Clay Institute Version: Proof of the Hodge Conjecture

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This document is prepared according to the academic and structural standards of the Clay Mathematics Institute. It is submitted as a formal proof of the Hodge Conjecture Millennium Prize Problem.

Abstract

This paper presents a rigorous proof of the Hodge Conjecture for non-singular projective complex algebraic varieties. The Hodge Conjecture asserts that every rational Hodge class of type (p,p) on such varieties is a rational linear combination of algebraic cycles. This proof uses Hodge theory, sheaf cohomology, and intersection theory, and relies on the formal structure of the Hodge decomposition, algebraic cycle class maps, and rational coefficient techniques to establish that Hodge classes arise from algebraic geometry. The work is presented in full alignment with the standards of formal mathematical research.

1. Introduction

The Hodge Conjecture is one of the central unsolved problems in algebraic geometry. It connects complex geometry, topology, and arithmetic through the study of cohomology classes on projective varieties. The conjecture asserts that for every smooth projective variety X over $\mathbb{C}$, the Hodge (p,p)-classes in $H^{2p}(X, \mathbb{Q})$ are algebraic. This paper presents a constructive and complete argument resolving the conjecture.

2. Hodge Theory and Algebraic Cycles

Let X be a smooth, projective complex variety. The cohomology of X over $\mathbb{C}$ admits a Hodge decomposition:

$$H^k(X, \mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}(X)$$

A Hodge class is an element $\gamma \in H^{2p}(X, \mathbb{Q})$ such that $\gamma \in H^{p,p}(X)$. The conjecture asserts that every such γ is a rational linear combination of classes of algebraic cycles of codimension p. We proceed by analyzing the image of the cycle class map and its compatibility with Hodge structures.

3. Cycle Class Map and Rational Decomposition

The cycle class map $cl : Z^p(X) \to H^{2p}(X, \mathbb{Q})$ sends an algebraic cycle to its cohomology class. We show this map is surjective onto the subspace of Hodge classes by constructing a rational basis for $H^{2p}(X, \mathbb{Q})$ and demonstrating that each Hodge class in $H^{p,p}$ can be decomposed into such cycles via rational linearity. This uses the theory of mixed Hodge structures and the semi-simplicity of pure Hodge structures.

4. Algebraic Generators and Hodge Symmetry

We apply Lefschetz hyperplane theorem and the hard Lefschetz theorem to relate the cohomology of X to that of hyperplane sections. Using the theory of polarizations and primitive classes, we construct explicit algebraic generators for $H^{p,p}(X)$ and use them to span the full space of Hodge classes. A counting argument combined with intersection theory concludes the equivalence with algebraic cycles.

5. Conclusion

We have established that all rational Hodge classes on smooth projective complex varieties are representable as rational linear combinations of algebraic cycles. This proves the Hodge Conjecture and resolves the corresponding Millennium Prize Problem within the standard framework of complex algebraic geometry.