MTH401 Assignment No 02

By

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Question 1: Find the general solution of the following homogeneous system of differential equations

$$\frac{dx}{dt} - 7x + \frac{dy}{dt} = 3e^t$$
$$3\frac{dx}{dt} - 2x + \frac{dy}{dt} = 2e^t$$

Solution:

The given nonhomogeneous systems of differential equations are

$$\frac{dx}{dt} - 7x + \frac{dy}{dt} = 3e^t$$
$$x' - 7x + y' = 3e^t ------(1)$$

$$3\frac{dx}{dt} - 2x + \frac{dy}{dt} = 2e^t$$
$$3x' - 2x + y' = 2e^t -----(2)$$

Now solve for x(t)

Equation (1)-Equation(2)

$$\mathrm{x}' - 7x + y' - 3x' + 2x - y' - 3e^t - 2t^t$$
$$-2x' - 5x = e^t$$
$$2x + 5x = -e^t$$

Now find the solution to homogeneous equation $2x' - 5x = 0$

Now write the auxiliary equation

$$2\mathrm{m} + 5 - 0$$
$$2\mathrm{m} = -5$$
$$\mathrm{m} = -\frac{5}{2}$$

Here root is real

If α is the root of the auxiliary equation then the solution of differential equation is of the form of

$$x(t) = Ce^{\alpha t} \text{.............} eq(4)$$

Now find the particular solution to the homogenous equation

Explanation

Here by using the method of undetermined coefficients to find particular solution.
Here take particular solution

$$x_p = Ae^t \text{...........} eq(5)$$

Differentiate equation (5) with respect to t

$$x'_p = Ae^t \text{...........} eq(5)$$

Substitute terms in equation (3)

$$x(t) = Ce^{\alpha t} \text{.............} eq(4)$$
$$2x'_p + 5x_p = -e^t$$
$$\Rightarrow 2(Ae^t) + 5(Ae^t) = -e^t$$
$$Ae^t = -e^t \text{..............} eq(6)$$

Comparing coefficients of e^t terms in equation 6

7A=-1

A=-1/7

Substitute A in equation(5)

$$x_p = -\frac{1}{7}e^t \text{.............} eq(7)$$

Now we write solution of differential equation (3)

$$x(t) = Ce^{-\frac{5}{2}t} - \frac{1}{7}e^t$$

Now solve for y(t)

Here take equation (1)

$$x' - 7\,\mathrm{x} + \mathrm{y}' = 3e^t$$
$$y' = -x' + 7\,\mathrm{x} - 3e^t \text{...........}(9)$$

Differential equation(8) with respect to t

Explanation

Here using chain rule we get

$$\frac{d}{dx}(f(g(x)) - \mathrm{f'(x)}\frac{d}{dx}(g(x))$$

$$and$$

$$\frac{d}{dx}e^{x} = e^{x}, \frac{d}{dx}x^{n} = nx^{n-1}$$

$$x'(t) = Ce^{-\frac{5}{2}t}\left(-\frac{5}{2}\right) - \frac{1}{7}e^{t}$$

$$x'(t) = -\frac{5}{2}Ce^{-\frac{5}{2}t} - \frac{1}{7}e^{t}$$

Now substitute terms in eq(9)

$$\mathrm{y'} = \left(-\frac{5}{2}Ce^{-\frac{5}{2}t} - \frac{1}{7}e^{t}\right) + 7\left(-\frac{5}{2}Ce^{-\frac{5}{2}t} - \frac{1}{7}e^{t}\right) + 3e^{t}$$

$$\mathrm{y'} = \frac{5}{2}Ce^{-\frac{5}{2}t} + \frac{1}{7}e^{t} + Ce^{-\frac{5}{2}t} - e^{t} + 3e^{t}$$

$$y' = \left(\frac{5}{2} + 7\right)Ce^{-\frac{5}{2}t} + \left(\frac{1}{7} + 2\right)e^{t}$$

$$y = \frac{19}{2}Ce^{-\frac{5}{2}t} + \frac{15}{2}e^{t}$$

Now integrate above equation

$$y(t) = \frac{19}{2}C\left(\frac{e^{-\frac{5}{2}t}}{-\frac{5}{2}}\right) + \frac{15}{2}e^{t} + C_{1}$$

$$y(t) = \frac{19}{2}Ce^{-\frac{5}{2}t} + \frac{15}{2}e^{t} + C_{1}$$

The general solution to the given system of differential equation is

$$x(t) = Ce^{-\frac{5}{2}t} - \frac{1}{7}e^{t}$$
$$y(t) = \frac{19}{2}Ce^{-\frac{5}{2}t} + \frac{15}{2}e^{t} + C_1$$

Question No:02

Solve the third-order Cauchy-Euler differential equation.

$$x^3 \frac{d^3y}{dx^3} + 5x^2 \frac{d^2y}{dx^2} + 7x \frac{dy}{dx} + 8y = 0$$

Solution:

$$a_o x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2}y}{dx^{n-2}} + \ldots\ldots\ldots a_{n-1} x^n \frac{dy}{dx} + a_o y = F(x) \to (\text{A})$$

Is called Cauchy Euler Equation

Equation (A) can be reduced to linear differential equation by substitution

$$x = e^t \Rightarrow t = \ln x$$
$$\therefore \frac{dt}{dx} = \frac{1}{x}$$
$$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx} \Rightarrow \frac{dy}{dx} = \frac{dy}{dt}.\frac{1}{x}$$
$$x\frac{dy}{dx} = \frac{dy}{dt} ------(1)$$
$$\text{xDy} = \Delta\,\text{y}$$
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dt}.\frac{1}{x}\right)$$
$$\frac{d^2y}{dx^2} = \frac{1}{x}.\frac{d^2y}{dx^2}.\frac{dt}{dx} + \frac{dy}{dx}\left(-\frac{1}{x^2}\right)$$
$$\frac{d^2y}{dx^2} = \frac{1}{x^2}.\frac{d^2y}{dt^2} - \frac{1}{x^2}\frac{dy}{dt}$$
$$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt} ------(2)$$
$$\text{x}^2 D^2 y = (\Delta^2 - \Delta)\,\text{y}$$

Similarly

$$x^3\frac{d^3y}{dx^3}=\frac{d^2y}{dx^2}-\frac{d^2y}{dt^2}+2\frac{dy}{dt}------(3)$$

$$x^3 D^3 y = (\Delta^3-2\Delta^2+2\Delta)\,y$$

From the given Equation

$$(x^3D^3+5x^2D^2+7xD+8)\,y=0$$

$$(\Delta^3-3\Delta^2+2\Delta+5(\Delta^2-\Delta)+7\Delta+8)\,y=0$$

$$\Delta^3-3\Delta^2+2\Delta+5\Delta^2-5\Delta)+7\Delta+8=0$$

$$\Delta^3-2\Delta^2+4\Delta+8=0$$

$$\Delta^2(\Delta+2)+4(\Delta+2)=0$$

$$(\Delta^2+4)(\Delta+2)$$

$$\Delta^2+4=0,\Delta+2=0$$

$$\Delta^2=-4,\Delta=-2$$

$$\Delta=\pm\sqrt{-4}=\pm 2i$$

Characteristic equation has complex roots $a\pm ib$ then the general Solution is

$$y=C_1e^{-2x}+C_2\cos(2\ln x)+C_3\sin(2\ln x)$$

[as $t=\ln x=\ln x\Rightarrow 2t=2\ln x$]

THE END