Linear Algebra
Lesson 8

Part I The identity matrix and permutation matrices

Part II Nonsingular matrices

There are four homework problems for each part.

Part I The identity matrix and permutation matrices

Watch the Playlist 313F20-8-1to4 which includes homework as classwork that you work on in class with hints and I did not put photos of those hints below because not enough students are watching the videos.

2:06 AM Sun Sep 6

Linear Algebra

Lesson 8

Part I

Square Matrices

* Identity Matrix

Permutation Matrices

Part II

Nonsingular Matrices

Recall

we know how to multiply a matrix by a vector.

(review Lesson 7 yourselves)

83%

Linear Algebra

①

<u>Defn</u> A square matrix is a matrix with n rows and n columns

The set of $n \times n$ square matrices: $\mathcal{M}_{n \times n}$

Example

$$\mathcal{M}_{2\times 2} = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} : a_{ij} \in \mathbb{R} \right\}$$

$$= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\}$$

$a_{i,j}$ is useful for $\mathcal{M}_{n\times n}$ where n is large.

2:07 AM Sun Sep 6

Linear Algebra

Thm: If $A \in M_{n\times n}$ and $\vec{v} \in \mathbb{R}^n$ then we multiply

$$A\vec{v} \in \mathbb{R}^n$$

Pf $A\vec{v} \in \mathbb{R}^n$ where n = number of rows of A

Here $\vec{v} \in \mathbb{R}^n \leftarrow$ number of columns of A is also n.

QED

Pf when $n=2$:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} a_{11}v_1 + a_{12}v_2 \\ a_{21}v_1 + a_{22}v_2 \end{pmatrix} \in \mathbb{R}^2 \checkmark$$

QED

Linear Algebra

Defn An identity matrix, I, or, Id, is a square matrix with $\boxed{I_{ii} = 1}$ for $i = 1$ to n and $\boxed{I_{ij} = 0}$ if $i \neq j$

In $M_{2\times 2}$: $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

In $M_{3\times 3}$: $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Try $M_{4\times 4}$

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in M_{4\times4}$$

Classwork:

What is the null space of this I in $M_{4\times4}$?

Solve for Null space

$$\mathrm{Null}(I) = \{\vec{v} \in \mathbb{R}^4 \mid I\vec{v} = \vec{0}\}$$

null space of I

Solve the homogeneous system

$$\left(\begin{array}{cccc|c} \boxed{1} & 0 & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 \end{array}\right) \to \text{reduced Echelon Form}$$

already Echelon form

already reduced echelon form

$$\boxed{\begin{aligned} 1x_1 &= 0 \\ 1x_2 &= 0 \\ 1x_3 &= 0 \\ 1x_4 &= 0 \end{aligned}} \qquad \mathrm{Null}(I) = \{\vec{0}\}.$$

Thm: $\text{Null}(I) = \{\vec{0}\}$ in all dimensions.

Proof: Solve the system

$$\sum_{j=1}^{n} I_{ij} x_j = 0 \text{ for } i=1 \text{ to } n$$

is 0 when $j \neq i$ and is 1 when $j = i$

$$0+0+\dots+1x_i+0\dots+0=0$$

$1x_i = 0$ for $i=1$ to n.

$$\text{Null}(I) = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right\}$$

$$= \{\vec{0}\}$$ QED.

Note that

$$\left(I \mid \vec{0} \right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right)$$

always has n leaders and no free variables and is already in reduced Echelon form

Permutation Matrices $P_{n\times n}$

<u>Defn</u> are square matrices where each row has a single entry that is a 1 and the rest are 0's and they same for each column.

List all 2x2 permutation matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

← each row has a 1 and a 0

↑ each column has a 1 and a 0.

$$\begin{pmatrix} \boxed{1} & 0 \\ 0 & 1 \end{pmatrix}$$

is only one that starts with 1 up left

$$\begin{pmatrix} \boxed{0} & 1 \\ 1 & 0 \end{pmatrix}$$

← row also needs a 1

↑ this column also needs a 1

this forces a 0 here

All 2x2 permutation matrices

$$P_{2\times 2} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$$

$P_{3\times 3}$

Case I

start with a 1 here

$$\begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & & \\ 0 & & \end{bmatrix}$$

rest of row must be 0

what can be here?

rest of column must be 0

Case II

$$\begin{bmatrix} 0 & & \\ & & \\ & & \end{bmatrix}$$

later check starting with 0

pause + try

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

if we put 1 here

must be 0

must be 0

must be 1

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \boxed{0} & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

if we put a 0 here.

must be a 1

must have 0 here

must be 1

$P_{3\times 3}$

Case I

start with a 1 here

$$\begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & & \\ 0 & & \end{bmatrix}$$

rest of row must be 0

what can be here?

rest of column must be 0

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

if we put 1 here

must be 0

must be 1

must be 0

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \boxed{0} & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

if we put a 0 here.

must be a 1

must have 0 here

must be 1

Case II

$$\begin{bmatrix} 0 & 0 & 1 \\ & & \\ & & \end{bmatrix}$$

or

$$\begin{bmatrix} 0 & 1 & 0 \\ & & \\ & & \end{bmatrix}$$

later check starting with 0

pause + try

even more cases!

$$\begin{bmatrix} 0 & 0 & 1 \\ & & 0 \\ & & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 & 1 \\ \boxed{1} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 0 & 1 \\ \boxed{0} & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ & 0 & \\ & 0 & \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ \boxed{1} & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 1 & 0 \\ \boxed{0} & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$P_{3\times3} = \left\{ \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}, \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}, \begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}, \begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}, \begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}, \begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix} \right\}$$

$\boxed{HW1}$ Find $P_{4\times4}$

Hint

Case I
$$\begin{bmatrix}1&0&0&0\\ & & & \\ & & & \\ & & & \end{bmatrix}$$

Case II
$$\begin{bmatrix}0&1&0&0\\ & & & \\ & & & \\ & & & \end{bmatrix}$$

What the follows

Case III
$$\begin{bmatrix}0&0&1&0\\ & & & \\ & & & \\ & & & \end{bmatrix}$$

Case IV
$$\begin{bmatrix}0&0&0&1\\ & & & \\ & & & \\ & & & \end{bmatrix}$$

Each case will lead to a few more.

Linear Algebra

Case I

$$\begin{bmatrix}1&0&0&0\\ & & & \\ & & & \\ & & & \end{bmatrix}$$

what is forced on us?

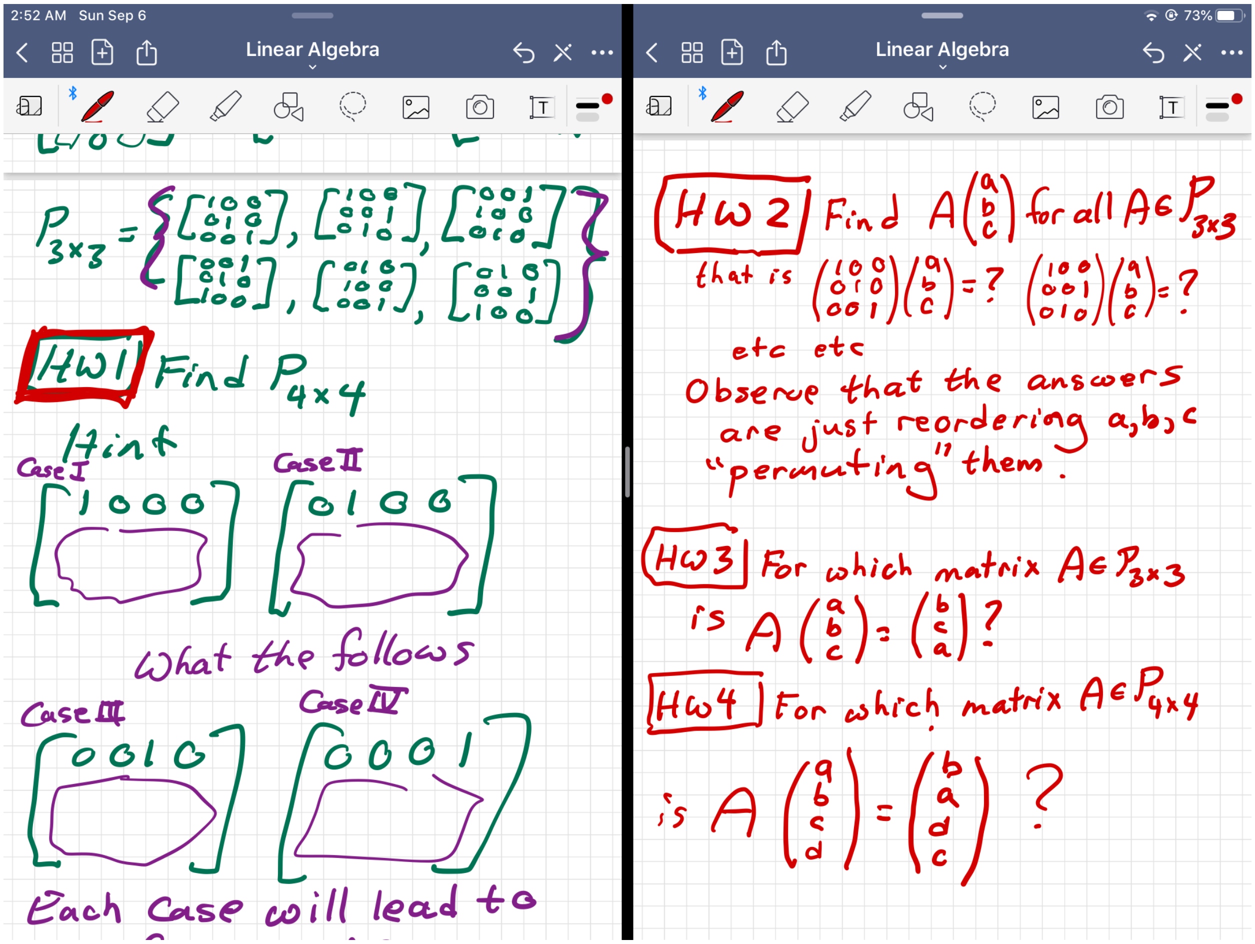
the rest of this column must be 0

$$\begin{bmatrix}1&0&0&0\\0& & & \\0& & & \\0& & & \end{bmatrix}$$

what possibilities fit here?

a 3x3 matrix with a single 1 in each row and a single 1 in each column.

6 possible ways to finish: each of the $P_{3\times3}$ work!



HW1 has a significant amount done within the videos.
HW2-4 have no hints in the videos.

Do all this homework before watching Part 2.

Part II Nonsingular matrices

Watch [Video 313F20-8-5]

3:17 AM Sun Sep 6

Linear Algebra

Lesson 8 Part 2

(5)

Nonsingular Matrices

Defn A nonsingular matrix is a square matrix whose null space is $\{\vec{0}\}$

$$\{A : \text{Null}(A) = \{\vec{0}\}\} \in M_{n \times n}$$

Example I is Nonsingular

$\text{Null}(I) = \{\vec{0}\}$

$$= \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right\}$$

Linear Algebra

Thm Permutation Matrices are Nonsingular Matrices

Proof Idea:

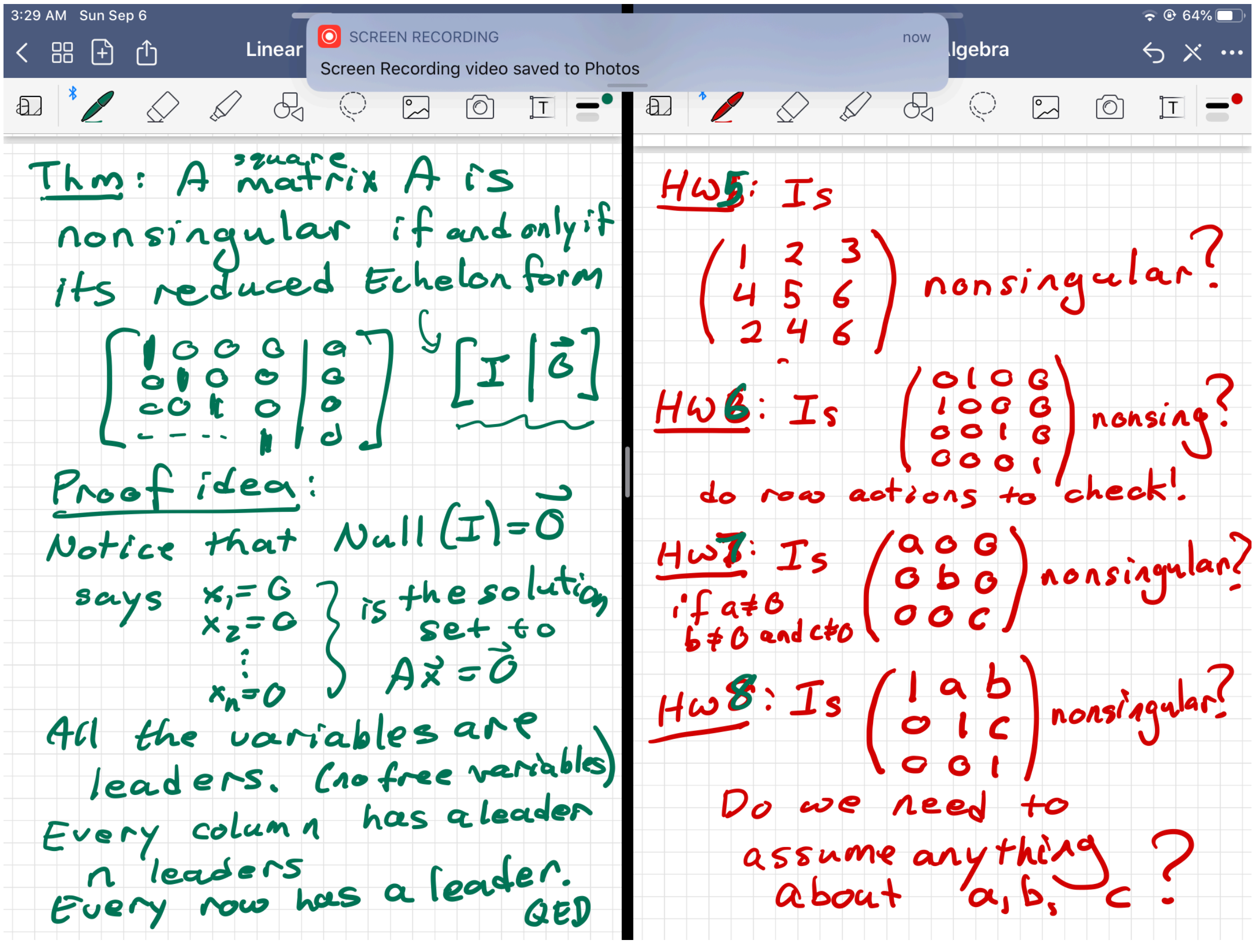
$$\left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

switches → switches →

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

Red Ech Form

Row reduction for a permutation matrix is just switches until we get $\left(I \,\middle|\, \begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right)$



Thm: A square matrix A is nonsingular if and only if its reduced Echelon form

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & \cdots & & 1 & 0 \end{array}\right] = [I \mid \vec{0}]$$

Proof idea:

Notice that $Null(I) = \vec{0}$

says $\left.\begin{array}{l} x_1 = 0 \\ x_2 = 0 \\ \vdots \\ x_n = 0 \end{array}\right\}$ is the solution set to $A\vec{x} = \vec{0}$

All the variables are leaders. (no free variables)

Every column has a leader

n leaders

Every row has a leader.

QED

HW5: Is $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 4 & 6 \end{pmatrix}$ nonsingular?

HW6: Is $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ nonsing?

do row actions to check!

HW7: Is $\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$ nonsingular?

if $a \neq 0$, $b \neq 0$ and $c \neq 0$

HW8: Is $\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$ nonsingular?

Do we need to assume anything about a, b, c?

Watch Video 313F20-8-6 for HW hints

HW1 has a significant amount done within the videos.
HW2-4 have some hints in the videos as well.

Julinda is the grader for these homework problems. Share your google doc with her julinda.pillati@gmail.com when you are ready for her to check it. She is only responsible to tell you if you are right or wrong, not tell you the answers. She will check that your proofs are written in the correct two column method taught in this course. She may request that you resubmit work.

If you had fun figuring out these questions then you might enjoy games like Sudoku as well. Here is a talk about the Mathematics of Games that you might enjoy:

Dr. Alexander Diaz Lopez
A (Mathematical) Journey through Popular Games
(speaking to Lehman College Students)
(the playlist is linked above at the title)

If you don’t have time for the whole video today, you can watch the first and then skip to the game you like. If you think you might like to become a mathematician, he talks about that at the end. Feel free to contact Dr. Alexander Diaz Lopez by clicking on his name above.