AP PRECALCULUS · TOPICS 1.9–1.10

Rational Functions: Vertical Asymptotes & Holes

Name: _____ **Date:** _____ **Class:** _____

Today I am learning to... analyze rational functions using factors, graphs, tables, and one-sided limits.

So I can... distinguish holes and asymptotes, describe behavior, and justify my conclusions.

I'll know I have learned it when... I can find domain restrictions, classify each discontinuity, locate holes, and write correct limit statements.

Essential Question: How does the denominator of a rational function help us predict its graph?

I · Warm-up and prerequisite skills

A rational function is a ratio of polynomials. Always exclude inputs that make the original denominator zero.

Key formula:

$$R(x) = \frac{N(x)}{D(x)}, \qquad D(x) \neq 0$$

W1. Factor the expressions

$$x^2 - 9 \qquad x^2 + 5x + 6$$

Difference of squares: $x^2 - 9 = (x - 3)(x + 3)$. **Product-sum factoring:** $x^2 + 5x + 6 = (x + 2)(x + 3)$.

W2. Find an excluded input

$$R(x) = \frac{x+2}{x-5}$$

Solve $x - 5 = 0$**, giving** $x = 5$. **Domain** $(-\infty, 5) \cup (5, \infty)$.

Check 1A. Why is division by zero not allowed? Answer: A function value cannot be defined by division by zero; exclude such inputs.

Check 1B. Find the excluded inputs of $\frac{3}{(x+2)(x-4)}$**. Answer:** $x = -2$ **and** $x = 4$.

Check IC. True/false: cancelling a factor puts its zero back into the domain. Answer: False. Original restrictions remain.

KEY IDEA: The original denominator gives the restrictions of the domain.

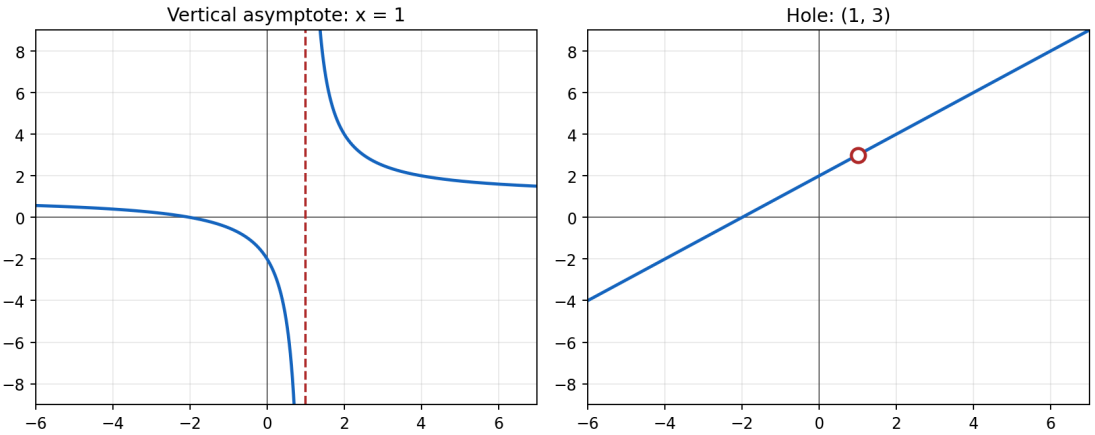
2 · Distinguishing a hole from a vertical asymptote

Vertical asymptote (VA): an excluded x-value where, after all common factors are removed, a denominator factor remains and the function becomes unbounded nearby. Hole: a missing point where the common denominator factor cancels completely and the simplified expression has a finite value.

Classify after factoring:

$$\frac{(x-a)A(x)}{(x-a)B(x)} = \frac{A(x)}{B(x)}, \quad x \neq a$$

If $B(a) \neq 0$, there is a hole at $x = a$. If a denominator factor remains at $x = a$ after cancellation, investigate a vertical asymptote.



Comparison of graph types

E1 · I DO. Hole versus asymptote

$$f(x) = \frac{(x-1)(x+2)}{(x-1)(x-4)}$$

Original denominator is zero at 1 and 4, so exclude both. Cancel $(x - 1)$ to get $(x + 2)/(x - 4)$ with $x \neq 1, 4$. Since 1 disappears completely, hole at $x = 1$, with $y = (1 + 2)/(1 - 4) = -1$, hence (1, −1). The factor $x - 4$ remains: VA $x = 4$.

E2 · WE DO. A second removable discontinuity

$$g(x) = \frac{x^2-9}{(x-3)(x+1)}$$

Factor $x^2 - 9 = (x - 3)(x + 3)$. Simplify to $(x + 3)/(x + 1)$, keeping $x \neq 3, -1$. The hole is at $x = 3$, with $y = 6/4 = 3/2$, thus $\left(3, \frac{3}{2}\right)$. VA at $x = -1$.

E3 · YOU DO. Work independently

$$h(x) = \frac{(x+2)(x-5)}{(x+2)(x-1)}$$

Cancel $x + 2$; original exclusions $-2, 1$. Simplified $(x - 5)/(x - 1)$. Hole at $(-2, (-7)/(-3)) = (-2, 7/3)$; VA $x = 1$.

Check 2A. When will an excluded input produce a hole? Answer: When its denominator factor completely cancels and the reduced function is finite there.

Check 2B. In $\frac{x-2}{(x-2)(x+3)}$, what is the hole coordinate? Answer: Reduced function $1/(x + 3)$; at $x = 2$, $y = 1/5$, so $(2, 1/5)$.

Check 2C. Correct the error: "Every denominator zero is a vertical asymptote." Answer: Some zeros become removable holes after cancellation.

KEY IDEA: Cancel completely → hole; denominator factor still present → vertical asymptote.

3 · Multiplicity and repeated factors

Multiplicity means how many times a factor repeats. Subtract the numerator and denominator powers for a common factor. Keep the original excluded input even if all copies cancel.

E4 · I DO. Repeated denominator factor

$$p(x) = \frac{(x-2)(x+3)}{(x-2)^2}$$

Cancel one $(x - 2)$, leaving $(x + 3)/(x - 2)$, with $x \neq 2$. A denominator copy remains; $x = 2$ is a VA, not a hole.

E5 · WE DO. All powers cancel

$$q(x) = \frac{(x+1)^3(x-4)}{(x+1)^2(x-2)}$$

Cancel two powers of $(x + 1)$ to obtain $(x + 1)(x - 4)/(x - 2)$, but $x = -1$ is still excluded: hole $(-1, 0)$. Remaining $x - 2$ gives VA $x = 2$.

E6 · YOU DO. Multiplicity challenge

$$r(x) = \frac{(x-3)^2(x+2)}{(x-3)^3(x+1)}$$

Reduce to $(x + 2)/[(x - 3)(x + 1)]$; VAs at $x = 3, -1$. No holes. Both 3 and −1 are excluded.

Check 3A. If numerator has $(x - 5)^2$ and denominator $(x - 5)^2$, what happens at 5 if other denominator factors are nonzero there? Answer: A hole at $x = 5$.

Check 3B. If numerator has $(x + 4)$ and denominator $(x + 4)^3$, what happens at −4? Answer: VA at $x = -4$ (two denominator copies remain).

Check 3C. Explain why you must compare powers before labeling a hole. Answer: Canceling one factor might still leave denominator copies, making a VA.

KEY IDEA: A common factor makes a hole only if its denominator power is zero after simplifying.

4 · One-sided limits and numerical evidence

The symbols $x \to a^-$ and $x \to a^+$ mean approach from the left and right. Outputs may grow without bound toward $+\infty$ or $-\infty$. These describe behavior; infinity is not a finite limit value.

Read the signs:

$$\lim_{x \to a^-} f(x) = +\infty, \qquad \lim_{x \to a^+} f(x) = -\infty$$

E7 · I DO. Different signs on each side

$$f(x) = \frac{3}{x-2}$$

On the left $x - 2$ is a tiny negative number, so $3/(x - 2) \to -\infty$. On the right it is tiny positive, so $3/(x - 2) \to +\infty$. VA $x = 2$.

E8 · WE DO. Same sign on each side

$$g(x) = \frac{2}{(x+1)^2}$$

The square is positive and tiny on either side, so both one-sided limits are $+\infty$; VA $x = -1$.

E9 · YOU DO. Sign investigation

$$h(x) = \frac{-5}{x+3}$$

Left: denominator tiny negative, so $-5/(negative) \to +\infty$. Right: denominator tiny positive, so $\to -\infty$.

Numerical table (calculator active). For $t(x) = \frac{x^2-16}{(x+4)^2}$, complete the table. Note: $x = -4$ is undefined.

x	−4.1	−4.01	−4.001	−4	−3.999	−3.99	−3.9
$t(x)$	+81	+801	+8001	undef.	−7999	−799	−79

Explanation: $\left(x^2 - 16\right)/(x + 4)^2 = (x - 4)/(x + 4)$ for $x \neq -4$. At -4^- numerator and denominator are both negative $\rightarrow +\infty$? Check: for $x = -4.1$, numerator $x - 4 = -8.1$, denominator $x + 4 = -0.1$, so +81. The numerical results above must therefore be +81, +801, +8001 on the left, and −7999, −799, −79 on the right. Thus $\lim_{x\to -4^-} t(x) = +\infty$ and $\lim_{x\to -4^+} t(x) = -\infty$.

Check 4A. For $\frac{1}{x-6}$, what is the left-hand behavior at 6? Answer: $-\infty$.

Check 4B. For $\frac{-2}{(x-1)^2}$, what are both one-sided behaviors at 1? Answer: Both $-\infty$.

Check 4C. Does $\lim_{x\to 2} \frac{1}{x-2}$ exist as a finite real number? Explain. Answer: No. Left is $-\infty$, right is $+\infty$.

KEY IDEA: Determine the sign of the numerator and denominator on each side separately.

5 · Coordinates, intercepts, end behavior and graphs

For a hole, substitute the excluded input into the simplified function to find its missing y-coordinate. For x-intercepts, set the uncancelled numerator to zero and check the original domain. To find horizontal end behavior, compare degrees and leading coefficients.

Degree test: Numerator degree smaller $\rightarrow y = 0$; equal degrees $\rightarrow$ ratio of leading coefficients; numerator degree larger $\rightarrow$ no horizontal asymptote (polynomial or slant behavior may occur).

E10 · I DO. Complete analysis

$$F(x) = \frac{(3x-13)(x-3)}{(x-2)(x-3)}$$

Cancel $(x - 3)$ to get $(3x - 13)/(x - 2)$ with exclusions 2,3. Hole $(3, (9 - 13)/(3 - 2)) = (3, -4)$. VA $x = 2$. Horizontal asymptote $y = 3$. x-intercept $3x - 13 = 0 \Rightarrow x = 13/3$ (allowed). y-intercept $F(0) = 13/2$. As $x \rightarrow 2^-$: numerator -7, denominator negative $\rightarrow +\infty$; $x \rightarrow 2^+$: $-\infty$.

E11 · WE DO. Another complete analysis

$$G(x) = \frac{(x+3)(x-4)}{(x+2)(x-4)}$$

Exclude $x = -2, 4$. **Reduce to** $(x + 3)/(x + 2)$. **Hole** $(4, 7/6)$; **VA** $x = -2$; **HA** $y = 1$; **zero** $x = -3$; $G(0) = 3/2$. **Signs:** $x \to -2^{-}$ **gives** $-\infty$, $x \to -2^{+}$ **gives** $+\infty$.

E12 · YOU DO. Graph features

$$H(x) = \frac{(x-2)(x+1)}{(x-2)(x-3)}$$

Reduce $(x + 1)/(x - 3)$ **with exclusions 2,3. Hole** $(2, -3)$; **VA** $x = 3$; **HA** $y = 1$; **zero** $x = -1$; **domain all real except 2,3.**

Check SA. Explain why the excluded zero of a canceled factor is not an x**-intercept. Answer: The original function is undefined there, so the graph cannot pass through the axis at that input.**

Check SB. Find the horizontal asymptote of $\frac{4x^2+1}{2x^2-3}$. **Answer:** $y = 2$ **because the degrees match and leading coefficient ratio is** $4/2$.

Check SC. Find the hole for $\frac{(x-1)(x+4)}{(x-1)(x-2)}$. **Answer: Hole at** $(1, (1 + 4)/(1 - 2)) = (1, -5)$.

KEY IDEA: A hole is an excluded/missing point; the limit can exist even when the function value does not.

6 · Construct a function and interpret descriptions

E13 · I DO. Create zeros, hole and VA

Create a rational function with zero at $x = 4$, **hole at** $x = 1$, **VA at** $x = -2$, **and horizontal asymptote** $y = 2$.

One answer is $f(x) = 2(x - 4)(x - 1)/[(x + 2)(x - 1)]$. $x = 4$ **is a zero;** $x = 1$ **cancels and gives a hole;** $x = -2$ **remains in denominator and gives VA; equal degrees give HA 2. Hole coordinate** $(1, 2(1 - 4)/(1 + 2)) = (1, -2)$.

E14 · WE DO. Opposite infinite behavior

Construct a function with hole at $x = 3$, $\lim_{x \to -1^{-}} f(x) = -\infty$, $\lim_{x \to -1^{+}} f(x) = +\infty$.

Choose $c = 1$ **(or any positive constant). For** $x \neq 3, -1$, $f(x) = 1/(x + 1)$. **Left of –1 is negative denominator** $\to -\infty$, **right is positive** $\to +\infty$. **Hole** $(3, 1/4)$.

EIS · YOU DO. Same-side behavior

Construct a function with a hole at $x = 2$ and $\lim_{x\to -4^-} f(x) = \lim_{x\to -4^+} f(x) = +\infty$.

One answer $f(x) = (x-2)/\left[(x-2)(x+4)^2\right]$. At 2 the factor cancels, yielding hole $(2, 1/36)$. Denominator $(x+4)^2$ is small positive on both sides of −4, giving $+\infty$.

Check 6A. Which factor creates a zero at $x = 7$? Answer: The numerator factor $(x-7)$, provided it is not canceled and denominator is nonzero at 7.

Check 6B. Which denominator factor creates a VA at $x = -5$? Answer: An uncanceled $(x+5)$ factor.

Check 6C. How would you create a hole at $x = 0$? Answer: Place an x factor in both numerator and denominator, ensure it cancels fully, retain $x = 0$ as excluded.

KEY IDEA: Build zeros in the numerator, asymptotes in the denominator, and holes with common/canceling factors.

7 · Independent practice · Three levels

P1 · Level 1. Find the domain and VA(s).

$$A(x) = \frac{x+5}{x^2-1}$$

Denominator $(x-1)(x+1)$. Domain excludes $-1, 1$. No cancellation; VAs $x = -1, 1$.

P2 · Level 1. Determine the hole and VA.

$$B(x) = \frac{(x+3)(x-2)}{(x+3)(x+4)}$$

Exclude $-3, -4$; reduced $(x-2)/(x+4)$. Hole $(-3, -5)$; VA $x = -4$.

P3 · Level 1. State both one-sided limits at $x = 5$.

$$C(x) = \frac{1}{(x-5)^2}$$

Both $+\infty$; a squared denominator is positive on both sides.

P4 · Level 2. Find all holes and VAs, showing cancellation.

$$D(x) = \frac{(x-1)^2(x+2)}{(x-1)(x+2)^2}$$

Reduced $(x-1)/(x+2)$ with original restrictions $x = 1, -2$. Hole $(1, 0)$; VA $x = -2$.

P5 · Level 2. Find domain, VA, hole, zero and end behavior.

$$E(x) = \frac{(2x+1)(x-4)}{(x+3)(x-4)}$$

Exclude $x = -3, 4$**; reduced** $(2x + 1)/(x + 3)$**. Hole** $(4, 9/7)$**; VA** $x = -3$**; zero** $x = -1/2$**; HA** $y = 2$**.**

P6 · Level 2. A student writes "hole at $x = -2$**" for the function below. Correct or justify.**

$$F(x) = \frac{x+2}{(x+2)^2}$$

Wrong: reduction leaves $1/(x + 2)$**, so** $x = -2$ **is a VA.**

P7 · Level 3. Write a rational function with zeros −1 and 3, hole at 2, VA at 4, and HA $y = 1$**.**

Function: _____

Choose $f(x) = \frac{(x+1)(x-3)(x-2)}{(x-4)^2(x-2)}$**.**

P8 · Level 3. Explain all discontinuities and one-sided behaviors.

$$J(x) = \frac{(x-2)(x+1)}{(x-2)(x+3)^2}$$

Hole at $x = 2$**, with** $y = 3/25$**. VA at** $x = -3$**; reduced numerator** $x + 1 = -2$ **near −3, denominator positive tiny, so both sides** $-\infty$**. Original domain excludes 2 and −3.**

P9 · Level 3. A table has values tending to −200, −2000, −20000 from the left of 1 and +200, +2000, +20000 from the right. Describe the VA and limit behavior.

Answer: _____

VA at $x = 1$**;** $\lim_{x\to 1^-} f(x) = -\infty$**,** $\lim_{x\to 1^+} f(x) = +\infty$**. A finite two-sided limit does not exist.**

8 · What I learned today

1. **domain restrictions 2. hole 3. vertical asymptote 4. one-sided 5. simplified expression.**

9 · Exit ticket

X1 · Basic. Classify the discontinuity at $x = 2$**.**

$$f(x) = \frac{x-2}{(x-2)(x+5)}$$

Cancel $x - 2$ **completely: hole at** $(2, 1/7)$**, with** $x = 2$ **excluded.**

X2 · Application. Find the domain, VA and hole.

$$g(x) = \frac{(x+1)(x-3)}{(x+1)(x-4)}$$

Domain excludes −1 and 4. Simplify to $(x-3)/(x-4)$. Hole $(-1,(-4)/(-5))=(-1,4/5)$; VA $x=4$.

X3 · Reasoning. Why is $x=-2$ a VA rather than a hole?

$$h(x)=\frac{x+2}{(x+2)^3}$$

Canceling one $(x+2)$ leaves $1/(x+2)^2$, whose denominator still vanishes at −2; both sides approach $+\infty$.

Teaching notes, source alignment and video links (teacher only)

Progression: Warm-up → excluded domain inputs → canceling and hole coordinates → multiplicity → limit signs and tables → graph features → constructing functions → three practice levels. This follows the main structures and exercise types of the three provided PDFs.

Common errors to address: (1) Forgetting original exclusions; (2) treating every denominator zero as a VA; (3) computing the hole's y-value from the original undefined form; (4) making both one-sided limits equal without checking signs; (5) treating infinity as a real limit value; (6) forgetting to remove canceled roots from x-intercepts.

Source note: The original materials have minor editorial inconsistencies. In particular, one model says an asymptote happens whenever a common factor fails to cancel, but repeated multiplicities must first be simplified. For unbounded behavior, write $\pm\infty$ as shorthand, not a finite number.

Video references supplied by the teacher

1. Teacher-supplied video 1 — suggested for review of this unit; exact content and timestamps could not be independently verified.
2. Teacher-supplied video 2 — suggested for review of this unit; exact content and timestamps could not be independently verified.
3. Teacher-supplied video 3 — suggested for review of this unit; exact content and timestamps could not be independently verified.
4. Teacher-supplied video 4 — suggested for review of this unit; exact content and timestamps could not be independently verified.

Additional verified topic resource: Graphing rational functions with asymptotes and holes — supplementary teacher demonstration.

Matching source files: "appc_1.9_packet.pdf" (graph, numerics, AP-style test prep); "APPC 1.9–1.10 Notes Rational Functions VA and Holes.pdf" (contrast and limit statements); "Notes_1.9–1.10R_Rational_Functions_Vertical_Asymptotes_Holes[1].pdf" (multiplicity, holes, graphical/numerical analysis, model construction).