Linear Algebra
Lesson 12:
Nullity, Rank, and Range

Please be sure to mark down the date and time that you start this lesson. Carefully take notes on pencil and paper while watching the lesson videos. Pause the lesson to try classwork before watching the video going over that classwork. If you work with any classmates, be sure to write their names on the problems you completed together. Please wear masks when meeting with classmates even if you meet off campus.

You will cut and paste the photos of your notes and completed classwork and a selfie taken holding up the first page of your work in a googledoc entitled:

MAT313F21-lesson12-lastname-firstname

and share editing of that document with me sormanic@gmail.com and with our graders. If you have a question, type QUESTION in your googledoc next to the point in your notes that has a question and email me with the subject MAT313 QUESTION. I will answer your question by inserting a photo into your googledoc or making an extra video.

Watch the Playlist 313F21-12-1to5

Nullity and Null Space of a Matrix

Defn: Nullity is the dimension of the nullspace.

Classwork: what is the nullity of $\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Start by finding null space

$$\left[\begin{array}{cccc|c} \boxed{1} & 0 & 2 & 1 & 0 \\ 0 & \boxed{3} & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{\rho_2 \to \frac{1}{3}\rho_2} \left[\begin{array}{cccc|c} \boxed{1} & 0 & 2 & 1 & 0 \\ 0 & \boxed{1} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 \end{array}\right]$$

So far we have three leaders
Since there are four columns
There is one free variable

$$\text{Null Space} = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{\text{position is } 0} + x_3 \underbrace{\begin{pmatrix} ? \end{pmatrix}}_{\text{direction}} \;\middle|\; x_3 \in \mathbb{R} \right\}$$

homog system

Null Space = span of directions
one free variable
only one direction

Dimension of null space = $\boxed{1 = \text{nullity}}$

Thm: Nullity of a matrix A = # of free variables when solving $[A \mid \vec{0}]$

(proof of this thm is difficult)

Rank and Range of a Matrix

Range of a Matrix is the Span of its columns.

$$\text{Range}\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \left\langle \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\3\\0\end{pmatrix}, \begin{pmatrix}2\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix} \right\rangle$$

Rank of a matrix = dim of the Range

Recall that the columns may not be linearly independent, use pivot columns to find linearly independent columns with the same span.

To find pivot columns do row reduction and find which columns have leaders.

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \cdots \to \left[\begin{array}{cccc|c} \boxed{1} & 0 & 2 & 1 & 0 \\ 0 & \boxed{1} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 \end{array}\right]$$

Echelon Form is enough

pivot columns have leaders

Columns 1, 2 and 4.

$$\underbrace{\text{Range}\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{original matrix}} = \left\langle \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\3\\0\end{pmatrix}, \underbrace{\cancel{\begin{pmatrix}2\\1\\0\end{pmatrix}}}_{\text{extra vector}}, \begin{pmatrix}1\\0\\1\end{pmatrix} \right\rangle = \left\langle \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\3\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix} \right\rangle$$

3 lin indep columns

dim of Range is 3

So rank = 3.

Thm: Rank of matrix A = # of leaders when solving $[A \mid \vec{0}]$

<u>Thm:</u> Rank + Nullity = number of variables $x_1, x_2 \dots \underbrace{x_m}$

If a matrix has m columns then rank + nullity = m.

Rank + Nullity = "dimension of domain"

A $\overset{n \text{ rows}}{m \text{ columns}}$ $A \in M_{n \times m}$

$$A\vec{x} = \vec{y}$$

$\vec{x} \in \mathbb{R}^m$ $\vec{y} \in \mathbb{R}^n$

input $\vec{x} \in \mathbb{R}^m$
output $\vec{y} \in \mathbb{R}^n$

Remember always follow the row reduction algorithm taught in the course.

Thm: Nullity of a matrix A = number # of free variables when solving $[A \mid \vec{0}]$

Proof Idea:

nullity = dim (Null(A))

$$Null(A) = \{ \vec{x} = \vec{0} + t_1\vec{v}_1 + t_2\vec{v}_2 + \dots + t_k\vec{v}_k \mid t_1, t_2 \dots t_k \in \mathbb{R} \}$$

where $t_1, \dots t_k$ are free variables and $\vec{v}_1, \dots \vec{v}_k$ are their directions

$$Null(A) = \langle \vec{v}_1, \vec{v}_2 \dots \vec{v}_k \rangle$$

$\dim(Null(A)) \leq k$ = number of free variables

we need to prove it equals k by showing that the direction vectors are linearly indepondant

Recall defn of lin indep

$$t_1\vec{v}_1 + \dots + t_k\vec{v}_k = \vec{0} \iff t_j = 0 \text{ for } j = 1 \text{ to } k.$$

You could check this every time, which would require you to do the row reduction and find the null space and then check the directions.

Find the nullity of

nullity = dim Nullspace

$$\begin{bmatrix} \boxed{1} & 2 & 0 & 3 & 4 & 0 & 0 \\ 0 & 0 & \boxed{1} & 5 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\boxed{1x_1} + 2x_2 + 0x_3 + 3x_4 + 4x_5 = 0$

$\boxed{1x_3} + 5x_4 + 6x_5 = 0$

$x_6 = 0$

$x_7 = 0$

leaders x_1, x_3, x_6, x_7

free x_2, x_4, x_5

nullity = number of free variables = 3

Why?

$\boxed{1x_1} = 0 - 2x_2 - 3x_4 - 4x_5$

$1x_3 = 0 - 5x_4 - 6x_5$

$x_2 = x_2$ $\quad x_4 = x_4$ $\quad x_5 = x_5$

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -3 \\ 0 \\ -5 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -4 \\ 0 \\ -6 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \;\middle|\; x_2, x_4, x_5 \in \mathbb{R} \right\}$$

direction for x_2 has a 1 in the 2^{nd} entry

but the other directions have 0 in the 2^{nd} entry

The direction for x_4 has a 1 in the 4^{th} entry

but other directions have 0's in the 4^{th} entry

direction for x_5 has a 1 in 5^{th} entry

others have 0 in fifth entry.

Because of this these three direction vectors are linearly indep.

$$\underline{x_2}\begin{bmatrix}-2\\1\\0\\0\\0\\0\\0\end{bmatrix}+\underline{x_4}\begin{bmatrix}-3\\0\\-5\\1\\0\\0\\0\end{bmatrix}+\underline{x_5}\begin{bmatrix}-4\\0\\-6\\0\\1\\0\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\\0\\0\\0\end{bmatrix}$$

$$\left.\begin{array}{l} x_2\cdot 1 + x_4\cdot 0 + x_5\cdot 0 = 0 \\ \quad \text{Thus } x_2 = 0 \\ 0x_2 + 1x_4 + 0x_5 = 0 \\ \quad \text{thus } x_4 = 0 \\ 0x_2 + 0x_4 + 1x_5 = 0 \\ \quad \text{thus } x_5 = 0 \end{array}\right\} \text{Thus the directions are lin. indep.}$$

This always happens with direction vectors!
They are always lin indep because the dir vector for free variable x_j has a 1 in the j^{th} entry and other directions have 0 in the jth entry because $\{x_j = x_j\}$ free!
So when we solve the system we will get $1x_j = 0$ and prove lin indep

"QED"

Classwork: Find the range + rank of our last example:

$$A = \begin{bmatrix} \boxed{1} & 2 & 0 & 3 & 4 & 0 & 0 \\ 0 & 0 & \boxed{1} & 5 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Quick answer for rank = number of leaders = 4

boxed the leaders

1st, 3rd, 6th, 7th columns

The range = span of the columns

$$= \left\langle \begin{pmatrix}1\\0\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}2\\0\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}3\\5\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}4\\6\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\\0\\0\end{pmatrix} \right\rangle$$

Classwork: find a basis of linearly independent vectors for the range (pause + try)

Recall that the pivot columns are linearly indep

These are the columns with leaders.

$$\text{Range} = \left\langle \begin{pmatrix}1\\0\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\\0\\0\end{pmatrix} \right\rangle$$

These vectors are a basis for range(A) because they are linearly indep

Classwork: Verify this basis is lin. indep.

Check

$$t_1 \begin{pmatrix}1\\0\\0\\0\\0\\0\end{pmatrix} + t_2 \begin{pmatrix}0\\1\\0\\0\\0\\0\end{pmatrix} + t_3 \begin{pmatrix}0\\0\\1\\0\\0\\0\end{pmatrix} + t_4 \begin{pmatrix}0\\0\\0\\1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\\0\\0\end{pmatrix}$$

has only one solution $t_j = 0$ for $j = 1$ to 4

$$\left(\begin{array}{cccc|c} \boxed{1} & 0 & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Do row reduction (easy demo for class)

no free variables so it IS lin. indep.

Homework: **For the following matrix A**

$$A = \begin{bmatrix} 1 & 0 & 4 & 2 & 5 \\ 0 & 1 & 2 & 0 & 6 \\ 1 & 1 & 6 & 2 & 11 \\ 2 & 0 & 8 & 5 & 11 \end{bmatrix}$$

Homework:

(Note Exam 2 is very similar to this homework but with easier matrices)

Part I of HW:

1) **Row Reduction to Echelon Form (boxing leaders)**
2) **Find Nullity and Rank**
3) **Write the Range as the span of a basis of linearly independent pivot columns.**
4) **Verify the basis of the Range is linearly independent**

Part II of HW:

1) **Continue Row Reduction of A to Reduced Echelon Form (boxing leaders)**
2) **Find Null Space of A**
3) **Check the directions in the null space using matrix multiplication**
4) **Write the Null Space as a span of a basis of linearly independent directions.**
5) **Verify the directions of the null space are linearly independent.**

After completing the homework check the homework solutions here. Email me your corrected solutions and ask any questions you have. You must also include a selfie with your work.

After that you may study and then practice the Sample Exam in a timed setting here. You do not submit your sample exam to me.