Analysis Lesson 8

Convergence and the Sandwich Theorem

This lesson has three parts.   You will start a new googledoc for all work leading to Exam 2 entitled MAT320-2-Lastname-Firstname.

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Part 1 Proving a sequence does not converge

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Watch Playlist No Limit

How to prove a sequence does not converge

Prove $\boxed{x_j = (-1)^j \text{ has no limit}}$

Classwork!

Indirect Proof by Contradiction

① Assume on the contrary that $x_j = (-1)^j$ has a limit L.

① Indirect Hypothesis.

② $\forall \varepsilon > 0\ \exists N_\varepsilon$ s.t $\forall j \geq N_\varepsilon\ |x_j - L| < \varepsilon$

② defn of limit

$d(-1,1) = |-1-1| = |-2| = 2$

What $\varepsilon = 1$?

$x_1 = (-1)^1 = -1$

$x_2 = (-1)^2 = 1$

$x_3 = (-1)^3 = -1$

not getting crowded!

Find a contradiction

Anything False not necessarily Archimedes

How to prove a sequence does not converge

Prove $\boxed{x_j = (-1)^j \text{ has no limit}}$

Classwork!

Indirect Proof by Contradiction

Statement	Reason
① Assume on the contrary that $x_j = (-1)^j$ has a limit L.	① Indirect Hypothesis.
② $\forall \varepsilon > 0 \ \exists N_\varepsilon$ s.t $\forall j \geq N_\varepsilon$ $\lvert x_j - L \rvert < \varepsilon$	② defn of limit
③ $x_j = (-1)^j$ is alternating $\begin{cases} x_{2n} = (-1)^{2n} = 1 \\ x_{2n+1} = (-1)^{2n+1} = -1 \end{cases}$	③ by given + powers
④ Let $\varepsilon = 1 > 0$ $\exists N_1$ s.t $\forall j \geq N_1$ $\lvert x_j - L \rvert < 1$	④ $\varepsilon = 1 > 0$ and step 2
⑤ Let $j = 2N_1 > N_1$ which is even $\lvert 1 - L \rvert < 1$	⑤ steps 3+4
⑥ $L \in (1-1, 1+1) = (0, 2)$	⑥ $\lvert a-b \rvert < R \Rightarrow b \in (a-R, a+R)$
⑦ Let $j = 2N_1 + 1 > N_1$ which is odd $\lvert -1 - L \rvert < 1$	⑦ steps 3+4
⑧ $L \in (-1-1, -1+1) = (-2, 0)$	⑧ $\lvert a-b \rvert < R \Rightarrow b \in (a-R, a+R)$

⊗ $L > 0$ in step 6 but $L < 0$ in step 8

Write final conclusion

To finish the classwork write the final conclusion "Thus the indirect…."

$\boxed{\text{HW1}}$ Prove $x_j = (-1)^j 5$ has no limit.

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Part 2 Sandwich Theorem

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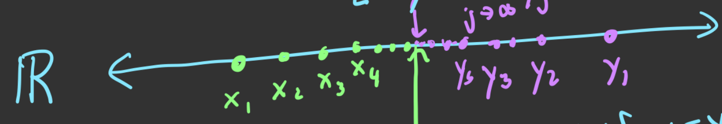
Watch [Playlist Sandwich]()

Convergence and the Sandwich Lemma

(also called the Squeeze Theorem)

$L = y = \lim_{j \to \infty} y_j$



$\lim_{j \to \infty} x_j = x = L$ What if $x = y$?

Sandwich Lemma: If $\lim_{j \to \infty} x_j = L = \lim_{j \to \infty} y_j$
and if we also have $x_j \le p_j \le y_j$

then $\lim_{j \to \infty} p_j = L$.

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Axiomatic Geometry

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Proving the Sandwich Lemma: If $\lim_{j\to\infty} x_j = L = \lim_{j\to\infty} y_j$ and if we also have $x_j \le p_j \le y_j$

then $\lim_{j\to\infty} p_j = L$.

Classwork!

Given: $x_j \to L$ $\forall \varepsilon>0\ \exists N_\varepsilon^x$ s.t $\forall j \ge N_\varepsilon^x$ $|x_j - L| < \varepsilon$

$y_j \to L$ $\forall \varepsilon>0\ \exists N_\varepsilon^y$ s.t $\forall j \ge N_\varepsilon^y$ $|y_j - L| < \varepsilon$

$x_j \le p_j \le y_j$

Show $p_j \to L$ $\forall \varepsilon>0\ \exists N_\varepsilon^p$ s.t $\forall j \ge N_\varepsilon^p$ $|p_j - L| < \varepsilon$

Proof Outline comes from Show

(1)

(2)

final

fill in the key steps

Show $p_j \to L$ $\forall \varepsilon > 0$ $\exists N_\varepsilon^p$ s.t. $\forall j \geq N_\varepsilon^p$ $|p_j - L| < \varepsilon$

Proof Outline comes from Show

(1) Given any $\varepsilon > 0$ Choose $N_\varepsilon^p = \boxed{}$

(2) $\forall j \geq N_\varepsilon^p$ we have ______

______

final $|p_j - L| < \varepsilon$

2:34 AM Mon Feb 22

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Show $p_j \to L$ $\forall \varepsilon > 0$ $\exists N_\varepsilon^p$ s.t. $\forall j \geq N_\varepsilon^p$ $|p_j - L| < \varepsilon$

(1) Given any $\varepsilon > 0$ Choose $N_\varepsilon = \boxed{}$

(2) $\forall j \geq N_\varepsilon^p$ we have

$\uparrow$?

$L - \varepsilon < p_j < L + \varepsilon$

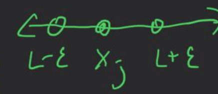
$-\varepsilon < p_j - L < \varepsilon$

final $|p_j - L| < \varepsilon$

final $|x| < R \Leftrightarrow -R < x < R$

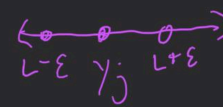
Given: $x_j \to L$ $\quad \forall \varepsilon > 0 \ \exists N^x_\varepsilon$ s.t $\forall j \geq N^x_\varepsilon$ $|x_j - L| < \varepsilon$

$L - \varepsilon < x_j < L + \varepsilon$



$y_j \to L$ $\quad \forall \varepsilon > 0 \ \exists N^y_\varepsilon$ s.t $\forall j \geq N^y_\varepsilon$ $|y_j - L| < \varepsilon$

$L - \varepsilon < y_j < L + \varepsilon$



$x_j \leq p_j \leq y_j$

Show $p_j \to L$ $\quad \forall \varepsilon > 0 \ \exists N^p_\varepsilon$ s.t $\forall j \geq N^p_\varepsilon$ $|p_j - L| < \varepsilon$

(1) Given any $\varepsilon > 0$ Choose $N_\varepsilon = \boxed{\quad\quad\quad\quad}$

(2) $\forall j \geq N^p_\varepsilon$ we have

$\uparrow$?

$L - \varepsilon < p_j < L + \varepsilon$

$-\varepsilon < p_j - L < \varepsilon$

final $|p_j - L| < \varepsilon$

final $|x| < R \iff -R < x < R$

Given: $x_j \to L$ $\quad \forall \varepsilon > 0 \ \exists N_\varepsilon^x$ s.t $\forall j \geq N_\varepsilon^x \ |x_j - L| < \varepsilon$

$L - \varepsilon < x_j < L + \varepsilon$

$y_j \to L$ $\quad \forall \varepsilon > 0 \ \exists N_\varepsilon^y$ s.t $\forall j \geq N_\varepsilon^y \ |y_j - L| < \varepsilon$

$L - \varepsilon < y_j < L + \varepsilon$

$x_j \leq p_j \leq y_j$

Show $p_j \to L$ $\quad \forall \varepsilon > 0 \ \exists N_\varepsilon^p$ s.t $\forall j \geq N_\varepsilon^p \ |p_j - L| < \varepsilon$

(1) Given any $\varepsilon > 0$ Choose $N_\varepsilon = \boxed{\quad\quad\quad}$

(2) $\forall j \geq N_\varepsilon^p$ we have

$j \geq N_\varepsilon^x$ and $j \geq N_\varepsilon^y$

$L - \varepsilon < x_j \leq p_j \leq y_j < L + \varepsilon$

$L - \varepsilon < p_j < L + \varepsilon$

$-\varepsilon < p_j - L < \varepsilon$

final $|p_j - L| < \varepsilon$

final $|x| < R \Leftrightarrow -R < x < R$

2:43 AM Mon Feb 22

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Given: $x_j \to L$ $\quad \forall \varepsilon>0 \ \exists N_\varepsilon^x$ s.t $\forall j \geq N_\varepsilon^x$ $|x_j - L| < \varepsilon$

$L-\varepsilon < x_j < L+\varepsilon$

$y_j \to L$ $\quad \forall \varepsilon>0 \ \exists N_\varepsilon^y$ s.t $\forall j \geq N_\varepsilon^y$ $|y_j - L| < \varepsilon$

$L-\varepsilon < y_j < L+\varepsilon$

$x_j \leq p_j \leq y_j$

Show $p_j \to L$ $\quad \forall \varepsilon>0 \ \exists N_\varepsilon^p$ s.t $\forall j \geq N_\varepsilon^p$ $|p_j - L| < \varepsilon$

(1) Given any $\varepsilon > 0$ Choose $N_\varepsilon = \boxed{}$

(2) $\forall j \geq N_\varepsilon^p$ we have

$j \geq N_\varepsilon^x$ and $j \geq N_\varepsilon^y$

$L-\varepsilon < x_j \leq p_j \leq y_j < L+\varepsilon$

$L-\varepsilon < p_j < L+\varepsilon$

$-\varepsilon < p_j - L < \varepsilon$

final $|p_j - L| < \varepsilon$

Choose Now
pause
try

final $|x| < R \Leftrightarrow -R < x < R$

Given: $x_j \to L$ $\quad \forall \varepsilon > 0 \ \exists N_\varepsilon^x$ s.t $\forall j \geq N_\varepsilon^x \ |x_j - L| < \varepsilon$

$L - \varepsilon < x_j < L + \varepsilon$

$y_j \to L$ $\quad \forall \varepsilon > 0 \ \exists N_\varepsilon^y$ s.t $\forall j \geq N_\varepsilon^y \ |y_j - L| < \varepsilon$

$L - \varepsilon < y_j < L + \varepsilon$

$x_j \leq p_j \leq y_j$

Show $p_j \to L$ $\quad \forall \varepsilon > 0 \ \exists N_\varepsilon^p$ s.t $\forall j \geq N_\varepsilon^p \ |p_j - L| < \varepsilon$

(1) Given any $\varepsilon > 0$ Choose $N_\varepsilon = \boxed{\max\{N_\varepsilon^x, N_\varepsilon^y\}}$	(1) Given $x_j \to L$ and $y_j \to L$
(2) $\forall j \geq N_\varepsilon^p$ we have $j \geq N_\varepsilon^x$ and $j \geq N_\varepsilon^y$	(2) defn of max
(3) $L - \varepsilon < x_j \leq p_j \leq y_j < L + \varepsilon$	(3) Given $\lvert x_j - L\rvert < \varepsilon$; Given $x_j \leq p_j \leq y_j$; Given $\lvert y_j - L\rvert < \varepsilon$
(4) $L - \varepsilon < p_j < L + \varepsilon$	(4) by step 3
(5) $-\varepsilon < p_j - L < \varepsilon$	(5) subtract L
final $\lvert p_j - L\rvert < \varepsilon$	final $\lvert x\rvert < R \Leftrightarrow -R < x < R$

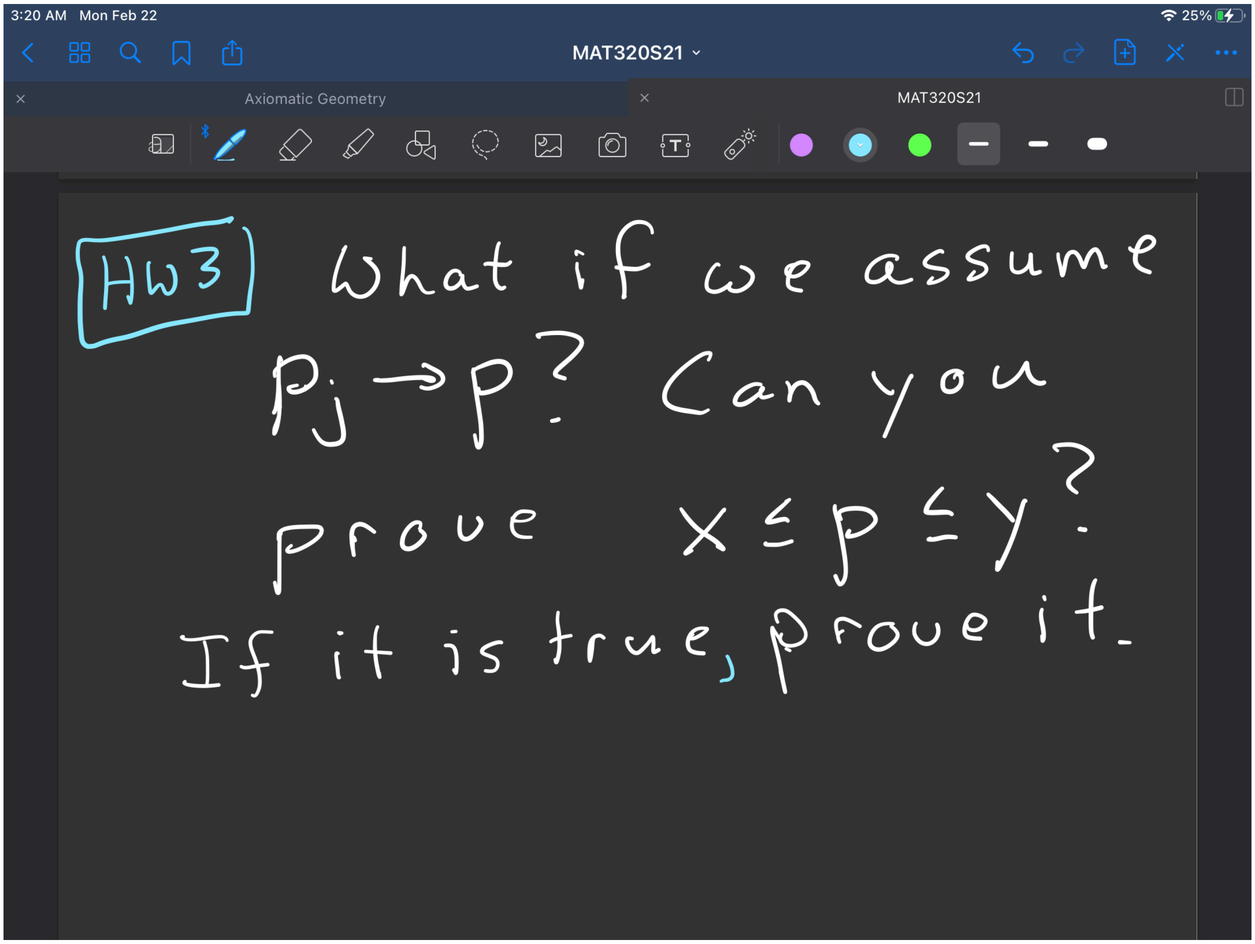
QED

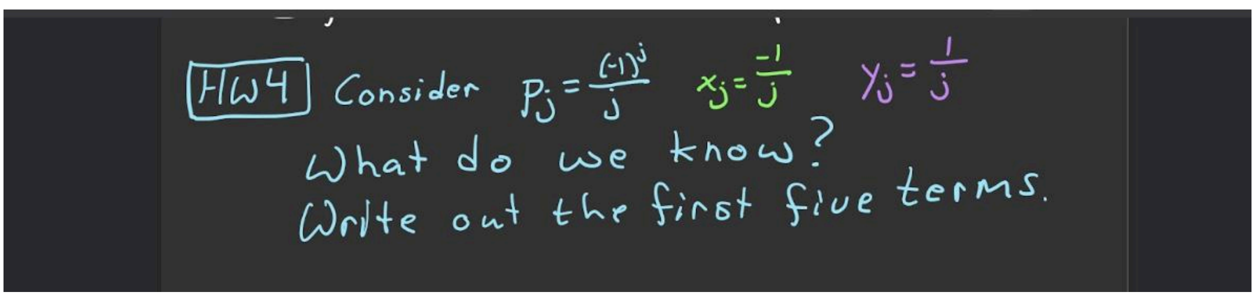
$\boxed{\text{HW2}}$ What happens if $x_j \to x$ and $y_j \to y$ and $x_j \le p_j \le y_j$

Imitate the proof above starting from the top down.

Can you say anything about p_j?

What about $x_j = -\frac{j+1}{j}$ $p_j = (-1)^j$ $y_j = \frac{j+1}{j}$?





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Part 3: Limits and Trig Functions

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It is important that everyone understand Unit Circle Trigonometry which is taught in precalculus at Lehman College. You should review this by watching the first two videos (at high speed is ok) and your classwork is to write the definition of sine and of cosine and of tangent using a unit circle:

[**LMK Defn of Trig Functions**](#)

[**LMK Graphing Trig Functions**](#)

This third video will help you with homework 5.
[LMK Values of Trig Functions]
We will need this to do the rest of the course.
These other LMK precalc [videos might also be useful] in the future.

Thm: If A={sin(x) | x in reals} then max(A)=1 and min(A)=-1

Extra Credit: Prove this theorem using the defn of sine
and the following trick

Homework: use this theorem and the sandwich lemma to complete HW5: