Physical Applications of Integration

Guided Notes

This resource is designed to help you actively engage with key mathematical concepts. By completing these notes, you'll establish connections between concepts, identify patterns, and develop a deeper understanding of the material.

Fill in the blanks, solve examples, and note questions as you go. Writing in your own words strengthens your memory and helps identify areas needing more attention. Bring your questions to class discussions where your instructor can address specific concerns. Don't focus on perfection—you'll learn math best through active engagement!

Mass and Density

Key Concept: Mass–Density Formula of a One-Dimensional Object

We can use integration to calculate the mass of a thin rod based on a **density function**. Let's consider a rod oriented along the x-axis from $x = a$ to x = b.

If the rod has constant density ρ, given in terms of mass per unit length, then the mass of the rod is: ______________

If the density varies along the rod, we use a linear density function $\rho(x)$.

To find the total mass:

1. Partition the interval $[a,b]$ into n segments, each of width Δx
2. The mass of a segment $[x_{i-1}, x_i]$ is:

$$m_i = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

3. The total mass is approximated by:

$$m \approx \underline{\hspace{5cm}}$$

4. Taking the limit as $n \to \infty$ gives the exact mass:

$$m = \underline{\hspace{5cm}}$$

Definition

Given a thin rod oriented along the x-axis over the interval $[a,b]$, let $\rho(x)$ denote a linear density function giving the density of the rod at a point x in the interval. Then the mass of the rod is given by:

$$m = \underline{\hspace{5cm}}$$

Practice: Finding Mass of a Rod with Variable Density

Consider a thin rod oriented on the x-axis over the interval $\left[\frac{\pi}{2}, \pi\right]$. If the density of the rod is given by $\rho(x) = \sin x$, what is the mass of the rod?

- Step 1: Set up the definite integral using the mass-density formula.

$$m = \underline{\hspace{4cm}}$$

- Step 2: Solve the definite integral.

$$m = \underline{\hspace{4cm}}$$

Think About It

If two rods have the same length and average density, could they still have different masses? Explain how this might be possible by discussing density distributions.

Key Concept: Mass–Density Formula of a Two-Dimensional Disk

We now extend this concept to find the mass of a two-dimensional disk of radius *r*.

We assume:

- The density is given in terms of mass per unit area (area density)
- The density varies only along the disk's radius (radial density)
- The disk is oriented in the *xy*-plane, with the center at the origin

To find the total mass of the disk:

1. Partition the interval [*0,r*] into *n* segments, each of width Δx
2. The area of a washer with inner radius x_i*-1* and outer radius x_i is:

$$Ai = \pi(x_i^2 - x_i - 1^2) = \pi(x_i + x_i - 1)\,\Delta x \approx \underline{\qquad\qquad\qquad}$$

3. The mass of this washer is approximately:

$$m_i \approx \underline{\qquad\qquad\qquad\qquad\qquad}$$

4. The total mass is approximated by:

$$m \approx \underline{\qquad\qquad\qquad\qquad\qquad}$$

5. Taking the limit as n→∞ gives the exact mass:

$$m = \underline{\qquad\qquad\qquad\qquad\qquad}$$

Definition

Let *ρ(x)* be an integrable function representing the radial density of a disk of radius *r*. Then the mass of the disk is given by:

$$m = \underline{\qquad\qquad\qquad\qquad\qquad}$$

Practice: Finding Mass of a Disk with Variable Density

Let $\rho(x) = \sqrt{x}$ represent the radial density of a disk. Calculate the mass of a disk of radius 4.

- Step 1: Set up the definite integral using the mass-density formula.

$$m = \underline{\hspace{6cm}}$$

- Step 2: Solve the definite integral.

$$m = \underline{\hspace{6cm}}$$

Think About It

In real-world applications, the mass distribution of an object can significantly affect its physical properties (like balance or moment of inertia). Can you think of scenarios where engineers might intentionally design objects with non-uniform density distributions? What advantages might this provide?

Work Done by a Force

Key Concept: Work Done by a Variable Force

In physics, **work** is related to force, which is often intuitively defined as a push or pull on an object. When a force moves an object, we say the force does work on the object.

The unit of force is the ________________, and work is measured in ________________.

When the force is constant, work can be expressed as: ______________________________

However, when the force varies, we need to:

1. Partition the interval [*a*,*b*] into *n* segments
2. For a variable force *F(x)*, the work done over each subinterval is approximately:

$$W_i \approx \underline{\hspace{6cm}}$$

3. Summing the work over all segments:

$$W \approx \underline{\hspace{6cm}}$$

4. Taking the limit as $n \to \infty$:

$$W = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

Definition

If a variable force $F(x)$ moves an object in a positive direction along the x-axis from point a to point b, then the work done on the object is:

$$W = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

If the force is constant, the integral evaluates to: ____________________________

Key Concept: Spring Systems

Consider a block attached to a horizontal spring on a frictionless surface. According to Hooke's law, the force required to compress or stretch the spring is:

$$F(x) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

where k is the spring constant, which depends on the spring's physical characteristics and is always positive.

Spring behavior:

- $x = 0$: ______________
- $x < 0$: ______________
- $x > 0$: ______________

Practice: Finding Work Done on a Spring

Suppose it takes a force of 10 N (in the negative direction) to compress a spring 0.2 m from the equilibrium position. How much work is done to stretch the spring 0.5 m from the equilibrium position?

- Step 1: Find the spring constant k.

  When $x = -0.2$, we know $F(x) = -10$, so:

$$F(x) = kx$$

$$k = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

- Step 2: Calculate the work done to stretch the spring.

$$W = \int_a^b f(x)\,dx = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

Work Done in Pumping

Key Concept: Work Done in Pumping

Consider the work done to pump water (or some other liquid) out of a tank. The process for calculating this work involves:

1. Define a frame of reference (typically with the origin at the top of the tank and the downward direction being positive)
2. Calculate the volume of a representative layer of water
3. Calculate the force needed to lift this layer (force = weight = volume × weight-density)
4. Determine the distance the layer must be lifted
5. Calculate the work for each layer and integrate to find the total work

Problem-Solving Strategy: Solving Pumping Problems

1. Sketch a picture of the tank and select an appropriate frame of reference.
2. Calculate the volume of a representative layer of water.
3. Multiply the volume by the weight-density of water to get the force.
4. Calculate the distance the layer of water must be lifted.
5. Multiply the force and distance to get an estimate of the work needed to lift the layer of water.
6. Sum the work required to lift all the layers to form a Riemann sum.
7. Take the limit as $n \to \infty$ and evaluate the resulting integral.

Practice: Pumping Water from a Cylindrical Tank

Assume a cylindrical tank of radius 4 m and height 10 m is filled to a depth of 8 m. How much work does it take to pump all the water over the top edge of the tank?

- Step 1: Define the frame of reference.

  Let *x* represent ____________________

  The water extends from x = __________ to x = __________

- Step 2: Calculate the volume of a representative layer.

  V = ____________________

- Step 3: Calculate the force needed to lift each layer.

  F = ____________________

- Step 4: Calculate the work to lift all layers.

  W = ____________________

- Step 5: Evaluate the integral.

  W = ____________________

Practice: Pumping Water from a Conical Tank

Assume a tank in the shape of an inverted cone, with height 12 ft and base radius 4 ft. The tank is full to start with, and water is pumped over the upper edge of the tank until the height of the water remaining in the tank is 4 ft.

How much work is required to pump out that amount of water?

- Step 1: Define the frame of reference.

  We need to partition the interval:

- Step 2: Find the radius of the disk of water at a given depth.

  Using similar triangles:

r_i = ______________________

- Step 3: Calculate the volume of a layer of water.

V_i = ______________________

- Step 4: Calculate the force needed to lift each layer.

$F_i \approx$ ______________________

- Step 5: Set up the integral for the total work.

W = ______________________

- Step 6: Evaluate the integral.

W = _____________

Think About It

Large municipal water systems must pump water to higher elevations for distribution. How might the shape of a water tower be designed to minimize the energy required for pumping? Would certain shapes be more efficient than others?

____

____

Hydrostatic Force and Pressure

Force is measured in newtons (metric) or pounds (English), and pressure is force per unit area, measured in ______________(metric) or ______________(English).

For a plate of area *A* submerged horizontally in water at a depth s:

- The force exerted on the plate is:

$$F = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

- The hydrostatic pressure is:

$$p = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

By Pascal's principle, the pressure at a given depth is the ___________________________ directions.

For vertically oriented surfaces, the depth varies from point to point, so we must integrate to find the _____________________.

Key Concept: Finding the Force on a Vertical Surface

For a thin vertical plate submerged in water:

1. Partition the plate into horizontal strips
2. The force on each strip is:

$$F_i = \_\_\_\_\_\_\_\_\_\_\_\_$$

3. The total force on the plate is:

F = ________________

Problem-Solving Strategy: Finding Hydrostatic Force

1. Sketch a picture and select an appropriate frame of reference.
2. Determine the ________________and ________________ functions, $s(x)$ and $w(x)$.
3. Determine the weight-density of the liquid (water: _____________ lb/ft³ or __________ N/m³).
4. Use the equation to calculate the total force.

Practice: Finding Hydrostatic Force

A water trough 15 ft long has ends shaped like inverted isosceles triangles, with base 8 ft and height 3 ft. Find the force on one end of the trough if the trough is full of water.

- Step 1: Define the frame of reference.

  Let the x-axis be oriented ________________ and the downward direction being ________________

  The origin corresponds to ________________

- Step 2: Determine the depth and width functions.

  $s(x)$ = ________________

  $w(x)$ = ________________

- Step 3: Set up the integral for the total force.

  F = ________________

- Step 4: Evaluate the integral.

  F = ________________

Think About It

Large dams must withstand enormous hydrostatic forces. The pressure increases linearly with depth, but the force depends on both the pressure and the surface area at each depth. How does this explain why dams are typically designed to be thicker at the bottom than at the top? What would happen if a dam were built with uniform thickness?

Center of Mass and Moments

The basic idea of the **center of mass** is the notion of a ______________________. For example, a plate balances perfectly on a stick when the stick is placed at a specific point called the ______________________of the plate.

Key Concept: Center of Mass of Objects on a Line

Consider a long, thin wire or rod of negligible mass resting on a fulcrum, with masses m_1 and m_2 placed at distances d_1 and d_2 from the fulcrum.

The masses balance each other if and only if:

$$m_1d_1 = \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_}$$

For point masses m_1 and m_2 located on a number line at points x_1 and x_2, the center of mass $\bar{x}$ is the point where the fulcrum should be placed to make the system balance.

Through algebraic manipulation: $\bar{x}$ = ____________________

The expression in the numerator, $m_1x_1 + m_2x_2$, is called the ____________________________.

The expression in the denominator, $m_1 + m_2$, is the ___________________________________.

For n point masses, the center of mass is given by:

$$\bar{x} = \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_}$$

Definition

Let $m_1, m_2, \ldots, m_n$ be point masses placed on a number line at points $x_1, x_2, \ldots, x_n$, and let $M = \sum_{i=1}^{n} m_i$ denote the total mass of the system. Then:

1. The moment of the system with respect to the origin is:

M = ________________________

2. The center of mass of the system is:

x̄ = ________________________

Practice: Finding Center of Mass on a Line

Suppose four point masses are placed on a number line as follows:

- m_1 = 30 kg, placed at x_1 = -2 m
- m_2 = 5 kg, placed at x_2 = 3 m
- m_3 = 10 kg, placed at x_3 = 6 m
- m_4 = 15 kg, placed at x_4 = -3 m

Find the moment of the system with respect to the origin and the center of mass of the system.

- Step 1: Calculate the moment of the system.

$$M = \sum m_i x_i$$

$$M =$$

- Step 2: Find the total mass of the system.

$$m = \sum m_i$$

$$m =$$

- Step 3: Calculate the center of mass.

$$\bar{x} = \frac{M}{m} = \underline{\hspace{6em}}$$

Think About It

When designing a car, engineers must consider the center of mass to ensure stability. What happens if a car's center of mass is too high? How might the center of mass change when a vehicle is loaded with cargo, and why might this matter for driving safety?

Center of Mass of Objects in a Plane

Key Concept: Center of Mass of Objects in a Plane

For a point mass m_1 located at point (x_1, y_1) in the plane:

- The moment M_x with respect to the x-axis is given by: ____________________
- The moment M_y with respect to the y-axis is given by: ____________________

Note that the x-coordinate is used to calculate the moment with respect to the y-axis, and vice versa, because:

- The x-coordinate gives the ____
- The y-coordinate gives the ____

Definition

Let $m_1, m_2, \ldots, m_n$ be point masses located in the xy-plane at points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, and let $m = \sum_{i=1}^{n} m_i$ denote the total mass of the system. Then:

1. The moments M_x and M_y of the system with respect to the x- and y-axes are:

$$M_x = \underline{\hspace{10em}}$$

M_y = ______________________________

2. The coordinates of the center of mass $(\bar{x}, \bar{y})$ are:

$\bar{x}$ = ______________________________

$\bar{y}$ = ______________________________

Practice: Finding Center of Mass in a Plane

Suppose three point masses are placed in the *xy*-plane as follows:

- m_1 = 2 kg, placed at (-1, 3)
- m_2 = 6 kg, placed at (1, 1)
- m_3 = 4 kg, placed at (2, -2)

Find the center of mass of the system.

- Step 1: Calculate the total mass of the system.

m = ______________________________

- Step 2: Find the moments with respect to the *x*- and *y*-axes.

M_y = ______________________________

M_x = ______________________________

- Step 3: Calculate the coordinates of the center of mass.

$\bar{x}$ = ______________________________

$\bar{y}$ = ______________________________

Think About It

If we have a collection of point masses in the *xy*-plane and we shift all masses by the same vector (*a*, *b*), how does this affect the center of mass? Try proving your conjecture mathematically.

Center of Mass of Thin Plates

When mass is distributed continuously across a thin sheet (called a ______________), we can find its center of mass or centroid.

For a rectangular lamina, the center of mass is where the ______________. This follows the symmetry principle.

Key Concept: The Symmetry Principle

If a region R is symmetric about a line l, then the centroid of R lies on ______________.

Finding the Center of Mass of a Lamina

Consider a lamina bounded above by the graph of a continuous function $f(x)$, below by the x-axis, and on the left and right by the lines $x = a$ and $x = b$.

To find the center of mass:

1. Divide the lamina into thin vertical strips
2. The mass of a strip is approximately: ____________________
3. The total mass is approximately: ____________________
4. Taking the limit as $n \to \infty$: m = ____________________
5. The moment with respect to the x-axis is: M_x = ____________________
6. The moment with respect to the y-axis is: M_y = ____________________
7. The coordinates of the center of mass are: $\bar{x}$ = ______________ and $\bar{y}$ = ______________

Key Concept: Center of Mass of a Thin Plate in the *xy*-Plane

Let R denote a region bounded above by the graph of a continuous function $f(x)$, below by the x-axis, and on the left and right by the lines $x = a$ and $x = b$. Let ρ denote the density of the associated lamina. Then:

1. The mass of the lamina is:

$m =$ ______________________________

2. The moments M_x and M_y of the lamina with respect to the x- and y-axes are:

$M_x =$ ______________________________

$M_y =$ ______________________________

3. The coordinates of the center of mass $(\bar{x}, \bar{y})$ are:

$\bar{x} =$ ______________________________

$\bar{y} =$ ______________________________

Practice: Finding the Center of Mass of a Lamina

Let R be the region bounded above by the graph of the function $f(x) = \sqrt{x}$ and below by the x-axis over the interval [0,4]. Find the centroid of the region.

- Step 1: Calculate the total mass (assuming $\rho = 1$ for simplicity).

$m =$ _______________

- Step 2: Compute the moments M_x and M_y.

M_x:

$M_x =$ _______________

M_y:

M_y = ______________

- Step 3: Find the coordinates of the centroid.

$\bar{x}$ = ____

$\bar{y}$ = ____

Think About It

When finding the center of mass of a lamina, we often assume constant density (ρ = 1) since the density cancels out in the calculation. Would this still be true if the density varied throughout the lamina? What modification would be needed in our formulas?

Center of Mass of a Region Bounded by Two Functions

Key Concept: Center of Mass of a Lamina Bounded by Two Functions

Now let's extend our approach to find centroids of regions bounded by two functions.

For a region bounded above by $f(x)$ and below by $g(x)$ over $[a,b]$:

1. The mass of the lamina is:

m = ____________________________

2. The moments M_x and M_y are:

M_x = ____________________________

M_y = ______________________________

3. The coordinates of the center of mass $(\bar{x}, \bar{y})$ are:

$\bar{x}$ = ______________________________

$\bar{y}$ = ______________________________

Practice: Finding the Center of Mass Between Two Curves

Let R be the region bounded above by the function $f(x) = 1 - x^2$ and below by the function $g(x) = x - 1$. Find the centroid of the region.

- Step 1: Find the points of intersection of the two functions.

- Step 2: Calculate the total mass (assuming $\rho = 1$).

m = _______________

- Step 3: Compute the moments M_x and My.

M_x:

M_x = _______________

M_y:

M_y = _______________

- Step 4: Find the coordinates of the centroid.

$\bar{x}$ = ______________________________

$\bar{y}$ = ______________________________

Think About It

For a region bounded by two functions, how does the location of the centroid change if we modify one of the bounding functions? For example, if we replace *g(x)* with *g(x) - c* where *c* is a positive constant, how would this affect the *y*-coordinate of the centroid?

The Symmetry Principle

Key Concept: The Symmetry Principle

The symmetry principle can greatly simplify finding centroids of symmetric regions.

Practice: Using the Symmetry Principle

Let *R* be the region bounded above by the function $f(x) = 4 - x^2$ and below by the *x*-axis.

- Step 1: Identify the symmetry in the region.

- Step 2: What does this symmetry tell us about the centroid?

- Step 3: Calculate the total mass (assuming $\rho = 1$).

m = ______________

- Step 4: Compute the necessary moment (M_x only).

  M_x:

M_x = ______________

- Step 5: Find the coordinates of the centroid.

  $\bar{x}$ = ____

  $\bar{y}$ = ____

Think About It

The symmetry principle states that if a region is symmetric about a line, the centroid lies on that line. What happens to the centroid when a region has multiple lines of symmetry? What can we conclude about regions with rotational symmetry?

Theorem of Pappus

Key Concept: Theorem of Pappus for Volume

The **theorem of Pappus** for volume allows us to find the volume of particular kinds of solids by using the centroid.

Definition

Let R be a region in the plane and let l be a line in the plane that does not intersect R. Then the volume of the solid of revolution formed by revolving R around l is equal to:

V = ____________________

where A is the area of R and d is the distance traveled by the centroid of R.

Practice: Applying the Theorem of Pappus

Let R be a circle of radius 2 centered at (4,0). Use the theorem of Pappus for volume to find the volume of the torus generated by revolving R around the y-axis.

- Step 1: Find the area of region R.

  A = ____________________

- Step 2: Identify the centroid of R.

  The centroid of R is the ________________________

- Step 3: Calculate the distance traveled by the centroid when R is revolved around the y-axis.

  d = ________________________

- Step 4: Apply the theorem of Pappus to find the volume.

  V = ________________________

Think About It

The theorem of Pappus relates the volume of a solid of revolution to the area of the generating region and the distance traveled by its centroid. What happens to this relationship if the axis of rotation intersects the region? Why doesn't the theorem apply in this case?

Key Takeaways

Summarize the most important concepts from each section:

Physical Applications:

- ____
- ____
- ____

Moments and Centers of Mass:

- ____
- ____
- ____

Questions to Ask in Class:

1. ____
2. ____
3. ____