About the Mathematics: Multiplication & Division Strategies

An overview of the progression of children's strategies in multiplication & division

Overview

Young children can solve multiplication and the two types of division problems starting in kindergarten. The organizational structure of each operation can be directly modeled and counted out[1]. Are kindergarteners thinking in multiples? No! Their work is additive. But, the structure of the contexts – placing/sorting things into equal groups, measuring out equal quantities, sharing equally – is accessible and relatable to the world around them. This book study addresses the cognitive shifts students progress through over time from additive to multiplicative ways of thinking. It focuses on *interpreting* student thinking as one observes and listens to their work. The ability to interpret influences how one *decides how to respond* instructionally.[2] Included are the problem types, the emerging strategies, and instructional approaches to cognitively guide students' transitions over time. These ways of thinking are not lessons mastered in a lesson or unit. These are dispositions and cognitive demands that mature over the grade levels.

Part I – What Makes Multiplication Different from Addition

While early multiplication strategies emerge from students' addition counting strategies, multiplication is fundamentally different in its attributes and conceptual underpinnings. When joining whole number sets of objects (addition) such as 'you have 5 apples and I give you 5 more apples,' you have apples plus more apples and you end up with 10 apples. The set got larger but you still have apples. In quantifying discrete sets in part-part-whole situations, you may have 5 apples and 5 oranges, but counting how many total objects there are, you end up with 10 pieces of fruit. There is a reclassification involved. However, the focus is on counting a similar class of objects. Students' earliest experiences with multiplication is through packaging of equal groups of objects. For example, I have 5 packages of cookies with 5 cookies inside each package. The first five quantifies the packages as the unit, the groups of objects. The second five are the contents within each of those packages as its unit. The resulting total, 25 cookies, doesn't even involve the packaging at all. *Multiplication involves unit transformation.* Addition does not. Additionally, when units involve rates such as if it takes me 1 hour to go 60 miles, my *speed* (a new composite unit) is 60 miles per hour. Tracking the unit transformations is an inherent aspect of multiplication that is not of concern in addition.

Multiplication involves the cognitive process of *making units of units* in order to use a composite unit that is more efficient. For example, when ordering school supplies for the whole school , I don't order individual pencils. I order boxes or packages of pencils. For larger quantities of pencils, I may need to order a carton that contains multiple boxes in which are individual pencils. If I buy three cartons and each carton contains ten boxes with 12 pencils inside each box, those cartons and boxes are *composite units* each containing subunits of individual pencils. As I shift from thinking about cartons to boxes to pencils, I mentally have to *coordinate units* to know which unit goes with which set of items. In the scenario just described, there were the quantities (3), (10), (12), (30), (120), and (360) either given or the result of some calculation. Which unit goes with each number and what detail or question does it answer? The *units transform* as one operates on the numbers thus creating the need to coordinate those units. Place value and measurement are both based on this multiplicative making of units of units so that the new composite unit is more efficient with which to work. As one moves back and forth across the composite and subunits, coordination of those units is a required cognitive demand. Additionally, multiplication involves thinking in scale; *n-times as many*. I ordered

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[1] Carpenter, T. P., Ansell, E., Franke, M. L., Fennema, E., & Weisbeck, L. (1993). Models of problem solving: A study of kindergarten children's problem-solving processes. *Journal for Research in Mathematics Education, 24*(5), 428-441.

[2] Jacobs, V.R., Lamb, L.L.C., Phillipps, R.A. (2010) Professional noticing of children's mathematical thinking. *Journal for Research in Mathematics Education. 41* (2), 169-202.

three times as many cartons at my school than yours. That information is mentally coordinated and understood distinctly different from what is required in addition.

"Multiplication as Repeated Addition"

The statement that multiplication "is repeated addition" is not an accurate statement in capturing what multiplication entails as an operation. While it is developmentally normal for young children to use repeated addition/additive strategies to begin solving multiplication structures early on in their learning, as educators we don't want them to stay at that level of thinking. Repeated addition involves additive thinking, not multiplicative thinking. 5 x 5 does have the same value as 5 + 5 + 5 + 5 + 5, and is an important connection for students to comprehend. However, the attributes described in the previous section of what makes multiplication distinct from addition are more sophisticated. The educational task is to cognitively guide students to develop these attributes of multiplication so that higher levels of mathematical concepts such as proportions, ratios, and functions can be developed. A better statement is *"Repeated addition is an important entry level strategy for students to use when first learning multiplication."*

Social Conventions & Language

Social conventions – The social convention being used here in organizing the quantities in a multiplication equation is as follows.

# of group x # inside each group allows you to determine the total objects in all the groups
The multiplier of the multiplicand results in the product

5 x 5 = 25, therefore, means there are 5 groups (the multiplier, the *scaling factor*)[3] with 5 objects inside each group (the multiplicand) for a total of 25 objects inside all of the groups (the product). Even though both the multiplier and the multiplicand are 5s, each represents a different unit of measure.

Language – Initially, students are encouraged to develop *groups of language* when reading a math expression such as 5 x 5, as in *what are five groups of five.* The language is more concrete than the *'times' language* that is typically used as in *five times five.* In order to keep to the social convention noted above, one needs to rephrase 'five times five' to *'I want 5 times as many fives',* or five times *'the* five*.'* Otherwise, "five *times five*" reverses the social convention order to be *I want five* (the multiplicand) *five times* (the multiplier). This is mathematically acceptable. It's just a different social convention. The important point is for students to comprehend which unit goes with which number so that as they progress with the task they are able to track which partial product provides which piece of information. However, one can't assume that students are interpreting the *groups of* and *times* language in identical ways.

Figure One
Five jars with five jelly beans inside each jar

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[3] Otto, A.D., Caldwell, J.H., Lubinski, C.A., Hancock, S.H. (2011). *Developing essential understanding of multiplication and division: Grades 3-5.* Reston, VA: National Council of Teachers of Mathematics.

Interim Summary I

Multiplication, and subsequently division, require cognitive demands that are distinct from addition and subtraction. While young children's initial approach to engaging with multiplication contexts can be successfully completed using repeated addition, the two operations have unique features from each other. Repeated addition is a powerful entry point for a child to engage with multiplication contexts, however, a child needs to mature over time to think beyond this level of conceptualization. The capacity to make units of units so that the newly formed composite units can be efficiently repeated, iterated, is an attribute of multiplication. Once one begins working with the newly formed composite unit, being able to simultaneously remember the scale increase of its subunits is required. Tracking how units transform as the focus changes across the work is a critical attribute to monitor and increase the need and capacity to coordinate units.

Multiplication has language features and social conventions that, if understood and explicitly used, allows tracking the units and various transformations more easily.

Reflection Point 1

Take a moment to complete and discuss with your reading partner(s) the following questions in order to summarize the ideas that were just presented. Remember to *follow the units* as one works. Think about what subunits each composite unit is built upon.

- ☐ Revisit the scenario of: *I buy three cartons of pencils. Each carton contains ten boxes with twelve pencils inside each box*. What unit of measure goes with each of the following quantities that are either given in the problem or are the result of some calculation: (3), (10), (12), (30), (120), and (360)
- ☐ Navigate to *At the Factory: Making & Boxing Colored Markers*. Complete this multi-step rich task (ideally with a partner to discuss out loud as you work) designed for fifth graders to engage in ideas of making units of units, unit transformation, and place value concepts.
    - ☐ Where do you find yourself productively struggling with monitoring which unit goes with which quantity?
    - ☐ If you explicitly label each quantity with its unit of measure, why do the units transform the way they do?

Part II – Multiplication & Division Problem Types

Problem types for multiplication and division vary in terms whether or not the contexts are symmetrical or asymmetrical, as well as influenced by the units of measure (equal groups, rates, price, comparison). This link provides specific examples to use as models to create situations for your own classroom.

Asymmetrical and Symmetrical Contexts & The Commutative Property of Multiplication

The commutative property of multiplication states that *'a' groups of 'b'* will have the same total of individual objects (the product) as *'b' groups of 'a'.* In asymmetrical problems, the commutative property of multiplication is not readily apparent to children.[4] Young students visualize real-world contexts differently whether the situation is a symmetrical or asymmetrical structure. Trying to carry 20 individual jars of jelly beans with 3 jelly beans inside each jar is a very different physical picture than trying to carry 3 jars with 20 jelly beans inside each jar. While both contexts result in the same purchase of 60 jelly beans, they are different in how students early in their development will model and count to determine the product.

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[4] Baroody, A.J., Wilkins, J.L.M., & Tiilikainen, S. (2003). The development of children's understanding of additive commutativity: From protoquantitative concept to general concept? In A.J. Baroody & A. Dowker (Eds.), *The development of arithmetic concepts and skills: Constructing adaptive expertise.* Mahwah, NJ: Lawrence Erlbaum Associates.

For a student at either the direct modeling, skipping counting stage, or early additive strategy stage, three jars of 20 jelly beans focuses on the count of 20. With twenty jars of 3, the focus of the count is on 3. It takes longer for students to realize that the commutative property will result in the same total of jelly beans. (Figure 2.)

3 jars of 20

20 jars of 3

Figure 2 – Commutative Property of Multiplication in Asymmetrical Contexts

In symmetrical problems, such as in area and array contexts, the commutative property is more readily seen as a matter of a perspective shift depending on where one first looks at the arrangement; as rows first with a certain number of objects in each row or columns first with a certain number of items within each column. Area and array contexts are symmetrical. 3 rows with 6 chairs in each row or 6 columns with 3 chairs in each column is an issue of perspective and orientation, not arrangement. (Figure 3.)

3 rows with six in each row

6 columns with three in each column

Figure 3 – Commutative Property of Multiplication in Symmetrical Contexts

As a result, *the commutative property, while relatively easy to conceptualize in area and array contexts, takes longer for students to transfer to asymmetrical contexts.* Understanding the commutative property in symmetrical contexts comes earlier than in asymmetrical contexts. A time lag between the two is developmentally normal. Asymmetrical packaging is what most students experience first when thinking of multiple quantities. Symmetrical contexts are less common in everyday life. So while the commutative property is easier to perceive in symmetrical contexts, it takes longer for that perception to readily transfer to asymmetrical contexts. Focused conversations for when it arises in either setting is where students can contemplate the property so that *the intuitive, implicit ideas arise to become explicit, conscious knowledge that can be used intentionally in one's work.*

Look at the following reconstructions of the work of Ashley and compare hers to that of Layna and Hallie (Figure 4). These samples represent strategies used early in a fourth grade classroom. The task was to visualize being in the bakery department of the local grocery store where there were *25 boxes of cookies with 6 cookies inside each box. How many cookies were used to fill all of the boxes?* The scenario is an asymmetrical, equal grouping context. Let's look at two student work samples to illustrate different approaches to solving the task.

Ashley literally illustrates the 25 boxes with six inside each box. Layna and Hallie demonstrate the use of the commutative property of multiplication and illustrate six groups of 25. Both sets of students arrive at the same answer of 150 cookies. Both use the same strategy of doubling to make new composite units. (More on this strategy in a moment.) However, much less work was needed by Layna and Hallie. The use of the commutative property, which in an asymmetrical context requires an abstract leap of faith, became a point in the public sharing when this arose. Why can one get away with switching the task around? Will this work all the time? In all operations? Processing publicly the decision making used by Layna and Hallie is key in moving the algebraic ideas from the *implicit* to the *explicit level* of understanding and usage throughout the whole learning community.

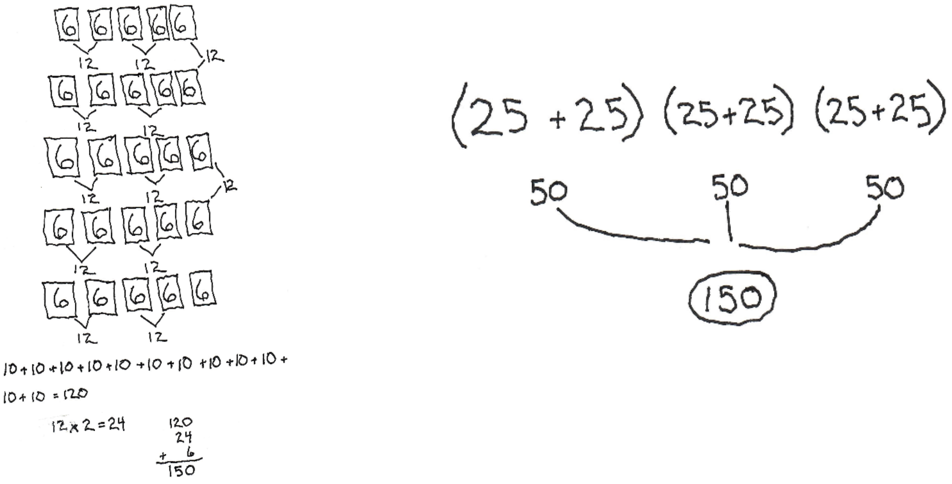

Figure 4 – Asymmetrical Equal Grouping Context

Context: 25 boxes with 6 cookies inside each box. How many cookies are in all of the boxes?

Ashley: Doubling Strategy – 25 sixes – modeled as structured in the context

Layna & Hallie: Doubling Strategy – 6 twenty-fives – Use of the commutative property of multiplication

Measurement and Partitive Division

To understand the difference in how students model and count solving different division contexts, it is useful to see the contexts through the lens of multiplication. Remember the social convention of the number of groups times the number inside each group. In the unit table below, each problem varies based on the information that is known and unknown.

	# of groups (jars)	**# in each group (jelly beans per jar)**	**Total in all groups (jelly beans in all jars)**
Multiplication	3	20	?
	I have 3 jars of jelly beans with 20 jelly beans inside each of the jars.. How many jelly beans will there be if I pour out all of the jelly beans into one bowl?		
Measurement Division	?	20	60
	I have 60 jelly beans in a bowl. If I put 20 jelly beans into a jar, how many jars will I need for all of my jelly beans?		
Partitive Division	3	?	60
	I have 60 jelly beans that I want to put into 3 jars. If each jar gets the same number of jelly beans, how many will end up inside each jar?		

In a *measurement division* problem, I know how many of the 60 jelly beans I want to put into a jar (20) if each jar gets the same amount. I just don't know how many jars I will end up using. Conversely, in a *partitive division* problem, I know how many jars I will be using (3) for my 60 jelly beans, but I am unsure how many will go into each jar if each jar gets the same amount. The modeling and the number sense initially used by young students is different for each of the scenarios.[5] Seeing the information as division expressions [$60 \div 20$ or $60 \div 3$] one doesn't explicitly see the difference in the question being asked. Seeing it as a missing factor multiplication equation does [$a \times 20 = 60$; $3 \times a = 60$].

If a student is confused about how to visualize and engage with a multiplication or one of the two division problem contexts, one instructional technique is to prompt the student to create a unit table capturing both the known and unknown information. It serves as a visual organizing tool that often helps the student to make decisions without dictating a particular strategy by which to solve the task.

Follow the Units

One "sound bite" to use with students is to *follow the units* while working. More than merely asking for students to "label your answer" – which is important – it is the interpreting of what each number means in the context and what unit the number represents which helps to focus students' attention as to both the meaning of the numbers as well as the units that are represented.

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[5] Carpenter, T.P., Fennema, E., Franke, M.L., Levi,L. & Empson, S.B. (2015). *Children's mathematics: Cognitively Guided Instruction.* Portsmouth, NH: Heinemann.

Video Stop 1

Pause and look at the following video from the book *Connecting Mathematical Ideas: Middle School Video Cases to Support Teaching and Learning* (Boaler & Humphreys, 2005[6]). The instructional moves used by the teacher while students solve a task known as The Border Problem provides a clear example of how students' attention is focused on interpreting what each number in their equations represents. Such conversations can begin in kindergarten. If a consistent practice across the grade levels, moving into generalized algebraic thinking becomes a natural part of the learning process.

Units as Abstractions

While it is straightforward to visualize 3 jars and the jelly beans inside each jar, if the units shift to price (a jar costs 75¢) or the speed at which a machine fills a jar (4 seconds), etc., while the numbers may be small and manageable, the unit of cents and seconds are more abstract than the tangible jars and jelly beans. Thus, different units of measure can make a problem context more or less difficult depending upon the students' ability to conceptualize the unit. The numbers may be within the students' range of experience. The unit, however, may not.

Integrating a variety of the multiplication and division contexts along with varying the types of units of measure into the mathematical diet, allows students to increase their capacity visualize the broad range of contexts in which multiplication and division can be applied as well as gain the flexibility in selecting effective strategies that can be used to solve the range of contexts. As noted previously, this link provides specific examples of how to word the different types of problem contexts.

Interim Summary II

Multiplication and division scenarios are visualized and modeled by students differently whether or not the context is symmetrical or asymmetrical. Students' initial experiences with multiplicative scenarios are with asymmetrical packaging contexts; things that come in equal groups, cost a certain amount of money, or increase at a particular rate. In equal grouping contexts, it is the number of objects inside each group that drives students' initial counting efforts. The commutative property of multiplication takes longer to develop and be applied in asymmetrical contexts than in symmetrical ones where the flexibility to consider objects in rows or in columns is a matter of perspective. The context drives the visualization.

There are two types of division depending upon the missing unknown unit to solve for. Direct modelers, skip counters, and those at the abstract number level solve the two division contexts differently from each other. As operations, multiplication and division are unit transforming. Having students articulate what each quantity represents is a means to help students sort through what information they are solving for. Some units are more abstract than others. Even if the quantities with which students are working, a problem can become more difficult due to the abstraction of the unit. Speed or pounds is more abstract to visualize than jars and jelly beans.

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[6] Boaler, J. & Humphreys, C. (2005). *Connecting mathematical ideas: Middle school video cases to support teaching and learning.* Portsmouth, NH: Heinemann.

Reflection Point 2

Review and explore the following before proceeding to the next section.

- ☐ Navigate to the link *The Art of Writing Story Problems*. Using the core ideas noted in what needs to be taken into consideration when crafting problems, *and* using the link to the *different types of multiplication and division problems types*, craft some problems drawing from various curriculum topics or literature already in use in the classroom that can be used with your students during math time.

Part III – Emerging strategies[7]

Consider the context of 8 groups with six objects inside each group. The following represents the emerging range of strategies students draw upon in order to determine the question being asked in the situation.

Direct modeling

These are the students who need to build with manipulatives, or pictorially draw out dots or tallies for each box of markers and count them all by ones.

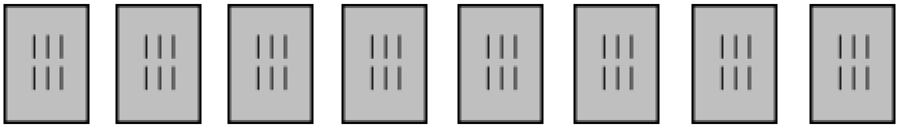

Figure 5 – Direct Modeling

Direct Modeling counting all starting at one

A progression in thinking would, instead of tallies or circles, is to write the number inside each box. (See Figure 4: Ashley solving 26 boxes of 6.) This transition typically leads to the next strategy.

Additive Skip Counting/Repeated Addition

Typically, all eight sixes are written out. The total is found by repeatedly adding or skip-counting up as far as one can go, then counting on by ones each of the remaining sixes. Typically, six fingers are re-used in any additional count to track that eight sixes are accounted for.

6 + 6 + 6 + 6 + 6 + 6 + 6 + 6

Says: 6, 12, 18… [19, 20, 21, 22, 23, 24]… [25, 26….]… 48

Figure 6 — Additive Skip Counting

Skipping counting as high as they can go and then counting one by ones each of the remaining sixes.

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[7] For more details on students' early strategies, read the following resources:

- Carpenter et al. 2015. *Children's mathematics: Cognitively Guided Instruction* (Chapters 4 & 6), Portsmouth, NH: Heinemann.
- Ambrose, R., Baek, J., Carpenter, T. P. 2003. Children's invention of multiplication and division algorithms. In A.J. Baroody & A. Dowker (Eds.) *The Development of Arithmetic Concepts and Skills: Recent Research and Theory*. Mahwah, NJ: Erlbaum.
- Baek, J. M. 2005. Children's mathematical understanding and invented strategies for multi-digit multiplication. *Teaching Children Mathematics, 12*, 242-247.

A progression in student's capacity to add on includes becoming more adept at adding chunks of decomposed elements of, in this case six, in order to more efficiently add on. The emphasis, however, remains on the repeated addition of the next six until all sixes are accounted for.

Says: 6, 12, 18… [+2] → 20 + [4] → 24 + [6] → 30 + [6] →36 + [4] →40 + [2] → 42 +[6] → 48
Skip counting demonstrating using landmarks of ten instead of counting all by ones

Skip counting exhibits an early form of multiplicative thinking. As one counts each six, a finger or tally is made to track that all the sixes have been counted. This is an early formation of making a unit of units (one 6 for six ones) so that the 6 becomes the more efficient countable unit. The tracking on one's fingers the number of sixes accounted for is an early attempt to coordinate the units.

Doubling and Complex Doubling
Using the double by pairing like numbers is done to reduce by half the numbers of quantities to be added. Initially, students may only double once and then add the numbers together as in the example on the left below. Complex doubling continues to make doubles until the answer is determined. Should an odd initial quantity be involved, say, 9 groups of six, then the outstanding amount is added at the end to the, in this example the 48.

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6 + 6 + 6 + 6 + 6 + 6 + 6 + 6
 \ /     \ /     \ /      \ /
 12  +   12  +   12   +   12

   10 + 10 + 10 + 10 = 40
       2 + 2 + 2 + 2 = 8
         40 + 8 = 48
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[a]

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6 + 6 + 6 + 6 + 6 + 6 + 6 + 6
 \ /     \ /     \ /      \ /
 12  +  12   +  12   +   12
    \     /          \     /
     24        +        24
        \              /
               48
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[b]

Figure 7 – Doubling & Complex Doubling
[a] Initial Doubling Strategy; [b] Complex Doubling Strategy

More multiplicative thinking is now evident. With each new double, a composite unit is formed from the previous sub-units. The need to coordinate units becomes more complex to track with each doubling effect. Each 12 represents 2 sixes. Each 24 represents two twelves as well as now 4 sixes, so that when one doubles again one can trust that all eight groups of six have been accounted for. The cognitive tracking is greater.

Additive Units of Units
The student writes only a portion of the sixes to find that subtotal and uses that subtotal as *a new composite unit* to figure out the rest of the total. These chunks over time begin to reflect coordination between the factors of the number of groups

Example one:

$$\begin{array}{r} 6 \\ 6 \\ \underline{+\ 6} \\ 18 \end{array} \qquad \begin{array}{r} 18 \\ \underline{+18} \\ 36 \\ \underline{+12} \\ 48 \end{array}$$

Example two:

$$\begin{array}{r} 6 \\ 6 \\ 6 \\ \underline{+\ 6} \\ 24 \end{array} \qquad \begin{array}{r} 24 \\ \underline{+24} \\ 48 \end{array}$$

Figure 8 – Additive Units of Units

Adding to a referent landmark then using that new composite unit to proceed

Similar to the cognitive load required in Doubling and Complex Doubling, the forming of new composite units out of smaller subunits and the need to coordinate units is essential so one does not lose track of the number of groups being accounted for. The difference between Doubling and Additive Units of Units is that the latter is decomposing the number of groups (8) into 2 x 3 + 2 x 6 [Example one] or 2 x 4 [Example two]. More *anticipatory thinking* is being enacted around decomposing the multiplier into what ultimately leads to factor combinations.

The distributive property of multiplication over addition

In this strategy, the student explicitly decomposes the number(s) into addends to use multiplication to determine the partial product combinations to figure out the total.

Example

8 x 6

8 x 6 = (5 + 3) x 6 = (5 x 6) + (3 x 6)

"I know 5 sixes is 30 and 3 sixes is 18; 30 plus 18 is 48"

Figure 9 – Distributive Property of Multiplication Over Addition

Decomposing one factor into a known combination

This is a fully multiplicative abstract strategy based on the anticipatory planning of how the numbers in the problem are to be decomposed for the most efficient use (to the solver) and represented in numerical form. [8 x 6 = 5 x 6 + 3 x 6]. The distributive property of multiplication over addition is based on the decomposition of numbers into addends, multiplying to determine the partial products, then adding the partial products to find the total. The act of decomposition creates subunits. Coordinating the various units is needed to know if all combinations have been multiplied and added. More of the calculation is internalized. Executing this strategy, being able to justify one's reasoning, is an indication of thinking multiplicatively.[8]

Factoring & the Associative Property

With the previous strategy using the distributive property of multiplication over addition, the numbers were decomposed into known addends, subunits, of the original quantity. In this next strategy, the quantities are

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[8] An individual can be taught the distributive property by rote and arrive at the total product without thinking multiplicatively. Typically such a student responds quickly the answer to 8 x 6 is 48 because it is memorized. A student who goes through the process of the example above demonstrates multiplicative thinking due to the *conscious act* of decomposing the 8 into 5 + 3 (or 6 + 2, etc.) and systematically calculating the total product. The decision is the students', not imposed from the outside. Learners who are allowed to develop this thinking go on to fluidly recall with understanding the various single-digit combinations. Follow this link to an article on Learning Single-Digit Combinations to explore the algebraic reasoning that develops when students are nurtured to develop *derived strategies* en route to fluently recalling the single-digit combinations. Fluent recall is different from rote memorization.

decomposed into factors. The factors are then potentially reordered (commutative property) and recombined (associative property) to determine the total product.

Example

8 x 6 = (2 x 4) x 6 = 2 x (4 x 6) = 2 x 24 = 48
"I know four sixes are 24 and then I doubled that to get a total of 48."

or

"I know two groups of eight are 16 and then I tripled that to get 48."
8 x 6 = 8 x (2 x 3) = (8 x 2) x 3 = 16 x 3 = 48

Figure 10 – Factoring & Associative Property of Multiplication

Decomposing quantities into their factors to then use the commutative and associative properties

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This is a fully multiplicative strategy. It involves anticipatory thinking in how to decompose numbers into subunits (factors), and coordinate the units as one works to make sure the original quantities are accounted for. Developing a capacity to think in scale ["I then tripled that."] emerges in a strategy such as this.

Interim Summary III

Beyond direct modeling, students initial number strategies are through using repeated addition. As the strategies evolve and mature, more cognitive shifts begin to occur and are tracked abstractly. New composite units are formed and counted in order to reduce the workload of ever growing number combinations. The need to track which unit goes with which quantity and coordinating all the various subunits simultaneously as one operates reflects the increasing cognitive demands necessary to master in order to think and reason multiplicatively. *These cognitive dispositions are not individual lessons, they are ways of thinking that mature over time and are often situational to the number quantities being worked with.* Focused conversations between and among students and the teacher over the arc of a school year and again across the grades is how these dispositions come into use.

Part IV – Maturing Cognitive Capacity

Let's revisit the strategies just outlined in the previous section but now focus more deeply on the cognitive load students engage in at each level. These ways of thinking, the shifting from additive to multiplicative thinking, matures over time. A teacher needs to be *attending* (observing and listening) to student moves and descriptions (verbal, drawn, and/or written) in order to *interpret* developmentally where a student is at. It is at this point *intentional instructional decisions* are made and enacted. This engagement with student thinking allows you, the teacher, to ask the questions or pose prompts that scaffolds students to clarify and elaborate on the choices they have made. The public sharing of strategies – those that are useful, contain errors, or even are dead ends – is where insights are gained and advances occur. *Making the implicit explicit* so that students can *anticipate* their choices is how thinking matures and becomes more multiplicative.[9]

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[9] From the "Teacher Noticing" and" Responsive Teaching" research of the following:

- Jacobs, V.R. and Empson, S.B., (2016), Responding to children's mathematical thinking in the moment: An emerging framework of teaching moves. *ZDM Mathematics Education, 48,* 185–197.
- Jacobs, V.R., Lamb, L.L.C., Phillipps, R.A. (2010) Professional noticing of children's mathematical thinking. *Journal for Research in Mathematics Education. 41*(2), 169–202.

Multiplication Context: *The head soccer coach has 8 bags of soccer balls for all of the teams to use in the league. There are 10 balls inside each bag. He also has one bag with only 2 balls in it. How many soccer balls does he have for all of the teams?*

Measurement Division Context: *The class basket of markers has 82 markers in it. At the end of the year, the teacher wants to place the markers back into boxes. If 10 markers fit in a box, how many full boxes will there be?*

Direct Modeling

The direct modeler is locked into following the structure of the problem. All sets are represented. In the bags of soccer ball scenario, the student would build either with manipulatives or pictorially something like the following.

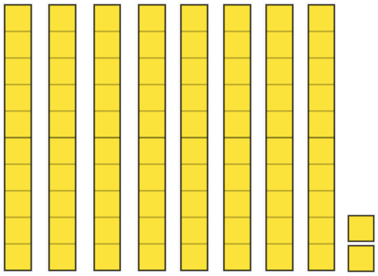
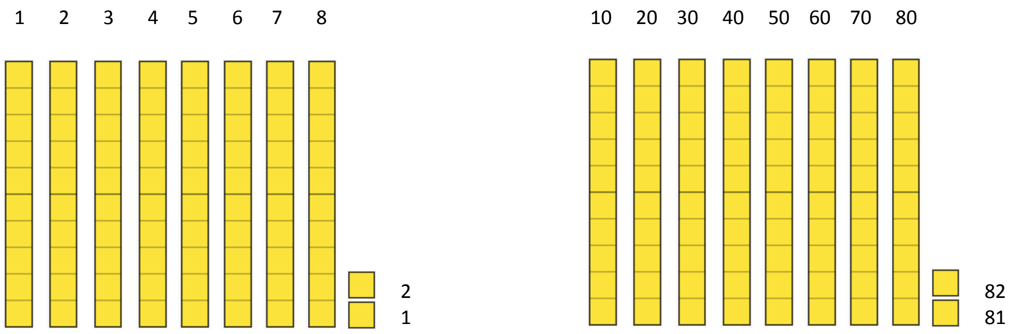


Figure 11a – Direct Modeling of 8 bags of 10 balls plus two more

In the building of this model, the direct modeler is cognitively focused initially on *tracking the bags*, as in counting 1, 2, 3, 4, 5, 6, 7, 8 bags then 1, 2 for the extra balls in the last bag. An early, very emergent direct modeler would need to count out the ten eight times in order to create each group. Example: 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 – That's one bag. 1, 2, 3, 4…10. That's two bags… A progression in thinking after experiences working with these manipulatives, the student is able to trust that there are ten in each rod, i..e., mentally holding that the one ten is also ten ones. However, cognitively, the student is *only focused on creating the 8 bags of 10 plus the two extra* as described in the problem context. Once that count is established, the direct modeler then switches the cognitive focus to recounting the collection to count the *total number of soccer balls* as in, 10, 20, 30, 40… 80, 81, 82. The earlier direct modeler would count everything by ones. The progression in cognitive load illustrated here features the student who holds that each rod is the same quantity and coordinates that one ten consists of ten ones.

- Richards, J. and Richardson, A.D., (2015). A review of the research on responsive teaching. In A D. Robertson, R.E. Scherr, & D. Hammer (Eds.) *Science and Mathematics Responsive Teaching in Science and Mathematics* (pp.36-55), Routledge, London.



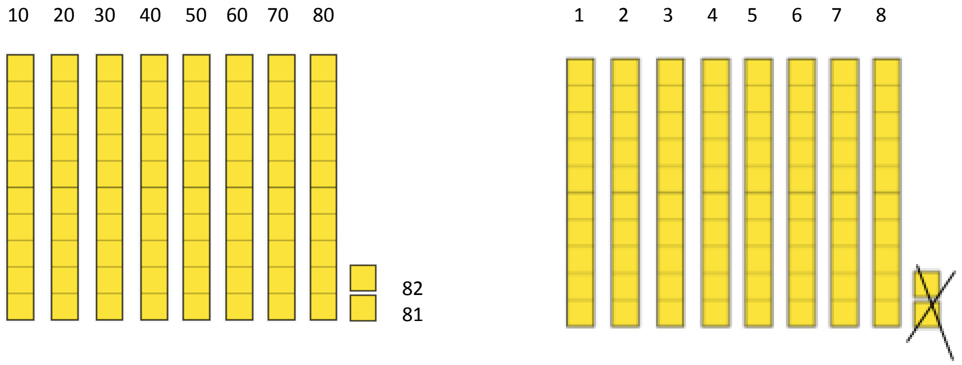
First count: The cognitive focus on the bags

Second count: The cognitive focus is on the balls

Figure 11b – Cognitive Focus of a Direct Modeler in a Multiplication Context

The important cognitive element to note is that students at this level can think in bags or in balls *but not both at the same time.* That requires *simultaneous thinking*[10] which is developmentally difficult at this level.

The same is true when the direct modeler solves a measurement division problem. (82 markers being put back into boxes of 10



First count: The cognitive focus is first on the markers

Second count: The cognitive focus is on the boxes

Note that, based on the question asked (How many full boxes), the remainder is ignored

Figure 11c — Cognitive Focus in a Measurement Division Context

The direct modeler can cognitively focus on the markers or the boxes, but not both at the same time.

Additive Skip Counting/Repeated Addition

Additive skip counting and repeated addition draw on the same cognitive level of thinking. A student will skip count each group up as high as the rote or recalled level will go and then count on by ones any remaining groups that need to be accounted for. The difference between the two is that in repeated addition, all the groups are typically visually written out before the counting begins. The skip counter uses fingers or tallies to keep track of how many groups or the number in each group is being accounted for as one goes along. What is interesting cognitively about the skip counter is the child is tracking two pieces of information at the same time. In solving the *8 bags of soccer balls with 10 in each bag plus a bag with only two balls in it*, a skip counter starts tracking *verbally* each group of 10 while putting up one finger to track the groups. Each finger

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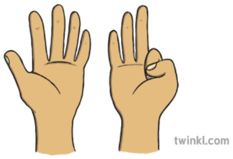
[10] Kamii, C. & Livingston, S.J. 1994. *Young children continue to reinvent arithmetic: 3rd grade.* New York, NY: Teachers College Press.

represents both a group of ten balls *and* one bag. The last number in the verbal count is the solution to the question asked. (Figures 12a & 12 b)

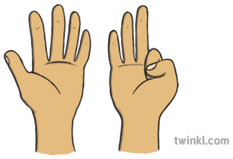


10, 20, 30, 40, 50, 60, 70, 80, 81, 82

The answer is 82 soccer balls.

Figure 12a – The Cognitive Focus in a Skip Counting Strategy in a Multiplication Context

Conversely, with the measurement division problem *82 markers being placed back into boxes of 10, how many full boxes will there be,* the tracking is reversed. *The number of fingers used* becomes the answer.



10, 20, 30, 40, 50, 60, 70, 80,

The answer is 8 full boxes.

Figure 12b – Cognitive Focus of a Skip Counting Strategy in a Measurement Division Context

Skip counting is an early form of multiplicative thinking in that the student is moving toward being able to track two pieces of information simultaneously. The reasoning remains additive in that each element is sequential and the tracking is an external visual.

Additive Units of Units [11]

The cognitive progression demonstrated by the student at this stage is the visual creation of a composite unit and the iterating (repeating) of that new more efficient unit to find a solution. An early form of additive units of units appears when students use doubling or complex doubling to determine a solution. Let's look at Ashley's work and Layna & Hallie's work once more. (Figure 13a-b)

Each of Ashley's new 12 represents two groups of six. She holds in her head that she has accounted for the 24 of the 25 sixes as she works with the twelve twelves. The same with Layna & Hallie. The three new fifties is the same as six twenty-fives. The creation of these new composite units requires more simultaneous thinking, more of the tracking of the information is abstracted internally, and the student's computational work is, in this case, cut in half.

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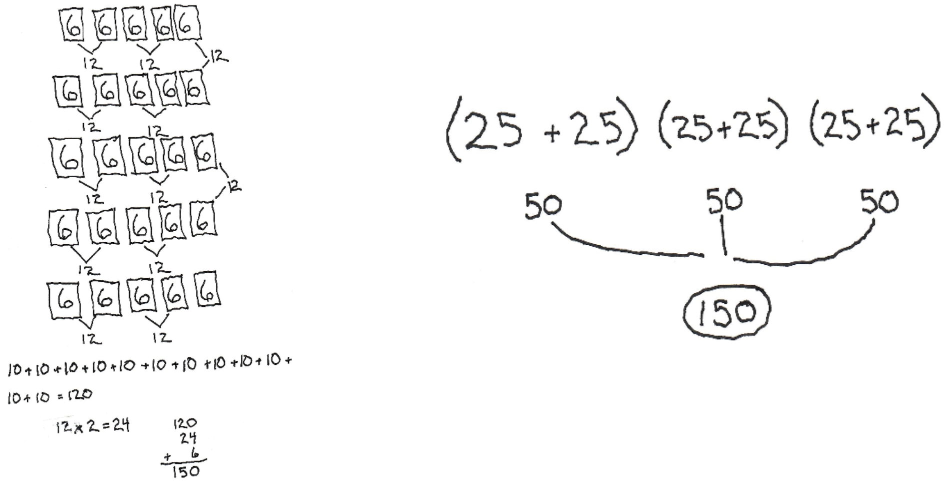
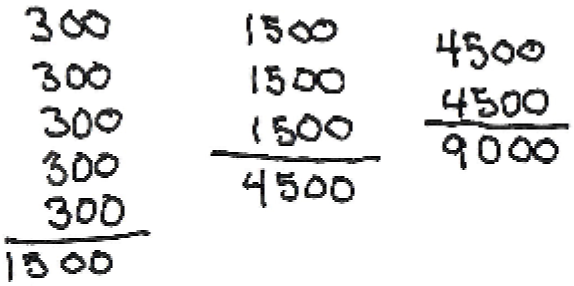
[11] Also described as "partitioning a single factor into non-decade numbers" (Ambrose, et al. 2003)



Context: 25 boxes with 6 cookies inside each box. How many cookies are in all of the boxes?
Ashley: Doubling Strategy – 25 sixes – modeled as structured in the context
Layna & Hallie: Doubling Strategy – 6 twenty-fives – Use of the commutative property of addition

Figure 13a-b – Doubling Strategy: An Early Emergence of Additive Units of Units

An even more sophisticated example of making additive units of units can be seen in Maddie's solution for 30 x 300 (Figure 14a). Reconstructed here from her original work, Maddie wrote an explanation of her thinking. "*The answer is 9,000. First I added 300 5 times because I was trying to get to 15 300s and since 3 x 5 is 15 I added 1500 3 times[. S]ince 15 is half of 30 I added 4500 2 times. The answer is 9,000.*"



Context: 30 x 300
Maddie: Additive Units of Units

Figure 14a – Additive Units of Units Strategy – Maddie's Thinking

In the description of her processing, how she created the composite unit 1500 from the five three hundreds and, subsequently, the composite unit of 4500 was created from the three 1500s captures the creation of the composite units used to strategically increase incrementally to a total of 30 three hundreds. Knowing that she was first at 5 three hundreds, then at 15 (because she knew 3 groups of 5 groups of 300 would get to 15 groups) she knew doubling 4500 would be the equivalent of 30 groups of three hundred. She visually added the amounts on paper while monitoring the number of groups abstractly in her head. While still additive in the calculations that she needed to enact at the various stages of her work, *she has developed the key concept that multiplication involves the abstraction of making units of units and the coordination of those units.*

Cognitively, what Maddie monitors internally as she works is a clear example of the emerging capacity to coordinate units simultaneously while working to find her solution. Looking again at her work (Figure 14b), what appears in red shows that as she creates each new composite unit, she simultaneously needs to coordinate internally that she is up to a sub-total of 30 groups of 300. 9000, therefore, is 2 groups of 4500 while simultaneously the same as 3 groups of 1500 which represents 15 groups of 300. Thirty is essentially factored into 5 x 3 x 2 and then reassociated with combinations of 300.

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30 x 300

30 x 300 = 2 x (3 x (5 x 300))

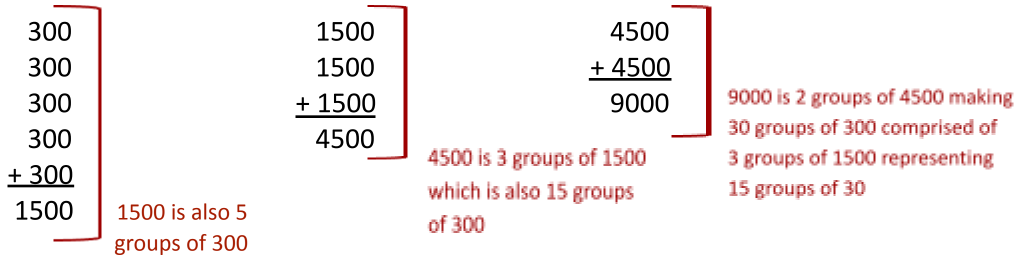

Figure 14b – Cognitive trace of Maddie's Additive Units of Units Strategy

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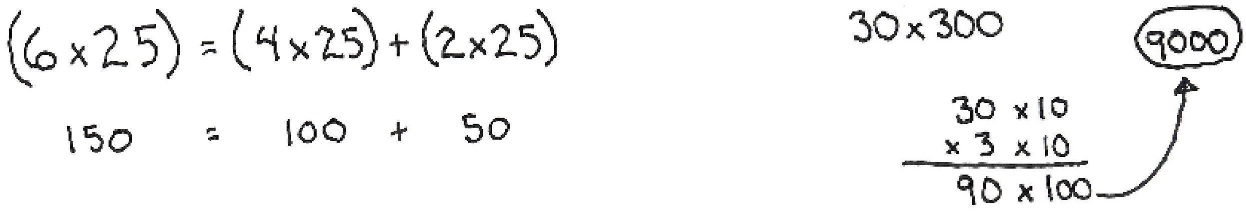
While Maddie's work is still additive in execution, she clearly is developing and demonstrating the core concepts that multiplication involves the making of units of units, the coordination of units, and the capacity to scale up while thinking simultaneously. Not letting the Maddies in one's classroom to pass through this stage hinders that foundational growth.

Distributive and Associative Properties of Multiplication

Both the distributive and associative properties of multiplication involve decomposing at least one set of numbers to make the task easier to calculate. The distributive property of multiplication over addition or subtraction involves decomposing the numbers into *addends* in order to multiply to obtain a partial product then adding all the partial products together to determine the total. To use the associative property, the decomposition of a number is into its *factors*. Potentially the commutative property is then used to realign the factors into a more efficient order to then reassociate the factors to determine the total. Both decomposition processes require *anticipatory thinking*, planning ahead into an efficient set of subunits that allow finding the total. In doing so, new units of units are formed, units are coordinated among various subsets simultaneously at the abstract level.

In the following examples, Jacob's use of the distributive property of multiplication over addition to solve the 25 boxes of six cookies by decomposing 6 into 4 + 2 or Sam's factoring in solving 30 x 300 into 3 x 10 x 30 x 10 demonstrates the preplanning (anticipatory thinking), the making of units of units, and the simultaneous coordination of units in order to find their respective totals. *The work occurs at the abstract multiplicative level rather than at the additive level of the previous strategies.* From both the written and accompanying verbal explanations of the two students, Jacob has, at least implicitly, used the commutative property of multiplication in that he is focused on multiples of 25 rather than multiples of 6 as the problem is originally structured. He knows 4 twenty-fives are 100 because of its relationship to money. Sam's decomposition of 300 into 30 x 10 rather than the adult instinct to use 3 x 100 is based on Sam knowing what 3 thirties are 90. 90 x 100 he articulates as being 9 x 10 x 100 since 10 x 100 is 1000 so 9 x 1000 is 9000. This factoring to use the powers of ten explicitly was done internally in his head.

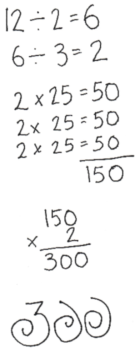


Context: 25 boxes with 6 cookies in each
Jacob: Distributive property of multiplication over addition

Context: 30 x 300
Sam: Factoring, Commutative property & the Associative property

Figure 15 – Strategies Using Either the Distributive and Associative Properties of Multiplication

Sophie provides another example of how factoring and the associative property can be used to find the solution to a problem. Sophie is solving 25 boxes of cookies but with 12 cookies inside each box. She, too, commutes the problem being 12 boxes of 25 cookies to focus on multiples of 25 rather than on multiples of 12. Expressing her work through division, she factors 12 into 2 x 6 and then factors 6 into 3 x 2. She then proceeds to do 2 groups of 25 three times to get 150 and then doubles that to get a total product of 300. In doing so, she makes a unit of units of 2 twenty-fives, triples that to make a new composite unit of 150 cookies and, finally, doubles that to arrive at 300 cookies. Internally, Sophie is coordinating the units to track that the initial doubling, then tripling, then doubling [2 x 3 x 2] totals 12 boxes of 25 cookies. This work is at the abstract level rather than the additive counting level.



Context: 25 boxes with 12 cookies inside each box
Sophie: Factoring, the commutative and associative properties
12 x 25 = (2 x 3 x 2 x 25) = 2 x (3 x (2 x 25))

Figure 16 – Factoring to Use the Commutative and Associative Properties of Multiplication

The Language of Value

What is foundational in the student work illustrated throughout this piece is that the students are *speaking in values rather than in digits.* There is no "carrying of one" as they proceed. They are expressing themselves in tens and hundreds, twenties and fifteens. This disposition allows them to accurately monitor the subtotals of the work. They are working mathematically and algebraically, not in a rote, mechanical manner that allows an answer to be arrived at but lacks meaning. For those who learn *merely by rote procedure*, they may produce answers but foundationally struggle to think proportionally and work with rates and ratios once they enter middle school mathematics.

Cognitively guiding student learning to develop the foundational underpinnings of multiplication and division to include the making of units of units, the coordination of units, and the capacity to scale up while thinking simultaneously allows the transition to thinking proportionally and in working with rates and ratios an easy one. A house of bricks is built rather than a house of either straw or twigs.

Progression from Additive to Multiplicative Thinking

Transitioning from additive to multiplicative thinking occurs over time. Attributes of this *progressive abstraction* are captured in a body of research.[12] The table below captures a range of those attributes.

<table>
<tr><td colspan="2"><i>Table 1</i>
<i>Attributes of Additive and Multiplicative Thinking</i></td></tr>
<tr><td colspan="2"><i>additive thinking</i> <i>multiplicative thinking</i>
<i>increase in progressive abstraction</i></td></tr>
<tr><td>Additive
(descriptors)</td><td>● Additive strategy progressions:
○ Direct modeling
○ Repeated addition
○ Skip counting
○ Doubling & Complex Doubling
○ Additive Units of Units
● Attributes of additive thinking
○ Smaller, incremental steps
○ Decomposition in smaller aggregations
○ Place value: ten as a unitary or counted unit
○ Minimal if no use of composite units
○ Confusion over/ignoring of units
○ Additive calculation as oppose to relational thinking
○ More grounded visually/physically than mentally</td></tr>
<tr><td>Multiplicative Thinking
(descriptors)</td><td>● Multiplicative strategies
○ Algebraic properties of operation
■ Distributive property of multiplication over addition and subtraction
■ Associative property of multiplication
■ Commutative property of multiplication
■ Equal sign as a comparative symbol
● Attributes of multiplicative thinking
○ Anticipatory thinking/planning ahead
○ Efficiency in decomposition of numbers (addends, factors)
○ Place value as a rate of ten, the capacity to re-unitize numbers</td></tr>
</table>

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[12] Brickwedde, 2011; Ambrose, Beck & Carpenter, 2003; van Putten, et al. 2005; Confrey, 1994; Confrey, et al., 2009; Kouba, 1989

	○ Making & Using of composite units ○ Coordination and transformation of units ○ Relational thinking ○ Use of scale factor or unit rate ○ Derived strategies ○ Mental abstraction
Qualifiers	● Accuracy ○ Both additive and multiplicative strategies can result in high levels of accuracy ○ Both could have inaccuracies in computation while using appropriate strategies ● Automaticity ○ Automaticity with algorithmic procedures may not be an indicator of multiplicative thinking, merely fluency with procedural steps while working additively ● Language of values ○ The use of the language of value is a strong indicator of conceptual understanding ○ Someone persisting using single digit language may have the capacity to think multiplicatively but one cannot rell without probing for understanding ● Place value ○ Interpreted only by a digit's location, and subsequent inability to re-unitize to other units of ten, may be a lack of exposure rather than an inability to think multiplicatively ● Formatting ○ Whether students represent their calculations vertically or horizontally is irrelevant to the efficiency or their ability to reason abstractly. That mathematics, the underlying algebraic principles, are the same. The mathematical ideas display differently across the two representations. Dienes' *Variability Principle*[13] augurs that working comfortably with both representations reinforces the mathematical structure rather than the procedural steps and social conventions required when using either representation.

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[13] Dienes, Z. (1971, fourth edition) *Building up mathematics.* London, UK, Hutchinson Educational Ltd.

Interim Summary IV

With each progression in solution strategy, from direct modeling through to the abstract use of the distributive and associative properties, more and more capacity in anticipatory thinking to decompose numbers, create new composite units, and to coordinate the various subunits is required. More and more of that work is internalized at the abstract level. The thinking moves from additive to multiplicative thinking. Multiplication as a distinct and unique operation is comprehended laying the foundation to begin to think in scale, proportionally, and consider how relations can co-vary.

Reflection Point 3

- ☐ Make a copy and solve the multistep rich task From Sheep to Baseball. As you work, keep close attention to the units that go with the various numbers and how they help you articulate your solutions. Think also how the need to coordinate information among the different parts of the tasks stretch your cognitive load. This type of task is intended to create an environment where monitoring multiple pieces of information and the need to label units is imperative.
- ☐ Solve 42 x 29 as if you were Maddie. What units of units make sense to create given these numbers? How does thinking in *groups of* language , as opposed to in *times* language help in creating those new composite units?
- ☐ Solve 42 x 29 horizontally using the distributive property over multiplication first over addition then over subtraction as in 42 x 29 = 42 x (30 – 1), i.e., the compensation strategy. *Watch your language* as you work!!! How does working horizontally expose the properties of operation more visibly than had you done the work using a vertical format?

Part V – Coordination of Units – Unit Confusion

In many ways, solving problems as a direct modeler is so much simpler in that everything is literally out on the table visible to be seen. While some abstraction of the student's thinking is projected onto the manipulatives, the modeling frees up much of the individual's working memory. The more abstract one becomes in operating with numbers, the more there is mentally to keep track of. In a cross-sectional study with third through fifth graders, as students began to move to more abstract and multiplicative ways of thinking, they entered a developmental messy period called *unit confusion*[14]*.* The term *unit confusion* captures those points in a student's thinking where an appropriate strategy is used to solve sub-parts of a mathematical task only to lose track of the appropriate units when reconfiguring a final solution. In other words, the student confuses elements of the measure space to arrive at a misconfigured solution as the units transform. Example: The two following fourth graders were asked the question *how many tens are in the number 783?* Nina answers, "*fifteen… because ten times ten is a hundred so it's then seventy plus eighty equals fifteen.*" Ezra answers, "70 tens [in 700]. Eight [tens in eighty]." [Teacher] "So how many tens altogether?" [Ezra] "150." Each student understood the individual elements, but when a final response is verbalized, a mix of various decomposed and re-unitized elements are mis-joined to create a solution. Their work was done abstractly in their heads and verbally conveyed to the teacher. No written work was created.

The mis-re-unitization of the elements within the task can be mapped as follows.

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[14] The term "unit confusion" comes from the research work described in Brickwedde, J. (2011). Transitioning from additive to multiplicative thinking: A design and teaching experiment with third through fifth graders." University of Minnesota Ph.D. dissertation. Accessed from http://conservancy.umn.edu/handle/115899

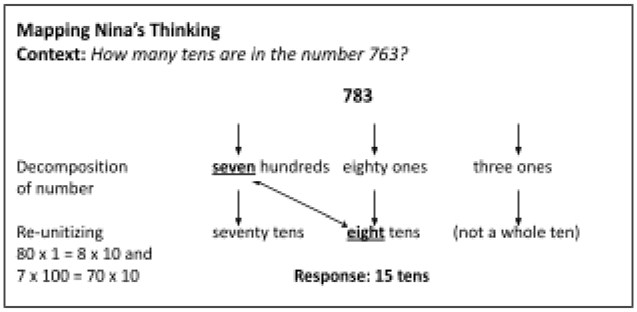


Figure 17a – Nina's unit confusion working at the abstract level

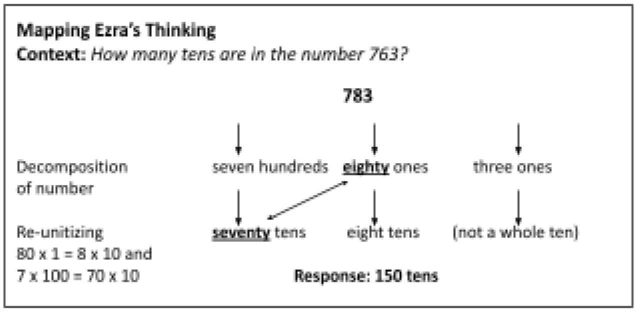


Figure 17b – Ezra's unit confusion working at the abstract level

Nina merges the seven from the *hundreds* and the eight *tens* to create 15 tens rather than eight *tens* and the seventy *tens.* Ezra merges the seventy *tens* with the eighty *ones* to create 150 tens instead of the eight *tens* and the seventy *tens.* This is after having accurately described all the individual elements.

Is each student merely misspeaking? There is a possibility that is true. Yet even from that perspective, it demonstrates how *as the student internalizes more of the work at the abstract level, there are more bits and pieces of information that the student must monitor and keep track of. Numbers can easily get separated from the units and get misconfigured in a final action.* In the study, as students matured in their thinking over time and grade level, this unit confusion disappeared.

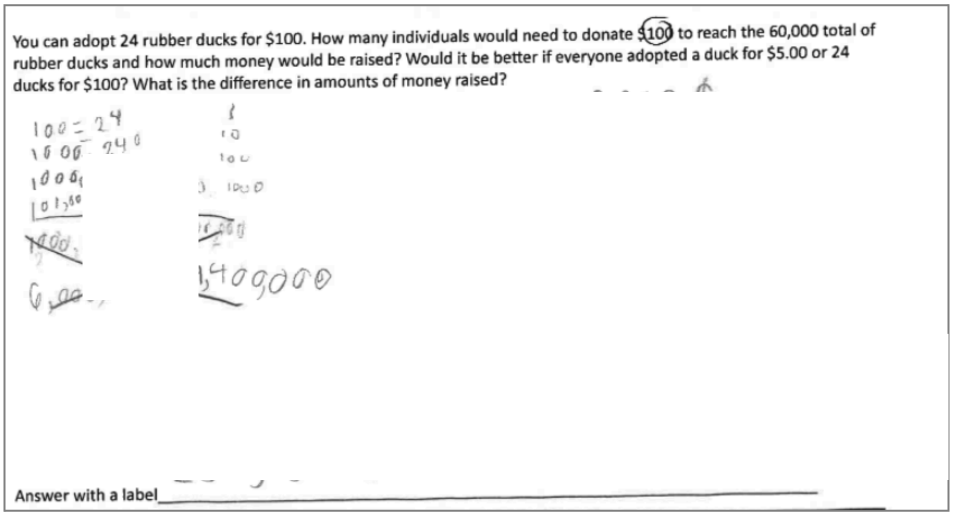
How to scaffold a student through this messy developmental phase? There are several instructional options. The first is to ask the student to repeat the mathematical sub-elements just described to see if the student self-corrects (a sign that they mis-spoke). Another scaffold is to re-voice what you heard the student verbalized and ask them to clarify what the solution should be. Hearing their own thinking from an outside source can many times get the student to recognize the error. Example: *So, what I heard you say that there are eight tens in the 80 ones and you said there were seventy tens in the seven hundred. So, how many tens are there altogether?*

Another scaffold is to help the students create a visual chart of their thinking so that the work moves from purely abstract mental mode to a visual numerical and text world. *Such visuals free up the working memory* and allow the brain to refocus and reflect on the now external information. With time, such visuals may not be necessary as the student has developed the capacity to monitor all the information needed to solve the task internally.

A Student Case – Lar Eh

Lar Eh was a fifth grader at the time he was asked to solve this multi-step rich task based on the Chicago Ducky Derby. The event is a Special Olympics fundraiser in Illinois. The nature of a rich task is that information from one stage of the task is potentially needed in successive tasks. These rich tasks were also intentionally designed so that paying attention to the unit transformations and subsequent unit coordination is central to the work. Lar Eh approached me as he was attempting to solve the second of what were four interrelated tasks.

When he approached me, nothing was crossed off at that point. I first had him explain to me his thinking process. He had scaled up $100 worth of rubber ducks (24 ducks) to 6,000,000 and he was perplexed about the number and what it meant and if he was on the right track. His ability to reason relationally was solid but he had lost track of what units went with each element. This represents a different version of unit confusion. He was not tracking which units went with which set of information. He was so absorbed in the calculation that he lost track of the ability to find the solution. Recognizing his own dilemma, he sought help. (Figure 18a)



You can adopt 24 rubber ducks for $100. How many individuals would need to donate $100 to reach the 60,000 total of rubber ducks and how much money would be raised? Would it be better if everyone adopted a duck for $5.00 or 24 ducks for $100? What is the difference in amounts of money raised?

100 = 24
1000 = 240

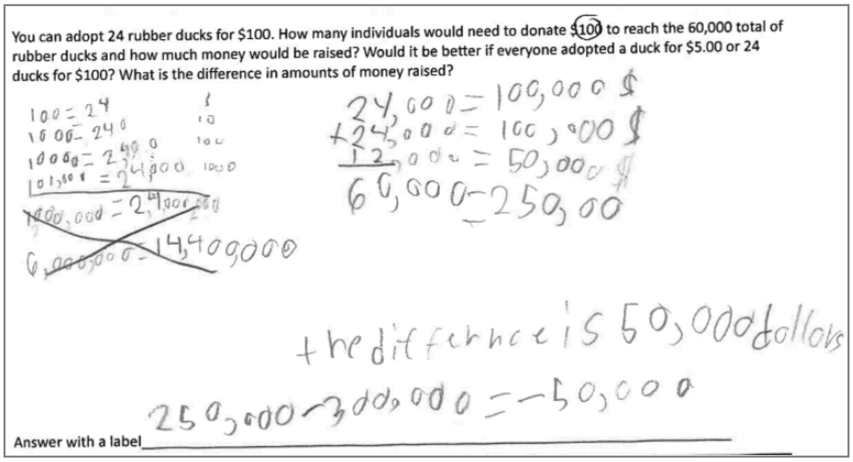
1
10
100
1000
1,400000

Answer with a label

Figure 18a – Lar Eh's initial work

I brought him back to his starting point and asked, *what did the 100 represent?* [The money] *What did the 24 represent?* [The number of rubber ducks] *What was the first question he was trying to find out?* [The number of individuals buying the rubber ducks] I directed him to add a third column allowing him to track this third unit of information. *How many individuals would pay $100 to buy 24 rubber ducks?* [One person]. That was written down in a new column off to the side. *When will you stop?* [When 60,000 rubber ducks were sold.] (A scaffold to have him consider the units to be tracked.)

He then proceeded to rethink his previous calculations. Once he got to 1000 *individuals* raising $100,000 by buying 24,000 rubber ducks, he realized his remaining work did not make sense. I directed him to cross off the unneeded calculations and posed the next question. *How can you use what you know about 1000 people buying 24,000 rubber ducks to figure out how much money would be raised by selling 60,000 rubber ducks?* (A scaffold to have him think relationally and in scale.) He proceeded on his own to determine that 60,000 rubber ducks would raise $250,000. He then answered the second question, based on the results from the first task in the series, that a deficit of $50,000 (–50,000) would result between the two sale combinations. (Figure 18b.)



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100 = 24
1000 = 240
10000 = 2400
100,000 = 24,000
~~1000,000 = 2,4000,000~~
~~6,000,000 = 14,400000~~

1
10
100
1000

24,000 = 100,000 $
+24,000 = 100,000 $
12,000 = 50,000 $
60,000 = 250,000

the differnce is 50,000 dollars

Answer with a label 250,000 - 300,000 = -50,000

Figure 18b – Lar Eh's succeeding work after scaffolded conversation with teacher

Intermediate grade curricular materials that predominantly focus on non-context, "naked" number multiplication arithmetic do a disservice to students in being able to deeply understand what is involved in being able to think multiplicatively. It is in context situations, and especially in multi-step rich tasks such as the Chicago Ducky Derby scenario, where one needs to track units of measure and monitor and coordinate the unit transformations. As students progress from additive to multiplicative ways of thinking en route to middle school mathematics, providing tasks and drawing out students' thinking in the public sharing arena where unit transformation and coordination is a prime focus of the work allow students like Lar Eh to have a solid foundation upon which to build one's mathematical understanding.

Interim Summary V

Multiplication as an operation is structured around the making of units of units, is unit transforming, and requires the ability to think simultaneously as one coordinate units as they are decomposed, re-unitized, and transformed. Multiplicative thinking requires the ability to think in scale (*n*-times as many), proportionally, and in covariance with another unit. This way of thinking is a disposition, not a lesson. It matures over time. It needs to be nurtured and scaffolded. Unit confusion appears to be a developmental messy phase that students move through as the transition from concrete and additive strategies to more internal abstract strategies. Supporting students with strategic prompts and suggested representational formats such as charts or tables, with specific labels, allow students mechanisms to keep track of their thinking and free up working memory. Cognitively guiding individual's learning and within the public sharing process draws learner attention to the units and the subsequent transformations. An overemphasis on naked number context and rote algorithmic practice leaves students with a house of cards that easily collapses in middle school.

Reflection Point 4

- ☐ Return to the last scenario for the multi-step rich task *At the Factory: Making & Boxing Colored Markers* where cartons are being loaded onto pallets. Create a series of equations that includes clearly labeled units that go with each quantity. Example: 5 cartons/row x 6 rows/layer. How do the units transform as you work through the problem? Why does the final solution reflect the units that it does? Without resorting to the tricks you may have been taught in unit analysis lessons in your high school chemistry class, how would you engage students meaningfully in conversation about how the units transform?

- ☐ Solve the Chicago Ducky Derby tasks yourself. Think like a fifth grader at various stages of additive or multiplicative thinking. As you solve the various tasks, where might labels be useful? When might a chart or table be useful as an organizer of information? What other representational options might be useful that could be strategically introduced or highlighted as students work or even who is selected to publicly share?

Part VI – Progression of Multi-Digit Multiplication Abstract Strategies

A trajectory of multi-digit strategies has been identified by various researchers[15]. It is important to know that the following trajectory is not a linear one. Students may shift across levels of abstract and additive strategies depending upon the number range involved in the problem. Therefore, it is important to monitor student thinking across several problems to comprehend the nature of their developmental range. You are monitoring "levels of thinking, not levels of students.[16]" While students' single-digit multiplication and division emerges from their direct modeling and counting strategies, *their multi-digit multiplication and division strategies are dependent on their flexible uses of multi-digit addition and subtraction strategies.* Flexibility with multi-digit addition and subtraction strategies provide potent entry points to multi-digit multiplication and division. The student work samples captured in Figures 13a and 14a provide examples with how students transition from purely additive to move forward with more abstract multiplicative strategies and levels of thinking.

Transitioning from Additive Units of Units to Partitioning Single Factor Into Non-Decade Numbers

Moving beyond repeated addition, doubling and complex doubling, and additive units of units, students begin to work at the abstract number level using more and more decomposing number strategies to make different combinations to form partial products that then lead to the total. While Additive Units of Units still falls within the additive level of thinking, it falls on the cusp of multiplicative thinking. Maddie's strategy in figures 14 a&b above demonstrated, elements of multiplicative thinking are beginning to emerge. Haley demonstrates how she moves beyond using additive units of units (Figure 19) to decomposing a single factor into non-decade numbers [38 x 7 = 38 x (5 + 2)]. This decomposition of a single factor grows out of work within the classroom around derived strategies when working with single-digit combinations as in 8 x 7 = 8 x 5 + 8 x 2[17]. Additionally, the element she adds in her work that is of notice is she creates a ratio table to organize her thinking. She does not iterate each 38 which would indicate a repeated addition strategy. She doubles the 38, then determines five thirty-eights, then adds the partial products together to find the total. More of her work is internalized at the abstract level. She is thinking multiplicatively by using the distributive property of multiplication over addition as her strategy.

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[15] Ambrose, R., Baek, J., Carpenter, T. P. 2003. Children's invention of multiplication and division algorithms. In A.J. Baroody & A. Dowker (Eds.) *The Development of Arithmetic Concepts and Skills: Recent Research and Theory*. Mahwah, NJ: Erlbaum.
Baek, J. M. 2005. Children's mathematical understanding and invented strategies for multi-digit multiplication. *Teaching Children Mathematics, 12*, 242-247.

[16] Battista, M. T. 2012. *Cognition based assessment and teaching of geometric shapes: Building on students' reasoning.* (p. 47) Portsmouth, NH: Heinemann.

[17] Brickwedde, J. 2022. Learning single-digit combinations: Developing important mathematical ideas. Project for Elementary Mathematics. https://www.projectmath.net/project-services/p-d-materials/professional-development-articles/basic-facts/

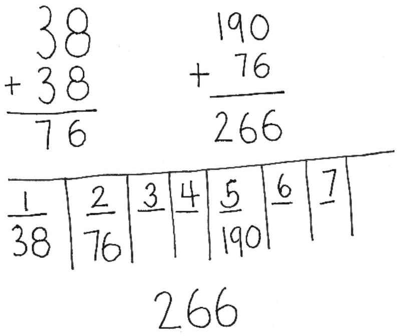


Figure 19 – Haley's Partitioning Single Factor into Non-Decade Numbers Strategy

Partitioning Both Multiplier and Multiplicand

Emma's strategy for solving 325 x 41 (Figure 20) is to decompose both the multiplier (number of groups: 325) and the multiplicand (number within each group: 41) into its place value components. [325 x 41 = (300+20+5) x (40 + 1)] then uses the distributive property of multiplication over addition to determine the partial products. She adds them all together to find the total product. Her work is largely internalized. This strategy using the distributive property is the same underlying mathematics of the multiplication standard algorithm. The difference is she starts with the larger elements to get closer to her final answer and works in values rather than in single digits.

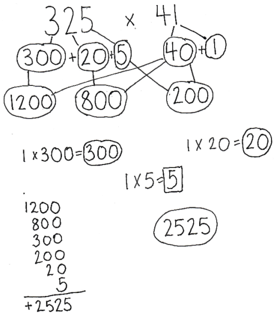
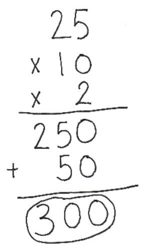


Figure 20 – Emma's Partitioning of Both Multiplier and Multiplicand

The numbers within a problem can influence a student's decision making as to how far the decomposition process needs to go. Jamie's strategy of solving 12 x 25 (Figure 21) has him only decomposing the multiplier (12) into 10 + 2. He multiplies 10 x 25, 2 x 25. The strategy is based on the distributive property of multiplication over addition. The numbers are just right (a power of ten and doubling) to save himself time and be more efficient. Emma in her work could benefit from more explicit conversation around the identity property of multiplication to realize that she could be more efficient in multiplying 325 x 1. Nurturing students' ability *to think relationally develops the capacity to take a step back before beginning to operate* to look at the expressions and equations in their entirety *to notice number relations among and within* them. Relational thinking *entails a flexible approach to calculation* in which the expressions are transformed or decisions about where to strategically begin are made up front before beginning to calculate. Such decisions are *based at least implicitly on the use of the properties of operations*.[18]



25
x 10
x 2
250
\+ 50
300

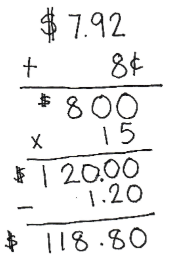
Figure 21 – Jamie's Partitioning of Either the Multiplier or Multiplicand

This is a case for how the classroom conversations around being efficient in how numbers are decomposed is beneficial to students as they progress across the school year and across grade levels. A teacher needs to determine whether or not a student thinks it's necessary to decompose 325 into 300 + 20 + 5 when multiplying by 1 or if one can look ahead to anticipate using the identity element in multiplication and be more efficient. A conceptual understanding of multiplication and the properties of operations supports such procedural efficiencies. The classic approach to the multiplication standard algorithm with its emphasis on procedural steps reinforces the student to multiply each partial product individually.

Compensation Strategy

A pair of socks cost $7.92. If the store that day sells 15 pairs of socks, how much money would the store earn? Celeste's strategy involves rounding the $7.92 to $8.00 as that is more efficient with which to work. (Figure 22.) She does so by adding 8¢ to the cost of each pair. She multiplies the $8.00 by 15 and then subtracts fifteen 8¢ from that total to find the exact total earned. Written out horizontally [15 x $7.92 = 15 x ($8.00 – $.08)] one can see how Celeste, at least at an intuitive level, understands the distributive property of multiplication over subtraction allows her to use this strategy to make her work in finding a solution more efficient.

Having students explore how this strategy works in public discussions is beneficial. If students already use compensation strategies in addition and subtraction, then connecting how the strategy works in multiplication is an easier exploration. If students have not explored compensation strategies previously, more time will need to be devoted to the mathematical underpinnings of how it works.



$ 7.92
\+ 8¢
$ 800
x 15
$ 120.00
− 1.20
$ 118.80

Figure 22 – Celeste's use of the Compensation Strategy

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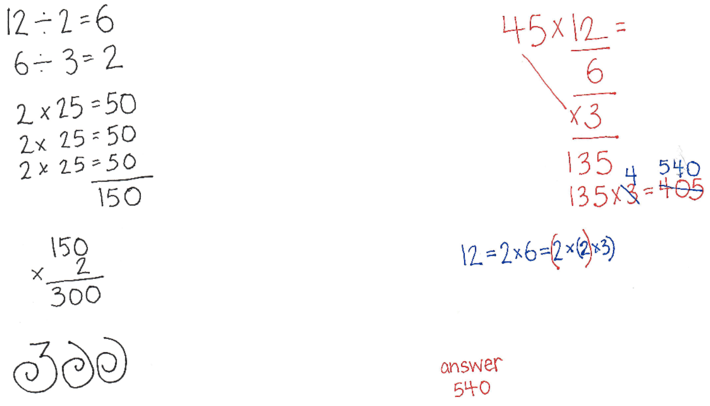
[18] Jacobs, V. R., Franke, M. L., Carpenter, T. P., Levi, L., & Battey, D. (2007). Professional development focused on children's algebraic reasoning in elementary school. *Journal for Research in Mathematics Education*, *38*(3), 258-288.

Factoring Strategy

The distributive property of multiplication over either addition or subtraction is structured on decomposing numbers in their *addends.* The two strategies presented below (Figure 23a&b) show how some students decompose numbers into *factors* in order to determine the product. Exploring decomposing numbers into factors arises through classroom conversations around strategies for single-digit multiplication combinations (Footnote 15) as well as in explicitly working with factors of ten when multiplying mathematical expressions such as 3 x 40 or 30 x 40. Sophie's solution to solving 25 boxes with 12 cookies inside each package, how many cookies in all is by factoring 12. First she divides 12 by 6 (12 = 6 x 2), then factors the 6 by dividing by 3. (12 = 2 x 3 x 2). She then calculates 2 twenty-fives three times, then doubles that amount to find the answer. (12 x 25 = ((2x25)x3) x2).



[a] Sophie [b] Chase

Figure 23a-b – Sophie and Chase's Use of the Factoring Strategy

Chase uses the same strategy to solve the same problem but self-selecting to work with 45 boxes of 12. Chase's initial work is in red ink. He factors 12 into 6 x 2 and 6 into 3 x 2. He then triples 45 to get 135. His next step results in an error where he loses track of the factors of 6 and slips into 6 being 3 + 3. He triples the 135 to get 405. Another student in the class challenges the answer which leads to retracing the factors (blue ink) to make the decomposition explicit. This allowed the final product of 540 to be arrived at.

The discussion working through Chase's error allowed both Chase and those in the class able to follow along to process how factoring as a decomposition option functions as well as conveying that, when exploring new strategies, errors are not unusual. Processing errors constructively and in full has been linked to student achievement gains.[19] Processing errors allows students to sort out the difference between which worthwhile mathematical ideas are valid and at which points new understanding needs to be worked through. Productive struggle is the byproduct of such episodes as students gain confidence in working through ideas through to appropriate endpoints.

Area Model

While the area model only shows up in students' work if its formatting has been formally introduced, the area model harkens back to its original formulation in 9th century C.E. Islamic mathematical roots. The concept of the distributive property of multiplication over addition was articulated by Muhammad ibn Musa al-Khwārizmī who is the progenitor of algebra. The partitioning of both the multiplier and multiplicand

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[19] Webb, N.M., Franke, M.L., Ing, M., Wong, J., Fernandez, C.H., Shin, N., and Turrou, A.C. (2014). Engaging with others' mathematical ideas: Interrelationships among student participation, teachers' instructional practices, and learning, *International Journal of Educational Research, 63,* 79-93.

strategy noted previously (Figures 20 and 21) is the area model, just not formatted in the same way. Example: 54 x 38. The 54 is decomposed into its place value components of 50 + 4; likewise the 38 into 30 + 8. The partial products are calculated and then combined to find the total. Figure 24 reflects the visual layout of the work. Part VII: Place Value as a Rate of Ten explores how to explicitly use the factors of ten to fluidly calculate the partial products with understanding.

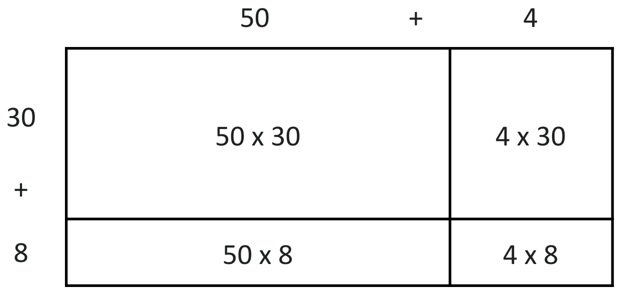

Figure 24 – Area Model Format for 54 x 38

Interpreting Frameworks – A Caution

There are some researchers who have created a trajectory for multiplication strategies who place the area model at less of a full abstract strategy.[20] While the distinction is not fully clear, I suspect it is because the graphical layout serves as a visual organizer. The layout serves as a bridge towards just working at the number level alone. Whether or not a student is dependent upon the use of the graphical layout or not, *the important assessment is how the student is processing the mathematical calculations.* The caution is not to focus on the format of a student's work. *The cognitive processing is key*. One can use the standard algorithm format and be very additive in one's processing. A student can use the area model as a visual anchor but is processing the calculations mentally at a very abstract level. The focus needs to be on are they decomposing numbers into addends or factors? Which properties of operations is the work based upon? Are the students working with values or single digits? These are the indicators by which to assess whether or not a student is working additively or multiplicatively. While Table 1 in this document does outline progressions in strategies, the emphasis of the attributes in the table focuses on *the progressive abstraction as students shift cognitively* to make units of units, coordinate units simultaneously, monitor the transformations of the units and shift to thinking in scale as elements covary with the operation.

Multiplication Standard Algorithm

The standard multiplication algorithm is based on the distributive property of multiplication over addition; the same mathematics articulated by al-Khwārizmī in 9th century C.E.[21] The mathematical underpinnings are sound ones. The limitation of the algorithm is that it limits students to the procedural need to strip out the place value factors of ten, the requirement to start in one's place, the use of single-digit language and placement of digits, and is only presented in a vertical format. The procedure moves digit-by-digit eliminating the relational thinking that Emma in Figure 20 could have benefited from in using the 1 x 325 or even the efficient decision made by Jamie in Figure 21 when he only decomposed the 12 into 10 + 2 in order to multiply 12 x 25. The limitations in the standard algorithm is that it obscures the algebraic underpinnings of the mathematics. If a student learns the procedure and uses it effectively, the algebraic underpinnings may not be realized or comprehended by that individual. The procedure is a rote one only. There are those

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[20] Vermont Mathematics Partnership Ongoing Assessment Project (OGAP). (2017). OGAP Multiplicative Framework. Retrieved: https://ogapmathllc.com/wp-content/uploads/2017/01/OGAP-Multiplicative-Framework-Color-7.7.20127v.2pdf.pdf

[21] Lattice multiplication is an older 9th century version of the standard multiplication algorithm. Lattice uses a condensed digitized version of the area model where the current standard multiplication algorithm uses a digitized placement in vertical formatting.

who advocate for the algorithm's use in that it reduces the cognitive load for the student in order to swiftly do the calculation.[22] The question remains when, if at all, should the standard algorithm be explored over the trajectory of a student's education. This topic will be further explored in Part IX: Putting it All Together.

Interim Summary VI

Students' initial single-digit multiplication strategies emerge from their direct modeling and counting strategies. However, their multi-digit multiplication strategies emerge from their flexible understanding and use of multi-digit addition and subtraction strategies. The capacity to partition numbers to efficiently multiply larger numbers emerges from the single-digit multiplication deriving strategies that they have developed. As students transition away from the more additive solution strategies, their work becomes more internalized. Their thinking becomes more anticipatory. The choices made in how to partition/ decompose numbers becomes more strategic, efficient, as they begin to coordinate units, work with composite units, and begin to think in scale. Students may fluctuate in the sophistication of their strategies depending upon the number range with which they are working. It is important to look at students' work over a range of problems to develop an accurate profile of their capacity. Assessment focus of any student's progress should consider the ability to make units of units, coordination of units, and to think in ever increasing units of scale.

Reflection Point 5

- ☐ Return to Table 1 on pages 17-18. Review the listed attributes that distinguish additive from multiplicative strategies.
- ☐ Use this link to analyze the work of two students
- ☐ Solve 24 x 36 first using an additive strategy.
- ☐ Solve 24 x 36 using the distributive property over addition
- ☐ Solve 24 x 36 by factoring and use of the associative property
    - ☐ Before you start, think relationally first by taking a big picture view of the quantities to determine the most strategic point of entry might be and where efficient steps might be undertaken.
    - ☐ As you work, speak in values, think about other properties of operations that may be tapped as you solve it, and think about where your need to work visually aids in your ability to track necessary pieces of information.

Part VII – Place Value as a Rate of Ten

Within and Across Place

We operate within a base ten system. A fourth grade Common Core State Standard reads, *Recognize that in a multi-digit whole number, a digit in one place represents ten times what it represents in the place to its right.* (4. NBT.1) And, at fifth grade this additional phrase is included, *Recognize that in a multi-digit number,... is 1/10 of what it represents in the place to its left.* (5.NBT.1) Those two statements succinctly capture the essence of place value at its core. Place value is a multiplicative rate of ten. Yes, there is a placeholder pattern in how the digits are manipulated and positions named. While this is important, it reflects the surface elements of place value only and fails to convey the rate of ten structure underlying those pattern features.

Look at the following number.

783

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[22] Fuson, K.C., Kiebler, S., & Decker, R. (2024). Accessible standard algorithms. *Mathematics Teacher: Learning and Teaching PK-12, 117* (4), 268-275.

Answer the following three questions.

- How many tens are in the number 783?
- How many tens are in the tens place?
- What is the difference between these two questions?

Both of the first two questions involve place value and base ten ideas. However, the first question is frequently never asked or is woefully underdeveloped. Many curricular materials and assessments think that if a student successfully answers the series of questions naming the digit by its place, its location, that the student understands place value. Yet when eventually asked the first question, students are stymied and can't answer it.

From a place value perspective, students need to see 783 flexibly as…

700 + 80 + 3
7 x 100 + 8 x 10 + 3 x 1
783 x 1
78.3 x 10
7.83 x 100

Unitizing/Re-unitizing, meaning cognitively reassigning a quantity from one unit of measure to another equivalent unit of measure, is an integral aspect of a robust understanding of place value[23]. It's a level of abstraction that is involved with the ability to hold the original and the newly assigned unit simultaneously in one's mind. Place value is built on the same conceptual attributes of multiplication.[24] It requires the making of *units of units* – For every ten ones there is one ten. There are 10 tens in 100 as well as 100 ones. *Units transform* as one re-unitizes from one unit of measure to another – Are you thinking in tens or in ones? This requires the *simultaneous thinking* necessary to *coordinate the units* as one re-unitizies across places. As noted in the Common Core math standards, *scaling up or down by a rate of ten* needs to be developed and understood. There are key places in operating on numbers where one needs to re-unitize a quantity across place. Examples include in graphing or in technical reports where instead of writing out 1,250,000,000 one writes 1.25 billion. Instead of working with 1.25 ones, one works with 125 hundredths to make it easier to compute. These places and corresponding unit transformations are often obscured with surface tricks or digitized language.

Reading Multi-digit Numbers

When we read multi-digit numbers beyond a thousand, the language used in English inherently reads across place. Consider the number 12,783. There is not a unique name to the leftmost place in the number. Its name is in reference to how many one thousands there are. There are 12 thousands. An alternative "place value grid" to use in classrooms that supports such recognition would look like the following to be able to read the number 136,459,783. Notice there one hundred thirty-six *million,* four hundred fifty-nine *thousand,* seven hundred eighty-three *ones.* The 'ones'/'units' are assumed in the language, just like in many other contexts in mathematics.

1	**3**	**6**	**4**	**5**	**9**	**7**	**8**	**3**
hundreds	tens	ones	hundreds	tens	ones	hundreds	tens	ones
millions			**thousands**			**ones/units**		

---

[23] Lamon, S.J. (1996). The development of unitizing: It's role in children's partitioning strategies. *Journal for Research in Mathematics Education. 27* (2), 170-193.
Lamon, S.J. (1994). Ratio and proportion: Cognitive foundation in unitizing and norming. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics. (pp. 89-122). Albany, NY: State University of New York Press.*

[24] Brickwedde, J. (2019). Place value as a rate of ten. *Teaching Children Mathematics*, *25* (1), 30-35.

Figure 25 – Alternative Place Value Chart

Multiplication & Measurement Division problem contexts to develop place value understanding

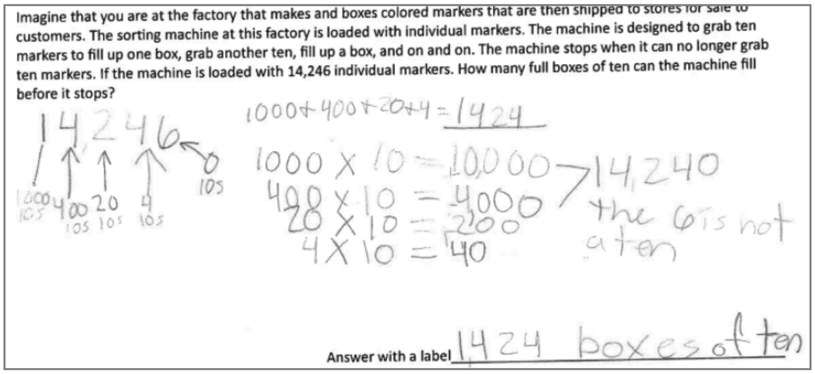
The following explores how to develop a more robust understanding of place value using multiplication and measurement division contexts.

Let’s return to the two contexts used to demonstrate emerging multiplication and division strategies found in Part III & IV of this piece.

> Multiplication Context: *The head soccer coach has 8 bags of soccer balls for all of the teams to use in the league. There are 10 balls inside each bag. He also has one bag with only 2 balls in it. How many soccer balls does he have for all of the teams?*
>
> Measurement Division Context: *The class basket of markers has 82 markers in it. At the end of the year, the teacher wants to place the markers back into boxes. If 10 markers fit in a box, how many full boxes will she have?*

The number choices in these two problems have ten as the organizing unit. As you review the strategies described in that earlier section, a direct modeler or a counter comes to understand that 82 is made up of 8 tens and 2 ones. Using multiplication and measurement division beginning in first grade with 10 as an organizing unit is a foundational means of building place value as a rate of ten concept.

Look at this fifth grader’s explanation for how many full boxes of ten colored markers can be filled by the sorting machine at the factory. [From the multi-step rich task *At the Factory: Making & Boxing Colored Markers*.] Notice the decomposition of number and the explicit use of the factors of ten needed to justify with evidence how the student reasoned through their thinking. The task could have been solved a lot quicker if the student had just been told “the trick” of covering up the ones’ place. However, demonstrating the trick and having the student practice just that surface pattern would have diminished the student’s underlying mathematical understanding. These tricks are like a house of cards that easily collapses when more sophisticated ways of thinking are necessary in middle school.



Imagine that you are at the factory that makes and boxes colored markers that are then shipped to stores for sale to customers. The sorting machine at this factory is loaded with individual markers. The machine is designed to grab ten markers to fill up one box, grab another ten, fill up a box, and on and on. The machine stops when it can no longer grab ten markers. If the machine is loaded with 14,246 individual markers. How many full boxes of ten can the machine fill before it stops?

1000+400+20+4 = 1424

14,246

1000 10s 400 10s 20 10s 4 10s 0 10s

1000 x 10 = 10,000

400 x 10 = 4,000

20 x 10 = 200

4 x 10 = 40

14,240

the 6 is not a ten

Answer with a label 1424 boxes of ten

Figure 26 – A Fifth Grader’s Justification Solving a Place Value Task

Explicitly Working With Factors of Ten

Kindergarteners through second graders can and should be exposed to multiplication and both forms of division as they can easily model the operations to solve such tasks. Using multiplication and measurement division to develop place value ideas as noted previously is a key addition to curricular materials at these grades. For additional in depth readings about developing place value ideas, read these articles. It is in third grade that multiplication and division become formal standards for learning and teaching. Strategies to work with single-digit combinations can be explored in the related work to this piece: Learning Single-digit Combinations: Developing Important Mathematical Ideas. What is explored here is what happens when a student encounters a mathematical expression such 3 x 40 for the first time.

3 x 40

A typical response from a student (and adults due to the history of how we were all taught) is to explain the answer based on a surface pattern.

I know 3 x 4 is 12 then I added a zero. The answer is 120.

To which I typically respond knowing my students and they knowing me…

But I thought 12 + 0 is still 12. Isn't a number plus zero still the number you started with?

[Student typically responding]: *But it's because I put the zero <u>there</u> that makes it 120.*

At this point, I steer the conversation to the mathematics behind the surface pattern that the student is attempting to articulate but that doesn't yet have the language to describe.

> *I know that knowing 3 x 4 makes determining a product easier. That's using what you know to figure out what you don't know. But let's look at how we need to mathematically break 40 apart to isolate the 4 in the forty. Forty is four what?*
>
> [Student] *Four tens*
>
> *Four tens. So, if we want to isolate the four in forty, mathematically we need to decompose the 40 into 4 x 10. So, our problem is changed into 3 x 4 x 10. Is this where the 3 x 4 comes from?*
>
> [Teacher transcription on the board capturing group conversation]
>
> 3 x 40
> / \\
> 3 x 4 x 10
>
> [Student] *Yes!*
>
> *Class, we call breaking a number such as 40 into 4 x 10 as decomposing into 'factors'. Many times we decompose numbers into 'addends' but we can also break numbers apart into their 'factors.' That's mathematically what is actually happening here. So, [student], what mathematically is left to do to find the total? We have multiplied the 3 x 4. What do we have left to do?*
>
> [Continuing written transcription capturing the group conversation]
>
> 3 x 40
> / \\
> (3 x 4) x 10
> \\ /
> 12

[Student] *Multiply by 10.*

[Continuing written transcription capturing the group conversation]

3 x 40
/ \
(3 x 4) x 10
\ /
12 x 10

So what would 12 tens total?

[Student] *Well, I know 10 tens is one hundred, and two tens is twenty, and 100 plus 20 is 120.*

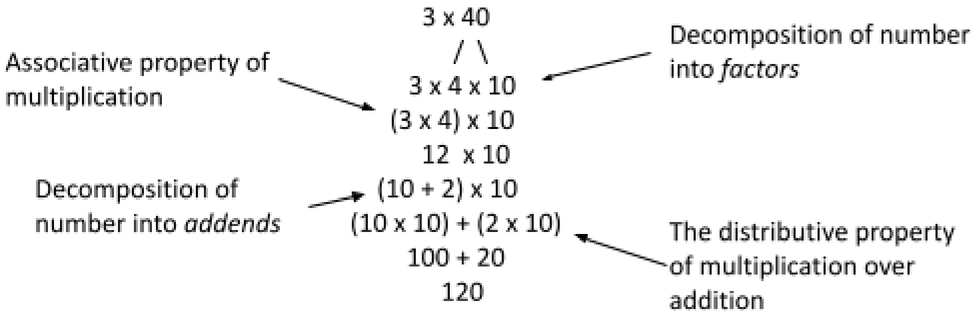
So, that's where the total of 120 comes from?

[Student] *Yes!*

[Continuing written transcription capturing the group conversation]

3 x 40
/ \
(3 x 4) x 10
\ /
12 x 10
10 x 10 + 2 x 10
100 + 20
120

Class, as we begin to work with numbers like 3 x 40, I want you to be able to explain how you are breaking the numbers apart and why the pattern the [student] noticed actually works. This is a "watch your language[25]" moment. You are not "adding a zero to the 12." You are multiplying the 12 by the remaining factor of ten. We decomposed the 40 into the factors of 4 x 10. But here, [pointing to the 12 x 10] *the 12 was decomposed into addends in order to find the total. As we move more and more into working with multi-digit numbers, we want to explore further when we need to work with a number's addends or its factors. Let's keep thinking about this further!*



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[25] "*Watch your language*" is a "sound bite" that I use with students to direct their mathematical language to speak in value rather than in digits. More will be explained about such teacher prompts in the Building Capacity section.

Figure 27 – Final visual transcription of scaffolded conversation

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Figure 27 captures the algebraic properties of operations that are at least implicitly being used in solving the expression. As conversations like this unfold, it is important to make explicit the operations so that in future tasks students can be intentional in their decision making when solving them.

The above vignette demonstrates how to cognitively guide the conversation in order to draw out *the mathematics of why a pattern actually works* rather than relying on the surface pattern alone. Guiding students to *explicitly work with the factors of ten* empowers them mathematically to think more multiplicatively as well as positions them in future grades to work with more complex number combinations from a place value perspective.

Consider the following mathematical expression.

$$2.6 \text{ x } 5.3$$

These numbers are where tricks and surface patterns are classically relied upon rather than exploring the actual mathematics underlying the operation, e.g., *"ignore the decimals points and pretend they are whole numbers."* A more mathematically sound approach is when the capacity to re-unitize *across place* is called upon. Two and sixth tenths ones (2.6) is equal to/the same as twenty-six tenths ($\frac{26}{10}$). Five and three-tenths ones (5.3) is equal to/the same as fifty-three tenths ($\frac{53}{10}$). Re-unitized, the place value factors of ten are now explicit in the following solution.

$$2.6 \text{ x } 5.3 = \frac{26}{10}\ x\ \frac{53}{10} = \frac{26\ x\ 53}{10\ x\ 10} = \frac{1378}{100} = 13.78$$

In executing that calculation, a student needed the capacity to re-unitize (from the unit of ones to tenths), to multiply using decomposition of number into both addends, e.g. 26 = 20 + 6, and factors, e.g. 20 x 50 = 2 x 10 x 5 x 10, in order to use the distributive, associative, and commutative properties. The factors of ten were explicitly used ($\frac{1}{10}\ x\ \frac{1}{10}$). The final point is when the student is faced with dividing 1378 by 100, $\frac{1378}{100}$. This is the core measurement division question drawing out place value understanding: *How many 100s are in the number 1378?* There are 13 whole hundreds in that number as well as seventy-eight hundredths of the next hundred. That's why the pattern and trick works the way it does. Students who explicitly know and have the fluid capacity to calculate with this level of understanding have a more secure foundation to move forward mathematically in middle and high school.

Is it easier and quicker to teach the trick and surface pattern? Absolutely! However, it leaves students impoverished in their mathematical understanding and leaves them vulnerable to failure. Better to take the time to build a house of bricks than a house of straw.

Part VIII - Decimals, the Distributive Property, & Relational Thinking

Extending the distributive property of multiplication over addition

In the previous section, solving 2.6 x 5.3 was solved by re-unitizing the numbers from the ones/units' place to the tenths' place in order to explicitly work with the factors of ten. This is the mathematics behind the 'trick' that most of us were taught in order to multiply such numbers. But that is not the only strategy that can be used. The distributive property of multiplication over addition or subtraction can be extended to such tasks. Number size may determine if this property of operation is more efficient than the prior strategy.

Knowing more than one strategy allows students to make that strategic decision upfront and not be limited to knowing only one way.

If asked to multiply 26 x 53, decomposing the two quantities into their place value components is likely automatic as in (20 + 6) x (50 + 3). Extend that same property to the quantities of 2.6 x 5.3. Multiplying the partial products is fairly simple. Two times five is easy. Two groups of three-tenths is not complicated. Five groups of six-tenths results in thirty-tenths which can be re-unitized as equivalent to three ones. (Note the implicit use of the commutative property of multiplication from sixth-tenths groups of five to five groups of sixth-tenths to make the calculation easier.) That leaves asking what is six-tenths of three-tenths which typically stops everyone in their tracks.

$$\begin{aligned} 2.6 \text{ x } 5.3 &= (2 + .6) \text{ x } (5 + .3) \\ &= (2 \text{ x } 5) + (2 \text{ x } .3) + (5 \text{ x } .6) + (.6 \text{ x } .3) \\ &= \quad 10 \quad + \quad .6 \quad + \quad 3.0 \quad + \quad ? \end{aligned}$$

There has been some very interesting work in elementary mathematics research around strongly emphasizing the unit fraction/decimal when developing an understanding of the relation quantity being described.[26] Example: three-fourths [$\frac{3}{4}$] is described as three, one-fourth pieces. This leads to later middle school work in rational numbers where three one-fourth pieces would be numerically expressed as 3 x $\frac{1}{4}$. That same focus on the unit fraction extends to working with decimals and relates directly to the previous section's focus on working explicitly with factors of ten.

$$\begin{aligned} .6 \text{ x } .3 \quad &= (6 \text{ x } .1) \text{ x } (3 \text{ x } .1) \\ &= 6 \text{ x } 3 \text{ x } .1 \text{ x } .1 \\ &= \quad 18 \text{ x } \quad .01 \end{aligned}$$

Factoring

Commutative & Associative Properties

18 x .01 = 18/100; place value understanding

Developing a visual and relational understanding of what a tenth of a tenth is [.1 x .1 or $\frac{1}{10} x \frac{1}{10}$] is important if the sense of the scale factor within place value is to be truly robust. Imagine having a set of one-tenth pieces and all of those were split into ten (an act of multiplication). Then ask yourself how much of the whole is one of those pieces (an act of division).[27] Thus .1 x .1 = .01 x 1. Spatially developing this relationship can anchor students in comprehending the relational sense of size as well as, in this instance, how a number to the right is one-tenth the size.

Interim Summary VII

Place value has more to do than naming a digit's value by its location. It is about comprehending the multiplicative relationship that *a digit in one place represents ten times what it represents in the place to its right and 1/10 of what it represents in the place to its left.* As a multiplicative rate of ten, place value requires an individual to be able to make units of units, note how units transform across place as well as within place (re-unitizing), coordinate units simultaneously, and be able to think in scale. These are the exact elements of multiplication as an operation. To build a robust understanding of place value as a multiplicative rate of ten, *the language of value* guides the neural images formed in the brain, the option of decomposing numbers

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[26] Empson, Susan & Jacobs, Victoria & Jessup, Naomi & Hewitt, Amy & Pynes, D'Anna & Krause, Gladys. (2020). Unit fractions as superheroes for instruction. *Mathematics Teacher: Learning and Teaching PK-12, 113*. 278-286. 10.5951/MTLT.2018.0024.

[27] Confrey, Jere & Maloney, A. & Nguyen, K. & Mojica, G. & Myers, Marrielle. (2009). Equipartitioning/splitting as a foundation of rational number reasoning using learning trajectories. *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education. 2.* 345-352.

into either addends or factors needs to be understood, and explicitly understanding the mathematics underlying the surface patterns need to be developed. This includes going beyond "counting zeros," "covering digits," and "pretending decimal points are not there" to get answers. It means explicitly working with the factors of ten to develop a robust comprehension of how numbers scale up to higher or lower magnitudes. It provides the generative foundation for understanding and explicitly tapping into the algebraic properties of operations and builds towards the multiplicative reasoning needed to think about rates, ratios, rational numbers and proportions in middle school mathematics and beyond.

Reflection Point 6

- ☐ Solve the third-grade multi-step rich task *At the Bakery: Peanut Butter Sandwich Crackers*. As you work through the task,
    - ☐ Notice how the multi-step multiplication and addition and the measurement division problem types are used to develop place value ideas.
    - ☐ Notice how attention to and coordination of units comes into play to make sure the correct selection of numbers are aligned with the correct set of units to find the totals asked for.
- ☐ How has your understanding of place value shifted in both reading this section as well as from solving the peanut butter sandwich cracker rich-task?
- ☐ Where and how frequently will you need to infuse such problem types and questions into the scope and sequence of your grade level?
- ☐ Work slowly through the slide deck of a Fraction of a Fraction. How does your understanding of the one expression, e.g. $\frac{1}{10} of\ a \frac{1}{10}$ piece, relate to relationally re-unitizing that same area in the grid to the whole grid, e.g. $\frac{1}{100}\ x\ 1$, or the whole?
    - ☐ Stretch yourself to now consider the same splitting and sharing visual activity but with odd number fractions, e.g. what a $\frac{.5}{5}$ (half of a one-fifth piece) is the same as how much of a whole piece.

Part IX – Division & Division Strategies

Problem Structure

As noted in Part II, there are two problem structures in division depending upon the known information.

	# of groups (jars)	**# in each group (jelly beans per jar)**	**Total in all groups (jelly beans in all jars)**
Multiplication	3	20	?
	I have 3 jars of jelly beans with 20 jelly beans inside each of the jars.. How many jelly beans will there be if I pour out all of the jelly beans into one bowl?		
Measurement Division	?	20	60
	I have 60 jelly beans in a bowl. If I put 20 jelly beans into a jar, how many jars will I need for all of my jelly beans?		
Partitive Division	3	?	60
	I have 60 jelly beans that I want to put into 3 jars. If each jar gets the same number of jelly beans, how many will end up inside each jar?		

In *measurement division, you know the number of items within each group*. You initially just don't know the number of groups that are possible. In *partitive division* – also known as fair sharing or equal sharing – *you know the number of groups you are starting with.* You just don't know how many items will be inside each group *if all the groups get the same amount.* Notice that last qualifier. All groups must end up with an equal amount. If such a statement is not declared, then a distribution between groups can be made; it's just they would not be of equal amounts.

Early Division Strategies

Direct Modeling Strategy Descriptions

At the direct modeling level, how a child uses manipulatives to solve the two division problems is distinctly different. In the scenario above, in a measurement division context the student would get out 60 cubes. The students would then begin to count off 20 cubes (That's one jar), another 20 cubes (That's two jars), and another 20 (That's three jars) until all the cubes are gone. Then the piles would be counted to determine that three jars are needed for all of the jelly beans. A counter would say 20, 40, 60 extending one finger up on a hand to keep track of the number of jars. Looking at the fingers at the end of the count, the answer would be three.

In contrast, partitive division feels much more like dealing out cards in a card game. While a trial and error distribution process is a typical response, many direct modelers would count out the sixty cubes and then start dealing into three groups as that is the known number of jars. One for you. One for you. One for me. The dealing would stop once all the cubes are distributed. Given the size of the number of jelly beans, a modeler with a stronger number sense might give out five or ten jelly beans right away as that would not use up all of the jelly beans but be more efficient than dealing the cubes out one-by-one. The remaining cubes might then be dealt out singly. A counter in a partitive division context has a harder time in that there is no clear countable cluster to initially use. As a result, a trial and error process is used to arrive at the exact amount.

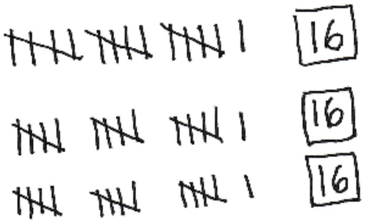


Figure 28 – Direct Modeling: Partitive Division Dealing Strategy– Split 48 into 3 equal groups

Skip Counting & Derived Strategies

Skip counting can be used with measurement division tasks as what is inside each group is the countable unit. With small quantities, students skip count starting from zero by the multiplier. Example: How many plates are needed if 3 cookies are placed on a plate and there are 48 cookies to put on the plates? [48 ÷ 3; or ? x 3 = 48]. A student may skip count, starting from zero, producing one finger or marking one tally to keep track of the unit. 0, 3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36, 39, 42, 45, 48. The answer is 16 groups of three. If a student has some combinations of three at a recall level, Saying, *I know 10 threes are 30,* the skip count can move forward from there with six more counts.

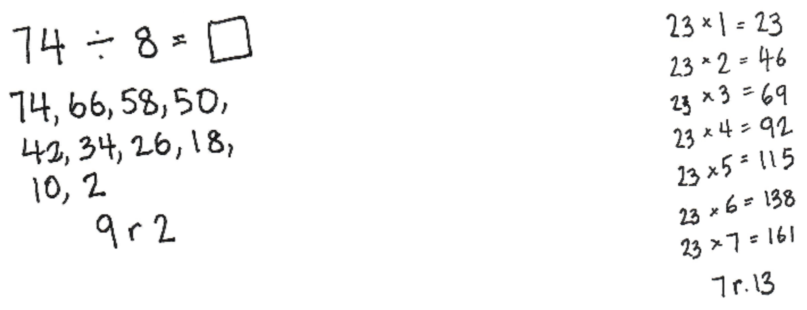
If the student reflects on what remains to be counted after 30 [30 + ? = 48 or 48 - 30 = ?] and knows that 6 threes are 18, then the strategy becomes fully derived using, at least implicitly, the distributive property of multiplication over addition. A focus on nurturing derived strategies with students with single-digit and lower quantities of single-digit times a multi-digit number builds the algebraic and relational thinking needed to move forward with multiplying and dividing ever larger quantities. (Read further about supporting the Learning Single-digit Combinations[28])

Caution: Skip counting is a natural, early developmental entry point for students to use with multiplication and division. As explored earlier in Part III of this document, skip counting demonstrates an early ability to coordinate units as each finger raised to track the count represents one group as well as the subunits of the count. Many teachers drill students in skip counting so that they can quickly recall the multiples of a number. While I cannot quote specific research studies on this issue, *it is conjectured by me from my experience working with students over the years that the more drilling students have experienced with skip counting, the harder it is to shift them to derived strategies which are more efficient and based on the algebraic properties of operations.* An intentional instructional focus on developing derived strategies in addition, subtraction, multiplication, and division helps students develop the same core strategies needed to operate with multi-digit numbers. The derived strategies also segue into the 'Building Up/Multiplying to Divide Strategies'.

Repeated Subtraction/Counting Back – Measurement Division

Division is often described as repeated subtraction. Like repeated addition in multiplication, these are not accurate statements. Repeated subtraction is an accessible entry level, developmentally appropriate solution strategy students initially use. But like its inverse operation multiplication, division is more complex and requires more abstract multiplicative levels of thinking.

Repeated subtraction is a strategy that is unique to the measurement division context; as in *how many sixes are in ninety-four.* Students approach partitive division using different number sense and strategies. Deshawn's strategy (Figure 29a) is to systematically subtract one six at a time, recording the count as he goes. His work demonstrates how it goes out of the counting back strategies with subtraction. The student work in Figure 29b demonstrates how an iterative building up count can be used to answer how many 23s are in 174. Without the student explaining how each step is calculated, the artifact by itself demonstrates the overall objective of adding one 23 at a time until one gets as close as possible to the target amount.



[a] [b]

Figure 29a-b – Repeated Subtraction/Counting Back & Iterative Building Up Strategies

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[28] Brickwedde, 2022. Learning single-digit combinations: Developing important mathematical ideas. Project for Elementary Mathematics. https://www.projectmath.net/project-services/p-d-materials/professional-development-articles/basic-facts/

Partitive Division

Partitive division is about sharing equally among a determined number of groups. At the direct modeling level, both a planned and trial and error distribution strategies are used by students. Planned distribution is like dealing cards in a card game; one for you, one for you, etc. Random distribution captures a level of number sense that a student thinks might be able to be given to everyone and then adjusted until all are equal. At the counting level, even skip counting is typically a trial and error process, as in "let me try threes. No, that didn't work. Let me try fours…"

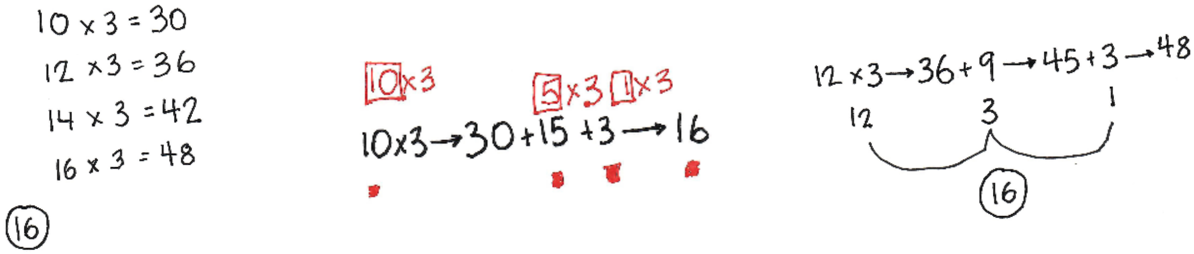
Moving beyond distributing one item at a time requires a number sense of relating the total items at hand and the number of groups the items need to be sorted into. Example, how many plates are needed to sort 52 cookies equally? As a student's number sense grows, and the individual recognizes sorting one-by-one is too tedious, starting with a larger known quantity first – say, 5 or 10 for each plate – allows an efficiency to be used in saving the amount of work needed to determine the result. Reflecting on what is still left to sort, allows the student to decide how best to continue the portioning out. In the case of 52 cookies placed in equal amounts on four plates, one could first pass out 10. That uses up 40 of the cookies. If I determine that 12 are left, and I know already how 12 can be distributed equally into 4 parts, the 3 more cookies can go on the plate. The first 10 plus remaining 3 cookies results in 13 cookies being on each of the plates. Such reasoning quickly moves from the earlier additive stages to this more abstract derived strategy level as the number sense is strong enough to navigate the distribution.

Abstract Strategies

Building Up/Multiplying to Divide Strategies

At the abstract level, as students bring their own sense making to measurement division contexts in particular, students initially *build up using multiplication*[29] to get to the total rather than using *division/ subtract back to/repeated subtraction* strategies. Figure 29b represents an iterative counting up strategy to determine how many twenty-threes are in 174 [174 ÷ 23].

Look at these reconstructed fourth grade partitive division work samples from early in the year. The prompt from the teacher was, *I know you could pass things out one-by-one, but what is the largest amount that you can safely start with in order to save yourself some time?* This prompt was designed to move students off an inclination to skip count their way up to determine the solution and use more derived number sense.



Devan: 48 ÷ 3
3 x ? = 48
[a]

Milda: 48 ÷ 3
3 x ? = 48
[b]

Chris: 48 ÷ 3
3 x ? = 48
[c]

Figures 30a-c – Student work samples solving 48 ÷ 3

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[29] Ambrose, R., Baek, J., Carpenter, T. P. 2003. Children's invention of multiplication and division algorithms. In A.J. Baroody, & A. Dowker (Eds.) *The Development of Arithmetic Concepts and Skills: Recent Research and Theory*. Mahwah, NJ: Erlbaum.

Devan articulates that he knew 10 threes would use up 30. He then moves up 3 groups of three in succession to get ever closer to using up all of the 48. He verbalized that he knew 11 x 3 = 33 and that 12 x 3 would just need to add three more. For 14 x 3, he used 12 x 3 plus six more. He did the same to determine 16 x 3.

Milda also starts with 10 x 3 = 30 but then does half as much for the next step to add 15. In her explanation to the class, she clarifies (the red notations) that the 15 comes from her knowing that 5 x 3 = 15. Knowing that she is not at 45 (30 + 15), she just needed to add one more group of three (1 x 3) to get to the total of 48.

Chris starts with 12 groups of 3, plus 3 groups of three for 9 more to get to 45 and then adds one more group of three to get to 48. He knows that the 12, the 3, and the 1 represent the groups. He adds them together, as the other two students had, to get the answer of 16.

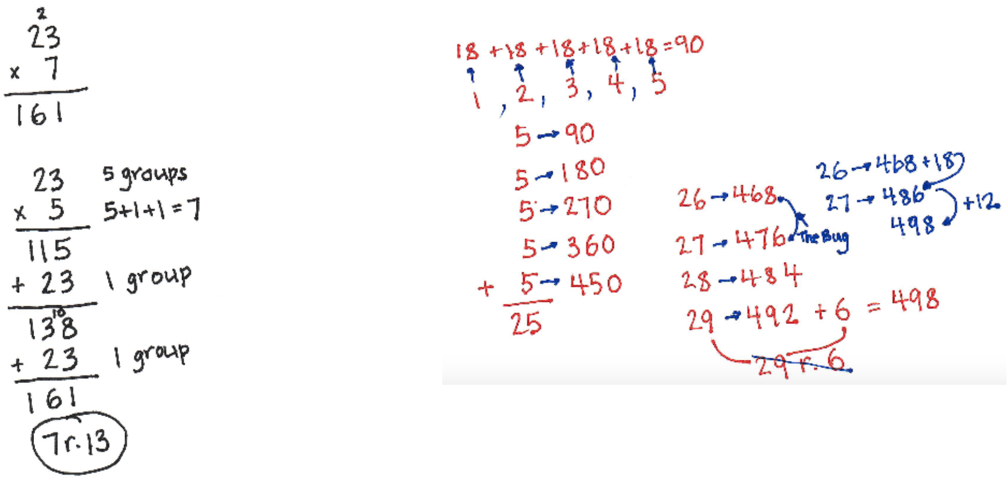
In each case, the student built up to 48. They just started with a different initial amount or dealt with the residual leftover using a range of number sense. Earlier attempts at the strategy typically result in a large initial jump followed by an "inching towards" the final target; typically afraid of going over. *A more developed number sense uses more anticipatory thinking* resulting in greater efficiency in spanning the remaining distance to the total. These students understand how division and multiplication are connected.

Partial Quotients – Decomposing the Dividend

As students' number sense grows, and in response to strategic prompts from the teacher, the efficiency in decision making begins to emerge in their work. As students begin to reflect on *what's left*, the focus shifts to decomposing the dividend as they work. Figures 31a and b provide examples of how decomposing the dividend emerges. Celeste [a] does 23 five times, finds the total, then proceeds one more 23 at a time to get as close to 174 without going over. She methodically tracks what each multiplier represents (number of groups). Ed [b] makes a unit of units with 5 groups of 18, iterates one group of 25 until he gets to 25 groups, then adds one 18 at a time until he gets as close to 498.

Ed's strategy, while built on making and using a composite unit of five 18s, demonstrates how the more iterative calculations a student makes, the more likely errors can occur. It was in processing Ed's strategy with the class that a calculation error was revealed. The blue writing reflects the added information by the teacher on the board as students, along with Ed, looked, found, and edited the original work. Processing errors completely and in full allows all students engaged in the conversation to see that editing and revising one's work is part of the mathematical process. Separating the strategy (an appropriate one) from the arithmetic allows the student(s) to know what to trust and continue to use and what requires editing and revising.

Emma (Figure 32) is also an example of partitioning the dividend with two noticeable differences from Celeste and Ed's. She sets out to solve how many groups of 18 are in 498. She first does 10 groups of 18, then, at least internally, sees that she has enough left over to double that amount. As she proceeds beyond 540 (The total of her now 30 groups), she specifically determines how much over she is to get to 498. This reflecting and calculating what's left shifts from a pure building up approach, as has been evident with many of the previous examples, to moving back towards zero. This is what distinguishes the larger category of partitioning the dividend with partial quotients.



[a]
Celeste
174 ÷ 3

[b]
Ed
498 ÷ 18

Figure 31a-b – Partitioning the dividend, Partial Quotients

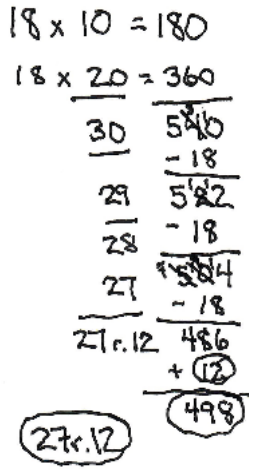


Figure 32 – Emma, Partitioning the Dividend/Partial Quotient: 498 ÷ 18

Compensation Strategy

The later part of Emma's work demonstrates how the compensation strategy emerges from students' explorations of strategies. Notice how Emma overshoots 498 with her 30 groups to end up at 540. Instead of abandoning her original work and starting over, she uses what she knows about the compensation strategy to work to determine the actual number of groups needed to find the solution. It is critical in the public sharing that Emma's strategy is understood by other students observing and listening. Reflecting on how rounding to an amount just over what one is looking to solve can be used effectively to determine a solution. Navigating the remainder in a context such as Emma's tends to be a sticky spot for students to initially navigate. Public discussion around such decision making allows students to connect Emma's choices to their own work should they in the future decide to use the strategy for themselves.

In the absence of a student's work sample such as Emma's, the compensation strategy can be drawn out by observing students' abandoned attempts at using a quantity that would go over the dividend. Students typically erase or cross out the original attempt and try a smaller amount that allows them to continue building up to the dividend instead. A prompt to ask after they have found the answer and explained their strategy is to say the following. *I saw that you tried [this amount] here. You said that it wouldn't work because it was too big. Now that you know what the answer is, could you have gone over the amount to find the same answer? How might we use that information as an actual strategy?* A small or large group scaffolded conversation regarding how the compensation strategy would work, elevates an intuitive idea to an explicit one. The conversation becomes a class reference point when others in the classroom attempt to use the strategy in the future.

A Reflective Prompt

For those students who build up to divide, there is a tendency for students to not reflect on 'what is left' after the largest multiples have been determined. In Emma's case, this was once she was up to 30. To make her next move more efficient over time, posing the following prompt when a student is at this point of decision-making helps them consider what total amount is still left. Ask, *Do you know how much you have left to go?* That prompts them to consider the total amount that remains instead of inching forward (or back) one group at a time.

Partial Quotients

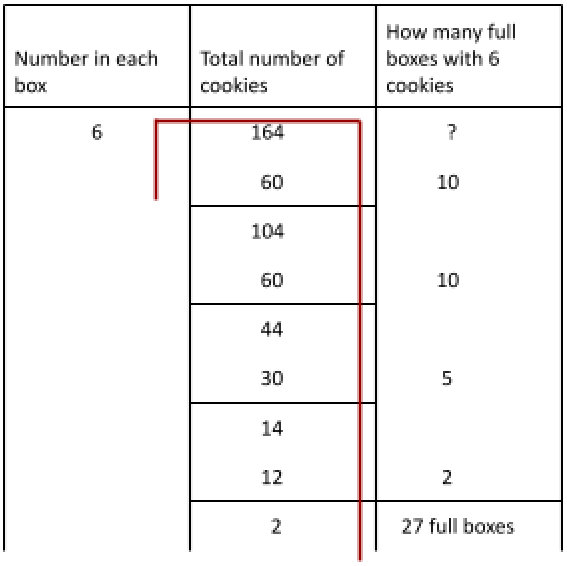
The formatting of partial quotients, and that of the standard division algorithm, needs to be formally introduced. If you look again at Figures 31a-b and 32, the 'guts' of the partial quotient and the standard algorithm is there. It's just the students are completing the mental calculations using a format that suits their ability to keep track of the information they need to monitor visually. Students can be introduced to the more conventional formatting under two key conditions. One, their own organizational structure is getting them confused so alternative formatting would be productive to track the information the student is generating. Two, they already have a sound conceptual understanding of division so the introduction of more conventional formatting allows the student to become familiar with how others would organize their work.

A Story: With one fourth grade class of students, they happened to have the textbook open, turned the page and saw that "Long Division" was the heading of the next chapter. A howl of complaints arose saying "division is hard," "I don't understand division," etc. The irony was the student students had been doing division problems intermixed with multiplication problems across the school year without any concern whatsoever. But the rumors of the terror of 'Long Division' is well entrenched and supported by the memories of older students, siblings, and even parents. So I said…

> [Teacher] "Okay! Let me give you a problem and let's just make a chart. I want you to imagine that you are working in the bakery up at [name of local grocery store] and there are 164 cookies that have just been baked. Your job is to put six cookies in clear plastic containers so they can be put out on the shelves for customers to buy. So my question to you is, how many full boxes of six cookies will you fill?
>
> Let's put this information in a chart so we can keep track of the information. I am going to make three columns (Figure 33). All right, then. What do we know?
>
> [Students] We know the total number of cookies.
>
> [Teacher] And that is?

> [Students] 164.
>
> [Teacher] I'm going to place that information in the center column as that will make tracking our work easier (Figure 33). What else do we know?
>
> [Students] There are 6 cookies in every box.
>
> [Teacher] I am going to put that information in the first column. So, what question are we looking to answer?
>
> [Students] How many boxes can be filled?
>
> [Teacher] I am going to put that question here in the last column. Your job is to fill the plastic boxes. Now, we could fill a box, make a tally mark, fill another box, make a tally mark until we are all done, but that is a tedious way of finding out how many boxes you might end up filling. What's the largest number of boxes you think you could fill to start with that would let you be more efficient?
>
> [Students] (After taking suggestions and settling on a relative consensus, typically at the beginning students tend to settle on…) Ten
>
> [Teacher] I will put ten here. Now how many cookies would be used up when you are done with ten boxes? How many cookies would be left to still put in boxes? (Working and recording through the subtraction/adding up to determine the difference between…)
>
> [Students] 104
>
> [Teacher] Could we do another ten boxes?
>
> [Students] Yes.
>
> [Teacher] (Recording the work on the board) How many cookies still need to go into boxes after we are done with those ten boxes?
>
> [Students] 44
>
> [Teacher] Can we do another ten boxes? (No) So what could be the next number of boxes we could fill without counting one box at a time?
>
> [Students] (In this instance, students settled on…) 5 boxes.
>
> [Teacher] How many cookies will that use up? [30] How many cookies are still left to put into boxes before you are done? [14] How many more boxes with six cookies can you fill? [2] How many cookies does that use up? [12] How many are left? [2] Can we fill any more boxes? [No] So, how many boxes with six cookies are ready to go out on the shelves? (Adding up the total in the third column…) [27 boxes]

At that point, I took a different colored marker and drew the partial quotient 'seven' shape over the chart and said to the students, "Do you know you just divided?" What still sticks in my memory to this day is what one student said out loud. "Is that all it is?" "Yes, that's all it is!" "Well, that's not so hard!" And that was the last complaint about long division that was uttered the rest of the year.



Number in each box	Total number of cookies	How many full boxes with 6 cookies
6	164	?
	60	10
	104	
	60	10
	44	
	30	5
	14	
	12	2
	2	27 full boxes

Figure 33 – Teacher work recording student responses

The scenario above is a measurement division problem [? x 6 = 164]. In another class, a student who found the chart with its labels highly useful was momentarily confused about how to set up the chart with a partitive division structure. (Example: 6 x ? = 164). Since this is a formatting issue only, not a conceptual one, I directly told him. “Oh, just switch the labels between the first and third column so that the question can go in the last column.” And off he went.

Follow the units

The benefits of having students build a unit chart in the early stages of using partial quotients is that it guides the students to *follow the units* of what each quantity means and what question is being asked. The numbers come with a unit of measure. Multiplication and division are unit transforming so *following the units* is essential. Just allowing students to consistently work with ‘naked numbers’, meaning numbers without a context, undermines the development of this conceptual understanding. Students should not arrive in high school chemistry class and be surprised by unit analysis. Unit analysis is a core concept of what makes multiplication and division the operations that they are.

Interpreting the answer

In division, the quotient, i. e., ‘the answer’, is always interpreted in relation to the divisor. There are 27 boxes of 6 cookies. Having students develop the habit of *recording the ‘answer’ with a label* aids them in interpreting the work accurately by the unit and how the unit has transformed. This is another reason to intentionally have students verbalize the meaning of each number with which they are working.

Interpreting the remainder

The question being asked shapes how the remainder is to be interpreted. *In the measurement division* scenario above, The question asked how many ‘full boxes’ would be ready to go out on the shelf for sale to customers. Students will often write the answer as 27 r. 2, but that does not answer the question. Students will demand what to do with the remaining 2 cookies. Helping students see the practical context of the bakery helps interpret what the workers will do with the extra cookies. But the question was how many full boxes will be made and put out on the shelf. The answer to that is 27 full boxes of six cookies. In this scenario, *the remainder is ignored.*

If the context was different and all items in question needed to be in boxes, *the remainder is interpreted as another whole box.* The answer to that question is 28. It's just that the last box is not a box of six.

In yet another context the question might be phrased such that the remainder *is the answer.* Then again, the remainder might require a fraction interpretation in relation to the whole number, e.g., 27 whole boxes of 6 with 2/6 of another box.

In *partitive division* contexts, the remainder nearly always is interpreted as a fraction or decimal. In the scenario where there are *14 pancakes being shared equally with four friends and all of the pancakes are eaten, how many pancakes will each friend get,* the remainder is shared in fractional terms. Each friend in the instance will get 3 and $\frac{2}{4}$ (or $\frac{1}{2}$ depending upon how the student partitions the remaining two pancakes). There is phrasing of the question within a partitive division structure where the items being sorted are not divisible fractionally. (Think about coins or markers as being shared equally.) In such cases, the remainder will be ignored. That said, in non-context equations, partitive division remainders are expressed either in fractional or decimal terms.

"Partial Quotients requires too many steps"

As numbers get bigger, and students work iteratively with 10 after 10, after 10 to find the solution, teachers often get frustrated with the use of the partial quotient format. This is a learning opportunity, and it is an important phase in the maturation of students' ability to begin thinking in multiples. As the numbers get bigger, strategic prompts provided by the teacher locally as well as in a whole class discussion sounds like the following. 364 ÷ 6. *Yes, you could start with 10, but are you close or far away from the total?* [Far away.] *If you are far away, could you double or triple the ten and save yourself some time?* That prompt, that scaffold, consistently verbalized across the classroom moves students towards *thinking in scale.* As discussed in Part VII, Place Value as a Rate of Ten, the scaling factor if I multiply 3 times as many tens I simultaneously get 3 times as many ones applies in this context as well. Guiding students to become more efficient through the use of strategic prompts is how the *n-times as many* aspect of multiplication begins to emerge. As students' number sense matures, partial quotients can be highly efficient. Like with derived strategies while learning to fluidly recall the single-digit combinations, thinking in scale by initially learning to double, triple, and quadruple two-digit numbers builds the capacity to become more efficient in division contexts.

How does partial quotients work with calculating decimals?

I have been in discussions with fifth and sixth grade teachers where the claim is that the partial quotients format does not work if students have to calculate the decimal amount of the remainder. The claim is "there is no room. Students have to write out a separate division problem just for the decimals." There are two easy solutions to this. First, there is no reason that the numbers need to come down the side. There are published curriculums where the partial quotients are stacked up above the line and then added together. The other solution is, if students are being asked to calculate to the hundredths place should there be a remainder, plan ahead and add the zeros to the dividend, e.g. 164.00. These are formatting issues only. Directly instruct students so they can adapt the formatting. It's a minor issue. It's not a mathematical one.

What is the conceptual mathematical issue is the language and re-unitizing students need to develop in order to calculate the decimal remainder. Altering the numbers used above in figure 33 to be 165 ÷ 6 so the result would be 27 r. 3, the classic instructional approach would be to *"bring down the zero, pretend that 3.0 is 30, divide, then count the number of decimal places starting from the right."* If students are to develop the cognitive ability to re-unitize numbers, it's important for students to navigate *across place* as well as *within place.* Meaning 3.0 is equal to 30-tenths. Don't pretend it's just thirty, mentally re-unitize it to be 30-tenths. 6 x 5-tenths = 30-tenths or $\frac{30}{10} \div 6 = \frac{5}{10}$. Re-unitized to the one's place, 30-tenths equals 3 ones. The final quotient then is 27.5 boxes of six cookies. Is it easier to pretend the decimal is not there and then count the decimal places? That works for a lot of people. It requires a lower demand of cognitive register. However, students fail to develop key foundational blocks that are needed when working with place value understanding as well as future topics such as rational numbers where these cognitive relationships are needed for understanding. (See the following for an explanation written for parents. Division Part 2)

The Division Standard Algorithm

Of the four standard algorithms, the long division algorithm is the most troublesome. The mathematics behind the addition, subtraction, and multiplication is sound. By simply talking in value and with slight modification of the formatting, those three algorithms are very functional. With the long division algorithm, once the value of language is introduced, the entire procedure falls apart. Consider 6)164 which were the numbers used above in the vignette and figure 33. Using the classic single-digit language used to start the procedure, one would say, *"6 does not go into 1, move over one place. Six goes into 16 two times, etc.* If value is placed into the script, it would change to *6 goes into 100.* You can't say 6 doesn't go into 100. It's mathematically incorrect. It becomes a completely different mathematical calculation.

Deep behind the single-digit language is, in fact, the partitioning the dividend/partial quotient strategy. The decomposition of 164 at its most efficient would be 6)120 + 42 + 2 from which 27 r. 2 emerges as the solution. If the goal is to teach conceptual understanding and procedural fluency, working to improve students efficiency with partial quotients leads to more advanced mathematical ideas that students will encounter in future grades. (See the following for an explanation written for parents. Division Part 1)

Interim Summary IX

Division grows out of students' multi-digit addition and subtraction strategies just as their multiplication strategies do. If left alone, students often approach division through the lens of multiplication where they build up to the total quantity in the context. There are two types of division problem structures: measurement (the number in each group is known but not the number of groups) and partitive (the number of groups is known but not the equal distribution among each group). Students' number sense is enacted differently between the two types of division problems. Students tend to flip building up to the total to working from the total back towards zero once they begin to intentionally reflect on what's left of the total as they work. The formatting of partial quotients or the standard division algorithm needs to be formally introduced. The partial quotient strategy is the more conceptually sound of these two approaches. To gain proficiency and efficiency with partial quotients, students need to be scaffolded by the teacher to think in more and more scaled quantities as they develop their number sense. The units of the various quantities in the context need to be followed and solutions need to be interpreted and labeled in relation to the divisor.

Reflection Point 7

- ☐ Given what you have previously considered in this piece so far, how do you interpret the following 5th-grade Common Core standard: *Fluently multiply multi-digit whole numbers using the standard algorithm.* (Notice that this is the first time the multiplication standard algorithm is mentioned; not at third-grade, not at fourth-grade, but for the first time at fifth-grade. Something else to ponder.)
  - ☐ The phrase *"including the standard algorithm"* is an interesting choice of words. Does that mean "must know", or "in the mix"?
  - ☐ What is the math behind the standard algorithms in each of the four operations?
  - ☐ Do the algorithms only work in a vertical format, or does it still work in a horizontal format?
  - ☐ The 4th grade Common Core Standards state:
    - ☐ *Multiply a whole number of up to four digits by a one-digit whole number, and multiply two two-digit numbers, using strategies based on place value and the properties of operations. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.*

- ☐ *Find whole-number quotients and remainders with up to four-digit dividends and one-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.*
- ☐ Notice the lack of mention of the standard algorithms for either multiplication or division. How does this change your expectations for yourself and/or for your students?
- ☐ How does the additional phrase "*using strategies based on place value and properties of operation"* change your interpretation about how the standard algorithm works and how it should be included in instruction?

Part X – Putting It All Together – Arithmetic through the lens of algebra

Many teachers find the early stages of development and the strategies students use as too messy, too tedious, and too time consuming to spend class time on. The urge is to jump quickly, if not immediately, to the standard algorithms. However, allowing students to move through the early stages of modeling, of complex doubling, and additive units of unit strategies builds understanding of key concepts and procedures. It allows students to build the capacity to explicitly work with various decompositions of numbers – both addends and factors. It establishes a more robust foundation of place value based on the factors of ten. Additionally, it allows for building a solid conceptual foundation to multiply and divide numbers fluidly based on the algebraic properties of operations.

Consider the following problem (Figure 34). Notice all of the algebraic thinking underpinning the work. Yes, one could just teach the algorithm with its single-digit language and have students arrive at a correct answer. However, an emphasis on the language of value and on making the algebraic properties explicit results in a level of mathematical understanding that is generative, one that builds vertically into more sophisticated mathematical work needed at middle school.

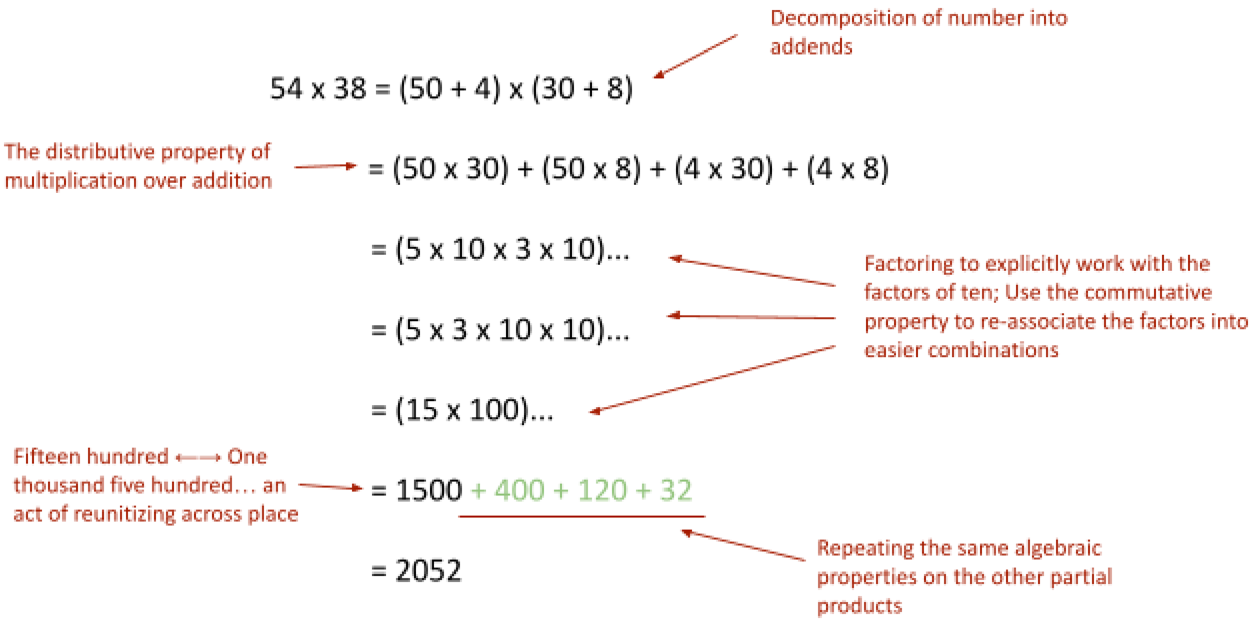


Figure 34 – The algebraic properties of operation in multiplying 54 x 38

With the knowledge and language of using these properties of operations fluidly, much of what is written out in detail like a high school algebraic proof can be consolidated as the work is internalized and doesn't need to be written out in full detail. Written out in vertical format, the mathematics is the same. It's only the format that has changed.

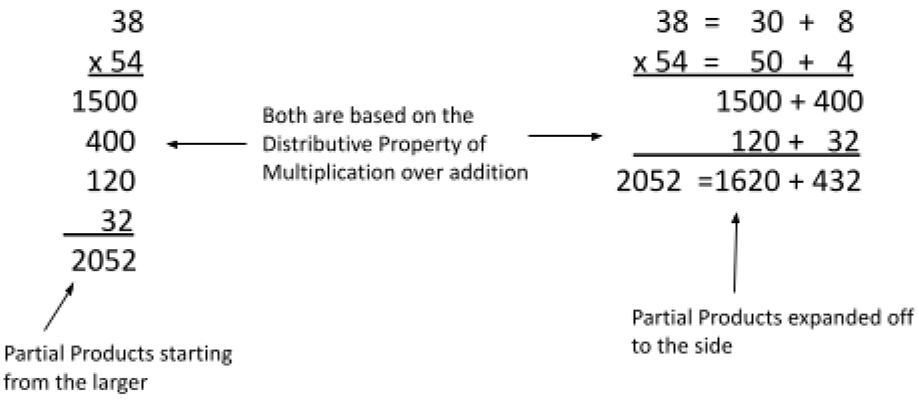


Figure 35 – Vertical formatting of partial products/distributive property of multiplication over addition

The standard algorithm in multiplication is based on the distributive property of multiplication over addition. It is just that the single-digit language and the insistence of starting in the ones' place obscures the mathematics that underlie the procedure. The benefit of starting with the larger – especially while using the language of value – allows one to know that my answer will at least be above 1500. The ability to estimate a final total amount is much closer to that endpoint. One does get the same answer starting in the ones' place (My answer will be larger than 32.) due to the commutative property of multiplication. However, 32 is significantly further away from the final answer than when one starts with 1500. The fourth grade Common Core Standard (4.NBT.5) reads: *Multiply a whole number of up to four digits by a one-digit whole number, and multiply two two-digit numbers, using strategies based on place value and the properties of operations. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models.* Students who use the above strategies are meeting the standard for place value (4.NBT.1) and for the properties of operations (4.OA). Students with this foundation of conceptual and procedural understanding also are well grounded to multiply two binomials when they arrive in middle school as the algebraic properties are aligned. Using the above expression of 54 x 38, and factoring out the place value rate of ten, the resulting binomials look like the following:

$$(5x + 4)\ (3x + 8) = 15x^2 + 40x + 12x + 32.$$

Make $x = 10$, and the explicit multiplicative place value rate of ten is clear. The arithmetic of the elementary grades is grounded in algebraic properties. The learning and teaching of these grades should be vertically aligned with the next levels of mathematics in middle school and beyond.

Part XI – Properties of Operations[30]

- For every real number…
    - Addition
        - Commutative Property: $a + b = b + a$
        - Associative Property: $(a + b) + c = a + (b + c)$
        - Identity Properties: $a + 0 = a;\ 0 + a = a$
        - Inverse: *For every real number* a, *there is a real number* $-a$ *such that* $a + (-a) = 0$
    - Subtraction
        - Zero Property: $a - a = 0,\ a - 0 = a$
    - Multiplication
        - Commutative Property: $a \times b = b \times a$
        - Associative Property: $(a \times b) \times c = a \times (b \times c)$
        - Identity Property: $a \times 1 = a;\ 1 \times a = a$
        - Inverse: For every real number a, $a \neq 0$, *there is a real number* $1/a$ *such that* $a \times 1/a = 1$
        - Zero Property: $a \times 0 = 0,\ 0 \times a = 0$
        - Distributive Property of multiplication over addition: $a \times (b + c) = (a \times b) + (a \times c)$
    - Division
        - Identity Property: $a \div a = 1$
        - Zero Property: $0 \div a = 0;\ a \neq 0$
    - Equality & Inequalities Symbols
        - The equal sign [=]: A comparison symbol of two equal value mathematical expressions
            - Reflexive Property:
                - $a = a$
            - Symmetric Property
                - $a = b + c;\ b + c = a$
                - $a + b = c + d$
            - Transitive Property: *If* $a = b$ *and* $a = c$, *then* $a = c$
        - Greater Than [>] and Less Than [<]

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[30] For further exploration of algebra and relational thinking in the elementary classroom, read Carpenter, T.P. Franke, M.L. and Levi, L. (2003). *Thinking mathematically: Integrating arithmetic and algebra in elementary school.* Portsmouth, NH: Heinemann.

- Read left to right: *a > b [a is greater than b]*
- Read left to right: *a < b [a is less than b]*

A little background on the number zero – Our international daily number system is organized as a base ten system. Computer systems are organized using the binary base two system. The base ten system matured over time from the counting boards/racks used to calculate quantities. The Babylonians in Mesopotamia understood zero as a placeholder over 5,000 years ago. But it didn't spread to other cultures and the idea died out. The Mayans independently developed the concept of zero as a placeholder in the 4th century C.E. but isolation from those in Europe and Asia left that knowledge unknown beyond the Mayans themselves. It was the introduction of the concept of zero growing out of the Indo-Hindu mathematics that zero as a placeholder took root as well as zero as null, an empty set. Further development in China and then in the Arabic Islamic mathematical world in the ninth century C.E. expanded the realm of mathematics even further. European mathematics only experienced the introduction of zero as a placeholder and empty set in the 12th century C.E. and it revolutionized Italian finance. One could say zero helped bring on the Italian Renaissance.

Concept of "cultural diffusion" captures how ideas take hold and are adopted/adapted across cultures. Example: While the Babylonians understood and had a symbol for zero, it did not spread to other cultures once that civilization collapsed. It appeared to have died with the collapse. The same with the Mayans. The Hindu-Indian conception of zero spread to China, to the Arab world, and then eventually to Europe.[31]

Part XII – Building Capacity

Integrating multiplication and division instruction together

Working with multiplication and both forms of division at the same time allows students to immediately understand *the interconnections between the two operations.* It also allows for the *cross-checking misconceptions* that may arise if only one operation is focused on for a substantial length of time before getting to the other. A unit where the priority is multiplication may have division tasks that include numbers within the students' comfort zone level making them more accessible while focusing on the instructional level with multiplication. A unit on division could include multiplication tasks that allow students to maintain skills just explored in a previous unit. In any division unit, *both measurement and partitive division need to be interspersed* so that students develop the comprehension skills and number sense needed to fluidly work within the two structures of the operation.

Thinking in Scale

Multiplication as *n-times as many* is a key foundational element in developing unit rate and scale factor. What is 'twice as many', 'three times as many' are relationships that capture this increase in value compared to the initial unit. Functions, rates, ratios, proportional relationships as well as spatial visual concepts involving sizing and similarities are all based on the capacity to think in scale. These are the underpinnings of middle school standards. Nurturing a sense of scale among intermediate elementary students allows for an easier transition to the middle school mathematical standards.

Multiplicative comparison problems begin to develop a sense of scaling up as the language used and the relations captured in the task nurtures that way of thinking.

> *Baby piglets weigh between 2 to 3 pounds at birth. At the end of the first week, they typically weigh 6 to 7-times as much. What is the range of weight at the end of the first week? How much might they weigh at the end of one month?*

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[31] https://en.wikipedia.org/wiki/0
https://geoalliance.asu.edu/sites/default/files/LessonFiles/Jenkins/Zero/JenkinsZeroS.pdf
https://www.history.com/news/who-invented-the-zero

The *'times as many'* language orients the scalar multiplicative relationship between the initial weight of the piglets to the end point. Measurement contexts also are conducive to comparative language situations, e.g. 'a traveler went three times as far as the other traveler.'

A *measure space*[32] captures the scalar relationship among a set of quantities. Imagine if I wanted to foster a sense of scale from a place value context. Consider the following number string where one line is presented and discussed before the next line is written.

10	10	100	[T: *Do you agree with me that there are 10 tens in 100?*]
__	10	300	[T: *How can you use that to figure out how many tens are in 300?*]

As the relationships are discussed among the students, adding in the arrows indicating the scalar relationship of how the elements co-vary focuses the attention of the students on the scale factor.

The implication for the students to focus on the covariance of the elements is as in *if you have three-times as many hundreds, you will have three-times as many tens.* This type of scalar thinking is also seen on how patterns and relations are captured in function tables, t-charts, and ratio tables. The example above, the use of a measure space with intermediate elementary students allows for the exploration of the multiplicative rate of ten in place value. An exploration that focuses on the simple functions of single-digit combinations can also be used, e.g., if 4 fours are 16, how many fours in 32?

Notice that no operation symbols are evident when writing the numbers out for students to view. The intent is to focus on the relationships between and among the quantities and to activate the necessary relational thinking needed.

Doubling, Tripling, & Quadrupling Numbers

Doubling numbers captures students' interest early in their development of number patterns. 2 + 2 is 4, 4 + 4 is 8, 8 + 8 is 16, and so on. Explicitly developing students' capacity to double, triple, and quadruple numbers builds the type of fluency when working with multi-digit numbers. Beginning in third grade, or if students have a limited prior experience in working at a mental level in the upper grades, starting with teen numbers provides an accessible entry point to the doubling process. Write the number 14 on the board. Ask students to *double 14* in their heads (appropriate accommodations acknowledged). Doubling 10 and doubling 4 are accessible for the majority of students. Recording a student's strategy of the sequence used to solve the task makes the elements visible to the other students in the group. Using an informal representation referred to as the *branching method*, the focus is on capturing the partial products, resulting end product, and the sequence used by the student sharing. This takes the focus off of the formal operation symbols and any need for it to be recorded either in vertical or horizontal formal notation. It's about capturing the landmarks of the working memory.

14
/ \
20 8
\ /
28

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[32] Vergnaud, G. (1994). Multiplicative conceptual field: What and why? In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics.* (pp.41-59). Albany, NY: State University of New York Press.

Public sharing focuses on the decomposition of the number, e.g. by place value components [10 + 4] or possibly even compensation by rounding to 15 [14 = 15 – 1]. Attention is also placed on which partial product the student used first, the larger [10] or smaller [4]. The resulting processing of the commutative property of multiplication emerges and conversations around the strategic benefit of starting from the larger can be explored [The answer is above 20 compared to the answer is larger than 8.]

As students gain confidence in doubling larger and larger numbers, returning to the teen numbers to triple 14 can begin. Again, this is because the teen numbers are readily accessible to students to operate on. Then as tripling multi-digit numbers becomes comfortable, present *quadrupling 14* begins.

The goal is to *develop efficiencies and fluency* in the amount of written calculations students need to take. This work is done in number talks/warm-ups early on builds the capacity for when in fourth grade, multiplying 52 x 38 one can prompt students to take that first 'big picture relational look' before beginning to calculate. Given 2 x 38 can potentially, by this time, be done in one mental step rather than in writing out two formal steps as in 2 x 30 + 2 x 8. Some students may be able to quintuple 38 and then multiply it by 10 [5 x 38 x 10] if they have been nurtured to work explicitly with factors of 10 (See Part VII).

When the numbers get bigger and more complex in division, where students are either building up or using partial quotients, the use of small increments becomes tedious. In the example of 664 ÷ 12, I could do 5, then 5, then 5… until I got close enough. I could be more efficient and do 10 at a time. However, if in number talks/warm-ups students have gained a capacity to double, triple, and quadruple numbers fluidly, *interrupting students with the prompt, 'Are you closer or far away? [Far away] Could you double or triple that amount and save yourself some time?* That prompt scaffolds the student's thinking to connect the number sense developed in one classroom context to the current one. It also scaffolds students to think more and more in scale *at an explicit level* while they are working.

Informal to formal notation/representations

The use of informal notations as students are first learning a set of skills or in expanding their mathematical range, allows the brain to just focus on the numbers generated mentally and the properties of operations being employed. Introducing informal representations is especially useful if students are very procedurally bound due to prior grade level restrictions on how the four operations are *'supposed to be'* recorded. The informal representations are a way to loosen them up, so to speak, to think about the mathematics first, the representation second.

Arrow notation with the *'leads to arrow'* helps students to not misinterpret and misuse the equal [=] sign. It allows the continuous flow of thinking to be captured without having to write out individual equations. Arrow notation can also be used to mimic the flow on a numberline prior to introducing the formal intricacies and structure of the formal number line. As with multiplication, the *'branching strategy'* can be used for *splitting and sharing* in partitive division tasks. The benefit for students with this representation is that it is clear that each group, each friend, gets the same portion as all of the other groups. In the formal division partial quotient or standard algorithm formatting, the single 'answer' needs to be interpreted to mean that each group, each friend, is apportioned that amount.

As the numbers get bigger, more complex, the informal strategies begin to become less useful. While how would three share 84 equally is readily recorded as represented below, imagine if the question was how would 12 share 84 equally? The recording process would be tedious and potentially more error prone.

$$\begin{array}{c} 84 \\ / \quad | \quad \backslash \\ 20\ 20\ 20 \longleftrightarrow 60 \\ \underline{\ 8}\ \underline{\ 8}\ \ \underline{\ 8} \longleftrightarrow \underline{24} \\ 28\ 28\ 28 \longleftrightarrow 84 \end{array}$$

It is at this point where introducing the more formal notation becomes productive in that it typically shortens the amount of space and tidies up information so that newer mathematical ideas can be explored and developed. The transition to the formal symbolic notations and formatting is more strongly grounded conceptually instead of merely capturing the procedural mechanics of the operation. The two arise together.

Working across representations

In middle school, one of the sets of standards is to interpret information moving from a table, to a graph, to an equation. Each representation captures the same mathematical ideas but displays them in different formats allowing the information to be interpreted and operated upon. In elementary classrooms, having students see how the mathematical ideas involved in the task before them are connected whether the information is in a picture, on a number line, in an area model, in number displayed vertically or horizontally, etc. Each representation provides a window into the mathematical ideas that are unique to that display. Explicitly drawing students attention to the interconnections reinforces the neurological pathways for students to work comfortably across the different representations. Dienes' *Variability Principle*[33] states that working comfortably across representations reinforces the mathematical structure rather than the procedural steps and social conventions required when using either representation. Working across different representations builds the capacity of students to work nimbly as they have a deeper toolbox of options with which to think through solutions and represent that thinking.

Interim Summary XII

The more emergent and additive strategies to solve multiplication and division tasks may appear tedious and messy to the untrained eye. Out of that tedium and messiness, however, arises the conceptual understanding of being able to think in multiples. It is a disposition that is developed along with becoming capable of finding a solution in an efficient way. If one looks at third, fourth, or fifth grade years from September through June as an arc of learning where ideas and the number sense are built over time rather than a production line where sub-skills are covered, instruction looks and sounds different. Focusing students' attention on the mathematical ideas, working initially with more informal representations and then transitioning to the formal symbolism, allows procedural fluency to arise from conceptual understanding. A students' cognitive processing is the focus of instruction and assessment rather than merely the production of answers. A jump to the formal representations and/or abstract standard algorithms too soon frequently diminishes a student's conceptual understanding. Answers may be able to be produced. However, that house of cards easily collapses when extensive multiplicative thinking is expected at the middle school level. The vertical progression and consistency in an approach to instruction across the grades is necessary if a conceptual *and* efficient procedural fluency foundation is to be built in the elementary grades. Out of the arithmetic arises the algebraic properties. This vertical progression is foundational to comprehending next levels of mathematics.

Reflection Point 8

- ☐ Solve the piglet problem. Solving using different representational formats, e.g. pictures, a number line, numbers, formal equations, etc. Think about how, as a teacher, you might lead a classroom conversation among the mathematical ideas captured in each of the representational forms.

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[33] Dienes, Z. (1971, fourth edition) *Building up mathematics*, London, UK, Hutchinson Educational Ltd.

    - ☐ *Baby piglets weigh between 2 to 3 pounds at birth. At the end of the first week, they typically weigh 6 to 7-times as much. What is the range of weight at the end of the first week? How much might they weigh at the end of one month?*
- ☐ Use the branching method for multiplication and partitive division to solve the following tasks.
    - ☐ Triple 167
    - ☐ Quadruple 259
    - ☐ Share 96 with four equally
    - ☐ Share 15 with four equally
        - ☐ How would a pictorial representation aid in the public sharing of this last task?
- ☐ How freeing is your thinking using these informal representations? What number sense and or algebraic properties underlie your work?

Part XIII – "Teaching" the Strategies – What does it mean to cognitively guide one's instruction?

Invariably, the topic arises about how to "teach" these strategies to children. Many curriculum teacher manuals organize specific lessons to have students engage in such strategies through direct/explicit instruction. In manuals reflective of "teaching" students multiple strategies, a specific strategy is introduced one at a time. Often the strategies come in rapid succession where one is introduced on Monday, another on Tuesday, leaving students and teachers with a jumble of ideas and confusion by the end of the week. The approach being taken here requires a combination of *purposefully selected problems,* with *carefully selected number combinations,* along with *capacity building number talks/warm-ups,* and *guided classroom conversations* that focus students' attention on *comparing and contrasting strategies, reflecting on one's work* to deeply comprehend the mathematics and to look for efficiencies. Students *unpack problems* before working as a means to *visualize the context.* The work is *frequently collaborative* in order to cross pollinate ideas. The teacher facilitates the public sharing of ideas both on an *interim basis* allowing students to discuss initial approaches and then to *edit and revise* one's work. *Post public sharing* after answers are determined allows for strategies to be analyzed, compared and contrasted. *Errors are processed constructively and in full* with attention to what works but needs revision and what doesn't work and should be discarded.

Instead of I do → We do → You do of the direct/explicit instruction, gradual release approach to learning mathematics, the shift is to Pose ⟷ Elicit ⟷ Engage ⟷ Share & Connect. Notice that in the former the arrows are in a one way direction; from teacher to student. In the latter, the arrows are two-way as there is a constant interchange between teacher and students, students and teacher, and students with their peers. The *attending* to student thinking in order to *interpret* where along the developmental spectrum the students' knowledge lies allows the teacher to be more strategic in *deciding how to respond* in-the-moment as well as in assessing and planning next steps in instruction. These *teacher noticing skills* are very different from the direct and explicit instruction based classrooms.

Such classrooms are language rich. Students engage in the language functions of explaining, describing, justifying with evidence, comparing and contrasting, summarizing, paraphrasing… These are skills that students need to be supported in developing. Every math teacher who seeks to become comfortable in classrooms that allow students to explore multiple strategies to solve problems needs to also see themselves as a language teacher. Building student voice and the concomitant active listening skills is as much a part of the lesson planning as is the mathematical content.

Much attention has been given to the research around *Building Thinking Classrooms.*[34] The practices found to increase student engagement are well worth exploring and implementing. The surface use of the practices outlined, however, is not enough. As found in other studies, it is pressing for details of students' thinking and making explicit connections between and among the other strategies that impact student achievement.[35] One of the Standards for Mathematical Practice is to *construct viable arguments and critique the reasoning of others.* If students are to meet the challenge of this standard, they need:

- To be in classrooms where multiple strategies are accepted
- Student thinking is explored, appreciated, and nurtured
- The language skills to express one's thinking is developed, and
- Explicit attention is given to the comparing and contrasting of ideas

The teacher skill set to facilitate the discourse practices may not match the practices of our upbringing. Teachers need to allow themselves the time to grow into the practices and become a learner along with their students.

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[34] Liljedahl, P., & Zager, T. (2021). *Building thinking classrooms in mathematics : 14 teaching practices for enhancing learning, grades K-12*. Thousand Oaks, CA: Corwin.

[35] Webb, N.M., Franke, M.L., Ing, M., Wong, J., Fernandez, C.H., Shin, N., and Turrou, A.C. (2014). Engaging with others' mathematical ideas: Interrelationships among student participation, teachers' instructional practices, and learning, *International Journal of Educational Research, 63,* 79-93.

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