\ -1\ \frac{1}{2}\ \right][0\ \theta\ ]} = -\ 6000\theta$$

Suppose $\theta = -1$, then the anticipated change $\Delta Z = -U^*MX^* = 6000$

Resolving revised problem shows objective function changes by 6000.

Solving and interpreting LP Models
Degeneracy

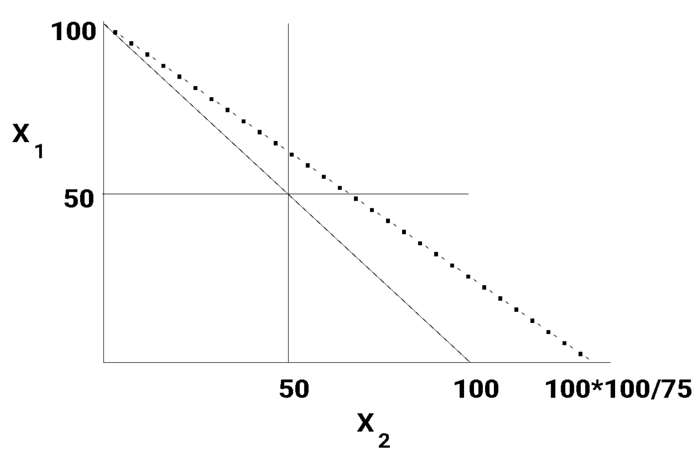
$$\begin{array}{lrcrcr} \text{Max} & 100X_1 & + & 75X_2 & & \\ & X_1 & & & \leq & 50 \\ & & & X_2 & \leq & 50 \\ & X_1 & + & X_2 & \leq & 100 \end{array}$$

The third constraint is redundant to the first two.

In the simplex solution X_2 can be entered in place of second or third slack. If X_2 is brought into the basis in the second row, the shadow prices determined are $(u_1, u_2, u_3) = (100, 75, 0)$.

If X_2 is brought in the third row the value of the shadow prices are $(u_1, u_2, u_3) = (25, 0, 75)$.

These differ depending on whether the second or third slack variable is in the basis at a value of zero. Thus, the solution is degenerate.



Solving an interpreting LP Models
Alternative Optimal

$$\begin{array}{lccccc} \text{Max} & 5X_1 & + & 5X_2 & & \\ & X_1 & + & X_2 & \leq & 50 \\ & X_1 & , & X_2 & \geq & 0 \end{array}$$

The objective is parallel to the binding constraints.

In the simplex solution X_1 and X_2 would have the same reduced cost.

If X_1 is brought into the basis, the solution is $X_1 = 50$ & $X_2 = S_1 = 0$ reduced costs = (0 , 5), shadow price = 5, obj = 250.

If X_2 is brought into the basis, the solution is $X_2 = 50$ & $X_1 = S_1 = 0$ reduced costs = (0 , 5), shadow price = 5, obj = 250.

An alternative optimal with a non basic variable having zero reduced cost.

Solving and interpreting LP Models
Shadow Prices and Bounded Variables

Linear programming codes impose upper and lower bounds on individual variables in a special way and this influences shadow prices. An example of a problem with upper and lower bounds is given below.

$$\begin{array}{llcll} \text{Max} & 3X_1 & - & X_2 & \\ \text{s.t.} & X_1 & + & X_2 & \leq 15 \\ & X_1 & & & \leq 10 \\ & & & X_2 & \geq 1 \end{array}$$

The second constraint imposes an upper bound on X_1, i.e., $X_1 \leq 10$, while the third constraint, $X_2 \geq 1$, is a lower bound on X_2. Most LP algorithms allow one to specify these particular restrictions as either constraints or bounds. Solutions from LP codes under both are shown in Table 3.2.

Table 3.2. Solutions under different model settings

Solution with Bounds Imposed as Constraints					
Variable	**Value**	**Margins**	**Equation**	**Slacks**	**Margins**
X_1	10	0	1	4	0
X_2	1	0	2	0	3
			3	0	-1
Solution with Bounds					
Variable	**Value**	**Margins**	**Equation**	**Slacks**	**Margins**
X_1	10	3	1	4	0
X_2	1	-1			

Solving and interpreting LP Models
Artificial Variables

These are variables added to the problem that permits an initial feasible basis but need to be removed before the solution is finalized.

$$
\begin{aligned}
\textbf{Max}\quad & CX \\
\text{s.t.}\quad & RX \le b \\
& DX \le -e \\
& FX = g \\
& HX \ge p \\
& \quad X \ge 0
\end{aligned}
$$

where, b, e, and p are positive.

We first add slack (surplus) variables into the model

$$
\begin{array}{llllll}
\text{Max} & CX & & & & \\
\text{s.t.} & RX + I_1S_1 & & & & = b \\
& DX & + I_2S_2 & & & = -e \\
& FX & & & & = g \\
& HX & & & - I_4W & = p \\
& \quad X, & S_1, & S_2, & W & \ge 0
\end{array}
$$

Artificial variables are entered into each constraint which is not satisfied when X=0 and does not have an easily identified basic variable.

In this example, three sets of artificial variables are required.

$$
\begin{array}{lllllllll}
\text{Max} & CX & & & & -MA_2 & - MA_3 & -MA_4 & \\
\text{s.t.} & RX + I_1S_1 & & & & & & & = b \\
& DX & + I_2S_2 & & & -I_2A_2 & & & = -e \\
& FX & & & & & + I_3A_3 & & = g \\
& HX & & & - I_4W & & & +I_4A_4 & = p \\
& \quad X, & S_1, & S_2, & W, & A_2, & A_3, & A_4 & \ge 0
\end{array}
$$

Here, A2, A3, and A4 are the artificial variables that permit an initial feasible nonnegative basis but which must be removed before a "true feasible solution" is present. Note that S_1, A_2, A_3, and A_4 can be put into the initial basis. But not S_2.

Solving and interpreting LP Models
Artificial Variables

Two approaches

1. phase 1 /phase 2
   Initially minimize sum of A

2. Big M
   Put -99*A in objective
   As follows

$$
\begin{array}{llllllllll}
\text{Max} & 3X_1 + 2X_2 & & & & -99A_2 & -\,99A_3 & -99A_4 & & \\
\text{s.t.} & X_1 + 2X_2 + S_1 & & & & & & & = & 10 \\
 & X_1 + \;\; X_2 & + S_2 & & & -A_2 & & & = & -2 \\
 & -X_1 + \;\; X_2 & & & & & + \;\; A_3 & & = & 3 \\
 & X_1 + \;\; X_2 & & - & W & & & + \;\; A_4 & = & 1 \\
 & X, \quad S_1, & S_2, & & W, & A_2, & A_3, & A_4 & \geq & 0
\end{array}
$$

Table 3.4. Solution to the Big M Problem

Variable	Value	Reduced Cost	Equation	Shadow Price
x_1	0	193	1	0
x_2	3	0	2	99
S_1	4	0	3	-97
S_2	0	-99	4	0
W	2	0		
A_2	5	0		
A_3	0	-2		
A_4	0	-99		