Exam 3 Review

Sec 5.3, 5.4, 5.6, 5.7, 6.1, 6.2, 7.1, 7.2, 7.3

The problems on this review sheet will help you practice some of the algorithmic and computational skills required for this exam. However, these technical skills are not a substitute for conceptual understanding! On the exam you may be asked to write and explain your thinking (not just solve problems).

What do you need to know? Some sample questions and important concepts:

1. Give the general form for the solution of a second order linear differential equation, and explain what each of the parts represents.
2. In solving a nonhomogeneous equation, one step is to make a guess about what a solution might look like. In the case of problem 2 below, based on the right hand side $-578\sin 5t$ we would guess a solution of the form $y(t) = A\sin 5t + B\cos 5t$. Why does our guess involve both sine and cosine (why not just sine)?
3. Consider the method of reduction of order. What type(s) of equation does it solve? What type(s) of solution does it produce? What information do you need to have in order to use it?
4. Consider the method of variation of parameters. What type(s) of equation does it solve? What type(s) of solution does it produce? What information do you need to have in order to use it?
5. Briefly describe the different components of a spring-mass system. When considering the various forces acting on an object suspended by a spring, which forces act in proportion to the position y of the object? Which are proportional to the object's velocity? Which are proportional to the object's acceleration? What is the meaning of the term $F(t)$ in the equation of motion for a mass-spring system?
6. Plugging a value of x into a power series results in the sum of infinitely many numbers, each given by a term in the power series. Is it possible to add up infinitely many numbers, and get a finite number for an answer? Explain.
7. Given a function f(x), the theory of MacLaurin and Taylor series shows us how to create polynomials that approximate f(x) near a point. We can make that approximation better by adding more terms, so nearby values of our polynomial get closer and closer to the actual value of f(x) (more decimal digits correct). Is it ever possible to obtain the exact value of f(x) using this strategy?

Review Problems

1. $y'' - 2y' - 3y = 3e^{2t}$
2. $y'' + 6y' + 9y = -578\sin 5t$
3. Use the method of reduction of order to find the general solution to $x^2y'' - xy' + y = x$ given that $y_1 = x$ is a solution to the complementary equation.
4. Use the method of reduction of order to find the general solution to $xy'' - (2x+2)y' + (x+2)y = 0$ given that $y_1 = e^x$ is a solution.

5. Find a particular solution of $x^2y'' + xy' - y = 2x^2 + 2$ given that $y_1 = x$ and $y_2 = \frac{1}{x}$ are solutions of the homogeneous equation.
6. Find a particular solution of $xy'' + (2 - 2x)y' + (x - 2)y = e^{2x}$ given that $y_1 = e^x$ and $y_2 = \frac{e^x}{x}$ are solutions of the homogeneous equation.
7. Find the general solution: $y'' - 2y' + y = 14x^{3/2}e^x$
8. A 96 lb object is suspended from a spring with spring coefficient 219 lb/ft. The whole system is suspended in a viscous liquid that has a damping coefficient of 18 lb*sec/ft and subjected to an external force $F(t) = 390e^{(-2t)}$.
    a. Find a general formula describing the position of the object at time t. You may use the fact that $mass = \frac{weight}{gravity}$, and acceleration due to gravity is $g = 32 ft/sec^2$.
    b. Suppose the object is initially moved up a distance of 9 ft, and is set in motion in the upward direction at a speed of 15 ft/sec. Find the formula giving the position of the object at time t.
    c. Express the answer to part b) using only a single trigonometric function (you are welcome to use exponential or other functions in your answer).
    d. Find the position of the object at time t=0.5 sec. Is it above or below its equilibrium position? How far?
9. Given the differential equation $y'' - xy' - y = 0$:
    a. Suppose that $y(x)$ has a Taylor series about $x = 0$,
    $$y(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + a_6x^6 + ...$$
    Substitute into the differential equation and simplify by grouping together terms with similar powers of x.
    b. Given the initial conditions $y(0) = 16,\ y'(0) = 15$, find the first five terms of the Taylor series solution $y(x)$.
    c. Use the answer to part b to find an approximation of $y(2)$
10. Given the differential equation $y'' + x^2y = 0$ with initial conditions $y(0) = 1,\ y'(0) = 0$, use the first five terms of the Taylor series about $x = 0$ to find an approximate value of the solution at $x = 1.2$.
11. Suppose y is the solution to a given initial value problem and y is given to you in the form of a MacLaurin series, $y(x) = 11 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{7}{12}x^3 + \frac{51}{24}x^4 + ...$.
    a. Find the values of $y(0),\ y'(0),\ y''(0),\ y'''(0),\ y^{(4)}(0)$ (Note that the notation $y^{(4)}(x)$ indicates the fourth derivative of y).
    b. The next term in the MacLaurin series would be a_5x^5. Find the value of the coefficient a_5, given that $y^{(5)}(0) = \frac{3}{7}$ (that is, the fifth derivative of y evaluated at x=0 is $\frac{3}{7}$).

Exam 3 Review ANSWER KEY

If you discover an error please let me know, either in class, on the OpenLab, or by email to jreitz@citytech.cuny.edu. Corrections will be posted on the OpenLab.

Answers to Review Problems:

1. $y(t) = c_1 e^{-t} + c_2 e^{3t} - e^{2t}$
2. $y(t) = c_1 e^{-3t} + c_2 t e^{-3t} + 8\sin 5t + 15\cos 5t$
3. $y = c_1 x + c_2 x\ln(x) + \frac{1}{2}x[\ln(x)]^2$
4. $y = c_1 e^x + c_2 e^x x^3$
5. $y_p = \frac{2}{3}(x^2 - 3)$
6. $y_p = \frac{e^{2x}}{x}$
7. *HINT: First solve the homogeneous equation. Then use variation of parameters.*
   $y = \frac{8}{5}x^{7/2}e^x + c_1 e^x + c_2 x e^x,$
8. a. General position of the object at time t: $y(t) = c_1 e^{-3t}\cos(8t) + c_2 e^{-3t}\sin(8t) + 2e^{-2t}$
   b. Position of the object at time t with the given initial conditions::
   $y(t) = 7e^{-3t}\cos(8t) + 5e^{-3t}\sin(8t) + 2e^{-2t}$
   c. Expressed with a single trig function: $y(t) = e^{-3t}\cdot\sqrt{74}\cos(8t - 0.620) + 2e^{-2t}$
   d. At time $t = 0.5$, the position of the object is $y(0.5) = -1.13$ feet, or just over a foot below its equilibrium position.
9. a.
   $(2a_2 - a_0) + (6a_3 - 2a_1)x + (12a_4 - 3a_2)x^2 + (20a_5 - 4a_3)x^3 + (30a_6 - 5a_4)x^4 + ... = 0$
   b. Taylor series at $x = 0$ given initial conditions $y(0) = 16$, $y'(0) = 15$:
   $y(x) \approx 16 + 15x + 8x^2 + 5x^3 + 2x^4 + ...$
   c. $y(2) \approx 150$
10. Taylor series at $x = 0$: $y(x) \approx 1 - \frac{1}{12}x^4$. Approximate value at $x = 1.2$ is
    $y(1.2) \approx 0.8272$
11. a. $y(0) = 11$, $y'(0) = \frac{1}{2}$, $y''(0) = \frac{3}{4}$, $y'''(0) = \frac{7}{2}$, $y^{(4)}(0) = 51$
    b. $a_5 = \frac{1}{280}$