Pure 2 Ch1 - Proofs

Q1.

(i) A student states

"if x^2 is greater than 9 then x must be greater than 3"

Determine whether or not this statement is true, giving a reason for your answer.

(1)

(ii) Prove that for all positive integers n,

$$n^3 + 3n^2 + 2n$$

is divisible by 6

(3)

(Total for question = 4 marks)

Q2.

(a) "If m and n are irrational numbers, where $m \neq n$, then mn is also irrational."

Disprove this statement by means of a counter example.

(2)

(b) (i) Sketch the graph of $y = |x| + 3$

(ii) Explain why $|x| + 3 \geq |x + 3|$ for all real values of x.

(3)

(Total for question = 5 marks)

Q3.

A student attempts to answer the following question:

Given that x is an obtuse angle, use algebra to prove by contradiction that $$\sin x - \cos x \geqslant 1$$

The student starts the proof with:

Assume that $\sin x - \cos x < 1$ when x is an obtuse angle $$\Rightarrow (\sin x - \cos x)^2 < 1$$ $$\Rightarrow \ldots$$

The start of the student's proof is reprinted below.

Complete the proof.

(3)

Assume that $\sin x - \cos x < 1$ when x is an obtuse angle

$$\Rightarrow (\sin x - \cos x)^2 < 1$$

(Total for question = 3 marks)

Q4.

Given $n \in \mathbb{N}$, , prove that $n^3 + 2$ is not divisible by 8.

(Total for question = 4 marks)

Q5.

(a) Prove that for all positive values of *a* and *b*

$$\frac{4a}{b} + \frac{b}{a} \geqslant 4$$

(4)

(b) Prove, by counter example, that this is not true for all values of *a* and *b*.

(1)

(Total for question = 5 marks)

Q6.

Complete the table below. The first one has been done for you.

For each statement you must state if it is always true, sometimes true or never true, giving a reason in each case.

Statement	Always True	Sometimes True	Never True	Reason
The quadratic equation $ax^2 + bx + c = 0, \quad (a \neq 0)$ has 2 real roots.		✓		It only has 2 real roots when $b^2 - 4ac > 0$. When $b^2 - 4ac = 0$ it has 1 real root and when $b^2 - 4ac < 0$ it has 0 real roots.
(i) When a real value of x is substituted into $x^2 - 6x + 10$ the result is positive. (2)				
(ii) If $ax > b$ then $x > \frac{b}{a}$ (2)				
(iii) The difference between consecutive square numbers is odd. (2)				

(Total for question = 6 marks)

Q7.

Prove, using algebra, that

$$(n + 1)^3 - n^3$$

is odd for all $n \in \mathbb{N}$

(Total for question = 4 marks)

Q8.

A student was asked to prove that the square of any number can be expressed in the form $5n$ or $5n \pm 1$ where $n \in \mathbb{N}$

The start of the student's proof is shown in the box below.

Let $m = 5k$

Consider $m^2 = (5k)^2 = 25k^2 = 5 \times 5k^2 = 5n \quad \text{where } n = 5k^2$ ✓

Let $m = 5k + 1$

Consider $m^2 = (5k + 1)^2$

$= 25k^2 + 10k + 1 = 5(5k^2 + 2k) + 1 = 5n + 1 \quad \text{where } n = 5k^2 + 2k$ ✓

Let $m = 5k + 2$

Consider $m^2 = (5k + 2)^2$

$= 25k^2 + 10k + 4 = 5(5k^2 + 2k + 1) - 1 = 5n - 1 \quad \text{where } n = 5k^2 + 2k + 1$ ✓

(a) Identify and correct an algebraic error in the box above.

(1)

(b) Show the calculations and statements that are required to complete the proof.

(4)

(Total for question = 5 marks)

Q9.

(a) Prove that the difference of the squares of two consecutive **odd** numbers is always a multiple of 8

(3)

(b) Show that the difference of the squares of two consecutive **even** numbers is **not** always a multiple of 8

(1)

(Total for question = 4 marks)

Q10.

$$\text{“}n^3 + 4n \text{ is prime for } n \in \mathbb{N}\text{”} \qquad \text{(I)}$$

(a) Determine whether statement (I) is always true, sometimes true or never true.

You must fully justify your answer.

(2)

$$\text{“}n^3 + 5n \text{ is prime for } n \in \mathbb{N}\text{”} \qquad \text{(II)}$$

(b) Determine whether statement (II) is always true, sometimes true or never true.

You must fully justify your answer.

(2)

(Total for question = 4 marks)

Q11.

(i) Prove that for all $n \in \mathbb{N}$, $n^2 + 2$ is not divisible by 4

(4)

(ii) "Given $x \in \mathbb{R}$, the value of $|3x - 28|$ is greater than or equal to the value of (x – 9)."

State, giving a reason, if the above statement is always true, sometimes true or never true.

(2)

(Total for question = 6 marks)

Q12.

In this question p and q are positive integers with $q > p$

Statement 1: $q^3 - p^3$ is never a multiple of 5

(a) Show, by means of a counter example, that Statement 1 is **not** true.

(1)

Statement 2: When p and q are consecutive even integers $q^3 - p^3$ is a multiple of 8

(b) Prove, using algebra, that Statement 2 is true.

(4)

(Total for question = 5 marks)

Q13.

Prove, using algebra, that

$$n^2 + 5n$$

is even for all $n \in \mathbb{N}$

(Total for question = 4 marks)

Q14.

(a) Prove that for all positive values of x and y

$$\sqrt{xy} \leq \frac{x+y}{2}$$

(2)

(b) Prove by counter example that this is not true when x and y are both negative.

(1)

(Total for question = 3 marks)

Q15.

Prove, using algebra, that

$$n(n^2 + 5)$$

is even for all $n \in \mathbb{N}$.

(Total for question = 4 marks)

Q16.

(i) Given that p and q are integers such that

$$pq \text{ is even}$$

use algebra to prove by contradiction that at least one of p or q is even.

(3)

(ii) Given that x and y are integers such that

- $x < 0$
- $(x + y)^2 < 9x^2 + y^2$

show that $y > 4x$

(2)

(Total for question = 5 marks)

Q17.

(i) Show that $x^2 - 8x + 17 > 0$ for all real values of x

(3)

(ii) "If I add 3 to a number and square the sum, the result is greater than the square of the original number."

State, giving a reason, if the above statement is always true, sometimes true or never true.

(2)

(Total for question = 5 marks)