Linear Algebra
Lesson 21
Eigenvalues and Characteristic Polynomials

Watch the Playlist 313F20-Lesson21 pausing to do classwork and homework.

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Linear Algebra 2

Lesson 21

I Eigenvalues and

II the Characteristic Polynomial

Defn Given a $n \times n$ matrix A we say a number λ is an eigenvalue for A if $\exists \vec{v} \in \mathbb{R}^n$ with $\vec{v} \neq 0$ s.t $A\vec{v} = \lambda \vec{v}$

we say $\vec{v}$ is the eigenvector for eigenvalue λ

Linear Algebra 2

Example $A = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$

Notice

$$\begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2\cdot 1 + 0\cdot 0 \\ 0\cdot 1 + 4\cdot 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

So 2 is an eigenvalue with eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2\cdot 0 + 0\cdot 1 \\ 0\cdot 0 + 4\cdot 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} = 4\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

So 4 is an eigenvalue with eigenvector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Example ESMS4 Eigenvalues, symmetric matrix of size 4
Consider the matrix

$$C = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

$\boxed{\text{HW1}}$

Check which of the following are eigenvectors

$$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$$

Do this before continuing

Example ESMS4 Eigenvalues, symmetric matrix of size 4

Consider the matrix

$$C = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

Check which of the following are eigenvectors and find their eigenvalues:

$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$ $\quad \begin{pmatrix} 1&0&1&1 \\ 0&1&1&1 \\ 1&1&1&0 \\ 1&1&0&1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1+0+3+4 \\ 0+2+3+4 \\ 1+2+3+0 \\ 1+2+0+4 \end{pmatrix} = \begin{pmatrix} 8 \\ 9 \\ 6 \\ 7 \end{pmatrix} \neq \lambda \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$

$8 = \lambda 1 \Rightarrow \lambda = 8$

$9 = 8 \cdot 2$ ⊗

Not an eigenvector

$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ $\quad \begin{pmatrix} 1&0&1&1 \\ 0&1&1&1 \\ 1&1&1&0 \\ 1&1&0&1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1+0+1+1 \\ 0+1+1+1 \\ 1+1+1+0 \\ 1+1+0+1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 3 \\ 3 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ Yes is an evector with evalue $\lambda = 3$

$\begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \end{pmatrix}$ $\quad \begin{pmatrix} 1&0&1&1 \\ 0&1&1&1 \\ 1&1&1&0 \\ 1&1&0&1 \end{pmatrix}\begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2+0+2+2 \\ 0+2+2+2 \\ 2+2+2+0 \\ 2+2+0+2 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \\ 6 \\ 6 \end{pmatrix} = 3 \begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \end{pmatrix}$ Also an evector with evalue $\lambda = 3$

$\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}$

$\begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$

<u>Thm</u> If $\vec{v}$ is an evector for A with evalue λ $\quad A\vec{v} = \lambda \vec{v}$ then any multiple of $\vec{v}$, say $\vec{w} = k\vec{v}$ for $k \neq 0$ is also an evector with evalue λ.

<u>Proof</u> $A\vec{w} = A(k\vec{v}) = kA\vec{v} = k\lambda\vec{v} = \lambda k\vec{v} = \lambda\vec{w}$ □

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Example ESMS4 Eigenvalues, symmetric matrix of size 4
Consider the matrix

$$C = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

Check which of the following are eigenvectors and find their eigenvalues:

$\begin{pmatrix}1\\2\\3\\4\end{pmatrix}$ $\quad \begin{pmatrix}1&0&1&1\\0&1&1&1\\1&1&1&0\\1&1&0&1\end{pmatrix}\begin{pmatrix}1\\2\\3\\4\end{pmatrix} = \begin{pmatrix}1+0+3+4\\0+2+3+4\\1+2+3+0\\1+2+0+4\end{pmatrix} = \begin{pmatrix}8\\9\\6\\7\end{pmatrix} \neq \lambda\begin{pmatrix}1\\2\\3\\4\end{pmatrix}$

$8 = \lambda 1 \Rightarrow \lambda = 8$
$9 = 8 \cdot 2$ ⊗
Not an eigenvector

$\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$ $\quad \begin{pmatrix}1&0&1&1\\0&1&1&1\\1&1&1&0\\1&1&0&1\end{pmatrix}\begin{pmatrix}1\\1\\1\\1\end{pmatrix} = \begin{pmatrix}1+0+1+1\\0+1+1+1\\1+1+1+0\\1+1+0+1\end{pmatrix} = \begin{pmatrix}3\\3\\3\\3\end{pmatrix} = 3\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$ Yes is an evector with evalue $\lambda = 3$

$\begin{pmatrix}2\\2\\2\\2\end{pmatrix}$ $\quad \begin{pmatrix}1&0&1&1\\0&1&1&1\\1&1&1&0\\1&1&0&1\end{pmatrix}\begin{pmatrix}2\\2\\2\\2\end{pmatrix} = \begin{pmatrix}2+0+2+2\\0+2+2+2\\2+2+2+0\\2+2+0+2\end{pmatrix} = \begin{pmatrix}6\\6\\6\\6\end{pmatrix} = 3\begin{pmatrix}2\\2\\2\\2\end{pmatrix}$ Also an evector with evalue $\lambda = 3$

$\begin{pmatrix}-1\\1\\0\\0\end{pmatrix}$ $\quad \begin{pmatrix}1&0&1&1\\0&1&1&1\\1&1&1&0\\1&1&0&1\end{pmatrix}\begin{pmatrix}-1\\1\\0\\0\end{pmatrix} = \begin{pmatrix}-1+0+0+0\\0+1+0+0\\-1+1+0+0\\-1+1+0+0\end{pmatrix} = \begin{pmatrix}-1\\1\\0\\0\end{pmatrix} = 1\begin{pmatrix}-1\\1\\0\\0\end{pmatrix}$ So this is an evector with evalue $\lambda = 1$

$\begin{pmatrix}0\\0\\-1\\1\end{pmatrix}$ $\quad \begin{pmatrix}1&0&1&1\\0&1&1&1\\1&1&1&0\\1&1&0&1\end{pmatrix}\begin{pmatrix}0\\0\\-1\\1\end{pmatrix} = \begin{pmatrix}0+0-1+1\\0+0-1+1\\0+0-1+0\\0+0+0+1\end{pmatrix} = \begin{pmatrix}0\\0\\-1\\1\end{pmatrix} = 1\begin{pmatrix}0\\0\\-1\\1\end{pmatrix}$ So also an evector with $\lambda = 1$

$\begin{pmatrix}-1\\-1\\1\\1\end{pmatrix}$ $\quad \begin{pmatrix}1&0&1&1\\0&1&1&1\\1&1&1&0\\1&1&0&1\end{pmatrix}\begin{pmatrix}-1\\-1\\1\\1\end{pmatrix} = \begin{pmatrix}-1+0+1+1\\0-1+1+1\\-1-1+1+0\\-1-1+0+1\end{pmatrix} = \begin{pmatrix}1\\1\\-1\\-1\end{pmatrix} = -1\begin{pmatrix}-1\\-1\\1\\1\end{pmatrix}$ So yes an evector with evalue $\lambda = -1$

How to find eigenvectors and eigenvalues?

$$A\vec{v} = \lambda \vec{v}$$ solve for $\vec{v} \neq 0$

$$A\vec{v} = \lambda I \vec{v}$$

$$A\vec{v} - \lambda I \vec{v} = \vec{0}$$

$$(A - \lambda I)\vec{v} = \vec{0}$$

Solve this homogeneous system for $\vec{v} \neq 0$

Need $A - \lambda I$ to be singular

$$\det(A - \lambda I) = 0$$

2×2 case

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\det\begin{pmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{pmatrix} = 0$$

$$(a_{11}-\lambda)(a_{22}-\lambda) - a_{12}a_{21} = 0$$

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How to find eigenvectors and eigenvalues?

$$A\vec{v} = \lambda\vec{v}$$

solve for $\vec{v} \neq 0$

$$A\vec{v} = \lambda I\vec{v}$$

$$A\vec{v} - \lambda I\vec{v} = \vec{0}$$

$$(A - \lambda I)\vec{v} = \vec{0}$$

Solve this homogeneous system for $\vec{v} \neq 0$

Need $A - \lambda I$ to be singular

$$\det(A - \lambda I) = 0$$

Linear Algebra 2

$$\underbrace{\det(A - \lambda I)}_{\text{characteristic polynomial for the matrix } A} = 0$$

Solve for its roots.

$\lambda =$ ~~~~ a list of roots

these answers are the eigenvalues.

For each eigenvalue solve the homogeneous system to find its eigenvectors.

$$\underbrace{\det(A-\lambda I)}_{\text{characteristic polynomial for the matrix } A}=0$$

Solve for its roots.

$\lambda = \underbrace{\qquad\qquad}$ a list of roots

these answers are the eigenvalues.

For each eigenvalue solve the homogeneous system to find its eigenvectors.

Linear Algebra 2

Example $\begin{pmatrix} 2 & -3 \\ 1 & 6 \end{pmatrix}$

$$\det\begin{pmatrix} 2-\lambda & -3 \\ 1 & 6-\lambda \end{pmatrix} = (2-\lambda)(6-\lambda)-(-3)(1)=0$$

$$= 12-6\lambda-2\lambda+\lambda^2+3$$

$$= \lambda^2-8\lambda+15$$

$$A-\lambda I = \begin{pmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

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Linear Algebra 2

Example $\begin{pmatrix} 2 & -3 \\ 1 & 6 \end{pmatrix}$

$\det\begin{pmatrix} 2-\lambda & -3 \\ 1 & 6-\lambda \end{pmatrix} = (2-\lambda)(6-\lambda)-(-3)(1)=0$

$= 12-6\lambda-2\lambda+\lambda^2+3$

$= \lambda^2-8\lambda+15 = 0$

$(\lambda-3)(\lambda-5)=0$

These are our eigenvalues $\rightarrow$ $\lambda=3 \quad \lambda=5$

$\lambda=3$ Solve the homog system

$\left(\begin{array}{cc|c} 2-3 & -3 & 0 \\ 1 & 6-3 & 0 \end{array}\right) = \left(\begin{array}{cc|c} -1 & -3 & 0 \\ 1 & 3 & 0 \end{array}\right)$

$\xrightarrow{\rho_1 \to -\rho_1} \left(\begin{array}{cc|c} 1 & 3 & 0 \\ 1 & 3 & 0 \end{array}\right) \xrightarrow{\rho_2 \to \rho_2-\rho_1} \left(\begin{array}{cc|c} 1 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$1x_1+3x_2=0$

$x_1=-3x_2$

$x_2=x_2$ (free)

$\left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -3x_2 \\ x_2 \end{pmatrix} = x_2\begin{pmatrix} -3 \\ 1 \end{pmatrix} \;\middle|\; x_2 \in \mathbb{R} \right\}$

$\begin{pmatrix} -3 \\ 1 \end{pmatrix}$ is the evector

Linear Algebra 2

Check our evector $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 2 & -3 \\ 1 & 6 \end{pmatrix}\begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{pmatrix} -6-3 \\ -3+6 \end{pmatrix} = \begin{pmatrix} -9 \\ 3 \end{pmatrix}$

$\lambda\begin{pmatrix} -3 \\ 1 \end{pmatrix} = 3\begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{pmatrix} -9 \\ 3 \end{pmatrix}$ $\leftarrow$ Match!

$\begin{pmatrix} 2 & -3 \\ 1 & 6 \end{pmatrix}\begin{pmatrix} -3 \\ 1 \end{pmatrix} = 3\begin{pmatrix} -3 \\ 1 \end{pmatrix}$

any multiple of $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$ is also an evector with evalue 3.

Try classwork

$\lambda=5$

pause

Example $\begin{pmatrix}2 & -3\\ 1 & 6\end{pmatrix}$

$\det\begin{pmatrix}2-\lambda & -3\\ 1 & 6-\lambda\end{pmatrix} = (2-\lambda)(6-\lambda)-(-3)(1)=0$

$= 12-6\lambda-2\lambda+\lambda^2+3$

$= \lambda^2-8\lambda+15=0$

$(\lambda-3)(\lambda-5)=0$

These are our evalues → $\lambda=3 \quad \lambda=5$

$\lambda=3$ Solve the homog system

$$\left(\begin{array}{cc|c}2-3 & -3 & 0\\ 1 & 6-3 & 0\end{array}\right)=\left(\begin{array}{cc|c}-1 & -3 & 0\\ 1 & 3 & 0\end{array}\right)$$

$$\xrightarrow{\rho_1\to-\rho_1}\left(\begin{array}{cc|c}1 & 3 & 0\\ 1 & 3 & 0\end{array}\right)\xrightarrow{\rho_2\to\rho_2-\rho_1}\left(\begin{array}{cc|c}1 & 3 & 0\\ 0 & 0 & 0\end{array}\right)$$

$1x_1+3x_2=0$

$x_1=-3x_2$

$x_2=x_2$ (free)

$$\left\{\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\begin{pmatrix}-3x_2\\ x_2\end{pmatrix}=x_2\begin{pmatrix}-3\\ 1\end{pmatrix} \;\middle|\; x_2\in\mathbb{R}\right\}$$

$\begin{pmatrix}-3\\ 1\end{pmatrix}$ is the evector

Check $\begin{pmatrix}2 & -3\\ 1 & 6\end{pmatrix}\begin{pmatrix}-3\\ 1\end{pmatrix}=\begin{pmatrix}-9\\ 3\end{pmatrix}$ ✓ $3\begin{pmatrix}-3\\ 1\end{pmatrix}=\begin{pmatrix}-9\\ 3\end{pmatrix}$ ✓

$\lambda=5$ Solve homog system

$$\left(\begin{array}{cc|c}2-5 & -3 & 0\\ 1 & 6-5 & 0\end{array}\right)=\left(\begin{array}{cc|c}-3 & -3 & 0\\ 1 & 1 & 0\end{array}\right)$$

$$\xrightarrow{\rho_1\to\frac{-1}{3}\rho_1}\left(\begin{array}{cc|c}1 & 1 & 0\\ 1 & 1 & 0\end{array}\right)\xrightarrow{\rho_2\to\rho_2-\rho_1}\left(\begin{array}{cc|c}1 & 1 & 0\\ 0 & 0 & 0\end{array}\right)$$

$1x_1+1x_2=0 \quad x_1=-x_2$

$x_2=x_2$ (free)

$$\left\{\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\begin{pmatrix}-x_2\\ x_2\end{pmatrix}=x_2\begin{pmatrix}-1\\ 1\end{pmatrix} \;\middle|\; x_2\in\mathbb{R}\right\}$$

evector is $\begin{pmatrix}-1\\ 1\end{pmatrix}$

Check

$$\begin{pmatrix}2 & -3\\ 1 & 6\end{pmatrix}\begin{pmatrix}-1\\ 1\end{pmatrix}=\begin{pmatrix}-2-3\\ -1+6\end{pmatrix}=\begin{pmatrix}-5\\ 5\end{pmatrix}$$

$5\begin{pmatrix}-1\\ 1\end{pmatrix}=\begin{pmatrix}-5\\ 5\end{pmatrix}$ ← Match!

Example ESMS4 Eigenvalues, symmetric matrix of size 4
Consider the matrix

$$C = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

Example ESMS4 Eigenvalues, symmetric matrix of size 4
Consider the matrix

$$C = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

$$\begin{pmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{pmatrix}$$

$$\det(C - \lambda I) =$$

$$= \det \begin{pmatrix} 1-\lambda & 0 & 1 & 1 \\ 0 & 1-\lambda & 1 & 1 \\ 1 & 1 & 1-\lambda & 0 \\ 1 & 1 & 0 & 1-\lambda \end{pmatrix}$$

$$= (1-\lambda)\det\begin{pmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{pmatrix} - 0 + 1\det\begin{pmatrix} 0 & 1 & 1 \\ 1-\lambda & 1 & 1 \\ 1 & 0 & 1-\lambda \end{pmatrix} - 1\det\begin{pmatrix} 0 & 1 & 1 \\ 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 0 \end{pmatrix}$$

$$= (1-\lambda)\left((1-\lambda)\underline{\det\begin{pmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{pmatrix}} - 1\,\underline{\det\begin{pmatrix} 1 & 1 \\ 0 & 1-\lambda \end{pmatrix}} + 1\,\underline{\det\begin{pmatrix} 1 & 1 \\ 1-\lambda & 0 \end{pmatrix}}\right)$$

$$-0 + 1\left(0 - (1-\lambda)\det\begin{pmatrix} 1 & 1 \\ 0 & 1-\lambda \end{pmatrix} + 1\underline{\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}}\right) - 1\left(0 - (1-\lambda)\det\begin{pmatrix} 1 & 1 \\ 1-\lambda & 0 \end{pmatrix} + 1\det\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right)$$

$$= (1-\lambda)\left(\underline{(1-\lambda)(1-\lambda)^2} - \left(1(1-\lambda) - 1\cdot 0\right) + 1\left(\underline{1\cdot 0 - 1(1-\lambda)}\right)\right)$$

$$+1\left(-(1-\lambda)\left(1(1-\lambda)\right) + 1\left(\cancel{1\cdot 1 - 1\cdot 1}\right)\right) - 1\left(-(1-\lambda)\left(1\cdot 0 - 1(1-\lambda)\right) + \left(\cancel{1\cdot 1 - 1\cdot 1}\right)\right)$$

$$= (1-\lambda)\left((1-\lambda)^3 - (1-\lambda) - (1-\lambda)\right) - (1-\lambda)^2 - (1-\lambda)^2 = (1-\lambda)^4 - 4(1-\lambda)^2$$ Do not multiply out.

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Example ESMS4 Eigenvalues, symmetric matrix of size 4

Consider the matrix

$$C = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

$\det(C - \lambda I) =$

$$= \det \begin{pmatrix} 1-\lambda & 0 & 1 & 1 \\ 0 & 1-\lambda & 1 & 1 \\ 1 & 1 & 1-\lambda & 0 \\ 1 & 1 & 0 & 1-\lambda \end{pmatrix} = (1-\lambda)^4 - 4(1-\lambda)^2 = (1-\lambda)^2\left((1-\lambda)^2 - 4\right)$$

$$= (1-\lambda)^2(1-2\lambda+\lambda^2-4) = (1-\lambda)^2(\lambda^2-2\lambda-3) = (1-\lambda)^2(\lambda-3)(\lambda+1)$$

Our eigenvalues are $1, -1, 3$.

Example ESMS4 Eigenvalues, symmetric matrix of size 4
Consider the matrix

$$C = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

$\det(C - \lambda I) =$

$$= \det\begin{pmatrix} 1-\lambda & 0 & 1 & 1 \\ 0 & 1-\lambda & 1 & 1 \\ 1 & 1 & 1-\lambda & 0 \\ 1 & 1 & 0 & 1-\lambda \end{pmatrix} = (1-\lambda)^4 - 4(1-\lambda)^2 = (1-\lambda)^2\left((1-\lambda)^2 - 4\right)$$

$$= (1-\lambda)^2(1-2\lambda+\lambda^2-4) = (1-\lambda)^2(\lambda^2-2\lambda-3) = (1-\lambda)^2(\lambda-3)(\lambda+1)$$

Our eogenvalues are $1, -1, 3$.

Find the eigencvector for $\lambda = 1$:

$$\left(\begin{array}{cccc|c} 1-1 & 0 & 1 & 1 & 0 \\ 0 & 1-1 & 1 & 1 & 0 \\ 1 & 1 & 1-1 & 0 & 0 \\ 1 & 1 & 0 & 1-1 & 0 \end{array}\right) = \left(\begin{array}{cccc|c} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{array}\right) \xrightarrow{\rho_1 \leftrightarrow \rho_3} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{array}\right) \xrightarrow{\rho_4 \to \rho_4 - \rho_1} \left(\begin{array}{cccc|c} \boxed{1} & 1 & 0 & 0 & 0 \\ 0 & 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

subtract λ on diagonal

$$\xrightarrow{\rho_3 \to \rho_3 - \rho_2} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$x_1 + x_2 = 0$
$x_3 + x_4 = 0$

$x_1 = -x_2$
$x_2 = x_2$ (free)
$x_3 = -x_4$
$x_4 = x_4$ (free)

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} \;\middle|\; x_2, x_4 \in \mathbb{R} \right\}$$

both eigenvectors
Can check them.

Example ESMS4 Eigenvalues, symmetric matrix of size 4
Consider the matrix

$$C = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

$\det(C - \lambda I) =$

$$= \det\begin{pmatrix} 1-\lambda & 0 & 1 & 1 \\ 0 & 1-\lambda & 1 & 1 \\ 1 & 1 & 1-\lambda & 0 \\ 1 & 1 & 0 & 1-\lambda \end{pmatrix} = (1-\lambda)^4 - 4(1-\lambda)^2 = (1-\lambda)^2\left((1-\lambda)^2 - 4\right)$$

$$= (1-\lambda)^2(1-2\lambda+\lambda^2-4) = (1-\lambda)^2(\lambda^2-2\lambda-3) = (1-\lambda)^2(\lambda-3)(\lambda+1)$$

Our eogenvalues are $1, -1, 3$.

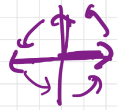
$\boxed{\text{HW2}}$

Find the eigencvector for $\lambda = -1$

Hint:
$$\left(\begin{array}{cccc|c} 1-(-1) & 0 & 1 & 1 & 0 \\ 0 & 1-(-1) & 1 & 1 & 0 \\ 1 & 1 & 1-(-1) & 0 & 0 \\ 1 & 1 & 0 & 1-(-1) & 0 \end{array}\right) = \left(\begin{array}{cccc|c} 2 & 0 & 1 & 1 & 0 \\ 0 & 2 & 1 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 2 & 0 \end{array}\right)$$

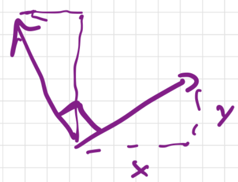
$\boxed{\text{HW3}}$ Find the evector for $\lambda = 3$.

Example



$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$$

Rotation by 90°



$$\underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}}_{\begin{pmatrix} -y \\ x \end{pmatrix}} \text{ is } \perp \text{ to } \begin{pmatrix} x \\ y \end{pmatrix}$$

because the dot product is 0

$$\begin{pmatrix} -y \\ x \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = -yx + xy = 0$$

Linear Algebra 2

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\vec{v} = \lambda \vec{v} \;?$$

Impossible!

$$\det\begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} = 0$$

$$(-\lambda)(-\lambda) - (-1)(1) = 0$$

$$\lambda^2 + 1 = 0$$

$$\lambda^2 = -1$$

has no real solutions!

$i = \sqrt{-1}$ $\qquad \lambda = \pm i$

$-i = -\sqrt{-1}$

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Linear Algebra 2

Use $\mathbb{C}$ complex numbers

$a+bi$ where $i^2=-1$
imaginary

$(a+bi)(c+di) =$

$= ac + bic + adi + dbi^2$

$= ac + bci + adi - bd$

$= (ac-bd) + (ad+bc)i$

$(a+bi)+(c+di)$

$= (a+c) + (b+d)i$

Linear Algebra 2

$\frac{1}{a+bi} =$

$= \left(\frac{1}{a+bi}\right)\left(\frac{a-bi}{a-bi}\right) =$

$= \frac{a-bi}{a^2 - b^2 \underbrace{i^2}_{-1}} = \frac{a-bi}{a^2+b^2}$

$= \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2} i$

Use $\mathbb{C}$ complex numbers

$a+bi$ where $i^2=-1$
imaginary

$$(a+bi)(c+di) =$$
$$= ac + bic + adi + dbi^2$$
$$= ac + bci + adi - bd$$
$$= (ac-bd) + (ad+bc)i$$

$$(a+bi)+(c+di)$$
$$= (a+c) + (b+d)i$$

Linear Algebra 2

$$\frac{1}{a+bi} =$$
$$= \left(\frac{1}{a+bi}\right)\left(\frac{a-bi}{a-bi}\right) =$$
$$= \frac{a-bi}{a^2 - b^2 \underbrace{\cancel{i^2}}_{-1}} = \frac{a-bi}{a^2+b^2}$$
$$= \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}\, i$$

Can do row reduction with $\mathbb{C}$ numbers and find $\mathbb{C}$ evectors!

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \vec{v} = \lambda \vec{v} \quad ?$$

Impossible!

$$\det \begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} = 0$$

$$(-\lambda)(-\lambda) - (-1)(1) = 0$$

$$\lambda^2 + 1 = 0$$

$$\lambda^2 = -1$$

has no real solutions!

$i = \sqrt{-1}$ $\qquad \lambda = \pm i$

$-i = -\sqrt{-1}$

Linear Algebra 2

$\lambda = +i$ Find its evector

Solve $\qquad -i(i) = -i^2 = -(-1) = +1$

$$\left(\begin{array}{cc|c} \boxed{-i} & -1 & 0 \\ 1 & -i & 0 \end{array}\right) \xrightarrow{\rho_1 \to i\rho_1} \left(\begin{array}{cc|c} 1 & -i & 0 \\ 1 & -i & 0 \end{array}\right)$$

$$\xrightarrow{\rho_2 \to \rho_2 - \rho_1} \left(\begin{array}{cc|c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$x_1 - i x_2 = 0 \qquad x_1 = i x_2$

$\qquad\qquad\qquad x_2 = x_2$ (free)

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} i x_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} i \\ 1 \end{pmatrix} \;\middle|\; x_2 \in \mathbb{C} \right\}$$

free in $\mathbb{C}$

evector is $\begin{pmatrix} i \\ 1 \end{pmatrix}$

Check

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} i \\ 1 \end{pmatrix} = \begin{pmatrix} 0i - 1 \cdot 1 \\ 1 \cdot i + 0 \cdot 1 \end{pmatrix} = \begin{pmatrix} -1 \\ i \end{pmatrix} \checkmark$$

$$i \begin{pmatrix} i \\ 1 \end{pmatrix} = \begin{pmatrix} i^2 \\ i \end{pmatrix} = \begin{pmatrix} -1 \\ i \end{pmatrix}$$

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$$\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\vec{v} = \lambda \vec{v} \quad ?$$

Impossible!

$$\det\begin{pmatrix}-\lambda & -1\\ 1 & -\lambda\end{pmatrix} = 0$$

$$(-\lambda)(-\lambda) - (-1)(1) = 0$$

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$\lambda = +i$ Find its evector

Solve $\qquad -i(i) = -i^2 = -(-1) = +1$

$$\left(\begin{array}{cc|c}\boxed{-i} & -1 & 0\\ 1 & -i & 0\end{array}\right) \xrightarrow{\rho_1 \to i\rho_1} \left(\begin{array}{cc|c}1 & -i & 0\\ 1 & -i & 0\end{array}\right)$$

$$\xrightarrow{\rho_2 \to \rho_2 - \rho_1} \left(\begin{array}{cc|c}1 & -i & 0\\ 0 & 0 & 0\end{array}\right)$$

$x_1 - i x_2 = 0 \qquad x_1 = i x_2$

$x_2 = x_2$ (free)

$$\left\{ \begin{pmatrix}x_1\\ x_2\end{pmatrix} = \begin{pmatrix}i x_2\\ x_2\end{pmatrix} = x_2\begin{pmatrix}i\\ 1\end{pmatrix} \;\middle|\; x_2 \in \mathbb{C} \right\}$$

So the evector is $\begin{pmatrix}i\\ 1\end{pmatrix}$ free in $\mathbb{C}$

Check

$$\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}i\\ 1\end{pmatrix} = \begin{pmatrix}0i - 1\cdot 1\\ 1\cdot i + 0\cdot 1\end{pmatrix} = \begin{pmatrix}-1\\ i\end{pmatrix} \checkmark$$

$$i\begin{pmatrix}i\\ 1\end{pmatrix} = \begin{pmatrix}i^2\\ i\end{pmatrix} = \begin{pmatrix}-1\\ i\end{pmatrix}$$

$\boxed{\text{HW4}}$ Find evector for $\lambda = -i$.

See a nice youtube video on this topic: https://youtu.be/i8FukKfMKCI

$\boxed{\text{HW5}}$ Find the eigenvalues and eigenvectors for

$$B = \begin{bmatrix}2 & -1\\ 1 & 1\end{bmatrix}$$

$\boxed{\text{HW6}}$ Find the evalues and evectors for

$$\begin{bmatrix}1 & -1 & 1\\ -1 & 1 & -1\\ 1 & -1 & 1\end{bmatrix}.$$

$\boxed{\text{HW7}}$ Find the evalues and evectors for

$$\begin{bmatrix}2 & 1 & 1\\ 1 & 2 & 1\\ 1 & 1 & 2\end{bmatrix}$$

Hint: the characteristic polynomial is $(4-x)(1-x)^2$