Differential Geometry
MAT432 and MAT733
Professor Sormani
Week 9 Implicit Function Theorem

All students must complete Week 8 Part I on the Inverse Function Theorem before starting this lesson because we need the topics. Students who are behind schedule see this checklist.

The first submission (with notes, questions, and attempts of homework) is due
Sun Oct 29 at 10pm
The resubmission (with corrections and completed homework) is due
Sat Nov 4 at 12 noon
Extra Credit may be submitted later

Googledocs: All work will be submitted by sharing your googledoc for this week with the professor using the correct title on that doc stating the course number, the week number and your name:

MAT432F23-Week9-YourNameHere

MAT733F22-Week9-YourNameHere

Please include a selfie.

Week 9 Implicit Function Theorems

This lesson has two parts.
All students should do both parts.

Part I: Implicit Function Theorem
Part II: Applications and Proof of the Implicit Function Theorem

Consult this checklist to see if you are ready for Week 9.

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Part I: Implicit Function Theorem
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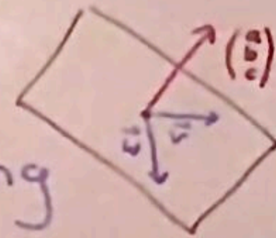
(Seven videos with classwork that is homework between the videos)

We review the implicit definition of a plane and an intersection of planes and how to find a parametric representation of the plane using leading and free variables as in linear algebra. Students are expected to already know linear algebra. This is just a review and an introduction to terminology. See Video Implicit Part 1.

Implicit Definition of a Plane

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : ax + by + cz = d \right\}$$



Parametrizing a plane using two free variables:

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = s \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + t \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} : s,t \in \mathbb{R} \right\}$$

Example $\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : 0x + 1y + 0z = 0 \right\}$

actually the xz plane

$\left(0 \ \boxed{1} \ 0 \,\middle|\, 0 \right)$

y is a leader

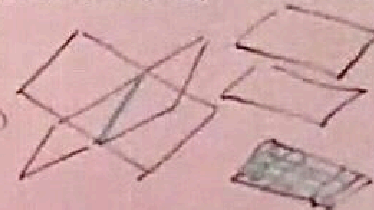
x and z are free

$y = 0$

$$\rightarrow \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ 0 \\ z \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} : x, z \in \mathbb{R} \right\}$$

Implicit Definition of an intersection of Planes

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} : \begin{matrix} a_{11}x_1 + \dots + a_{1n}x_n = d_1 \\ \dots \\ a_{m1}x_1 + \dots + a_{mn}x_n = d_m \end{matrix} \right\}$$

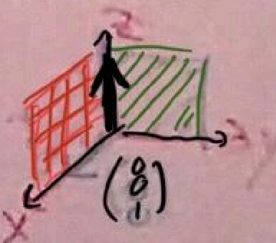


Solve the linear system

$$\left\{ \begin{pmatrix} x\\y\\z\end{pmatrix} = s\begin{pmatrix} v_1\\v_2\\v_3\end{pmatrix} \text{ or } = s\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix} + t\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix} \dots \right. : s,t\in\mathbb{R} \text{ free variables}$$

The same set parametrized by the free variables

Example $\left\{\begin{pmatrix}x\\y\\z\end{pmatrix} : \begin{matrix} 0x+1y+0z=0 \\ 1x+0y+0z=0\end{matrix}\right\}$

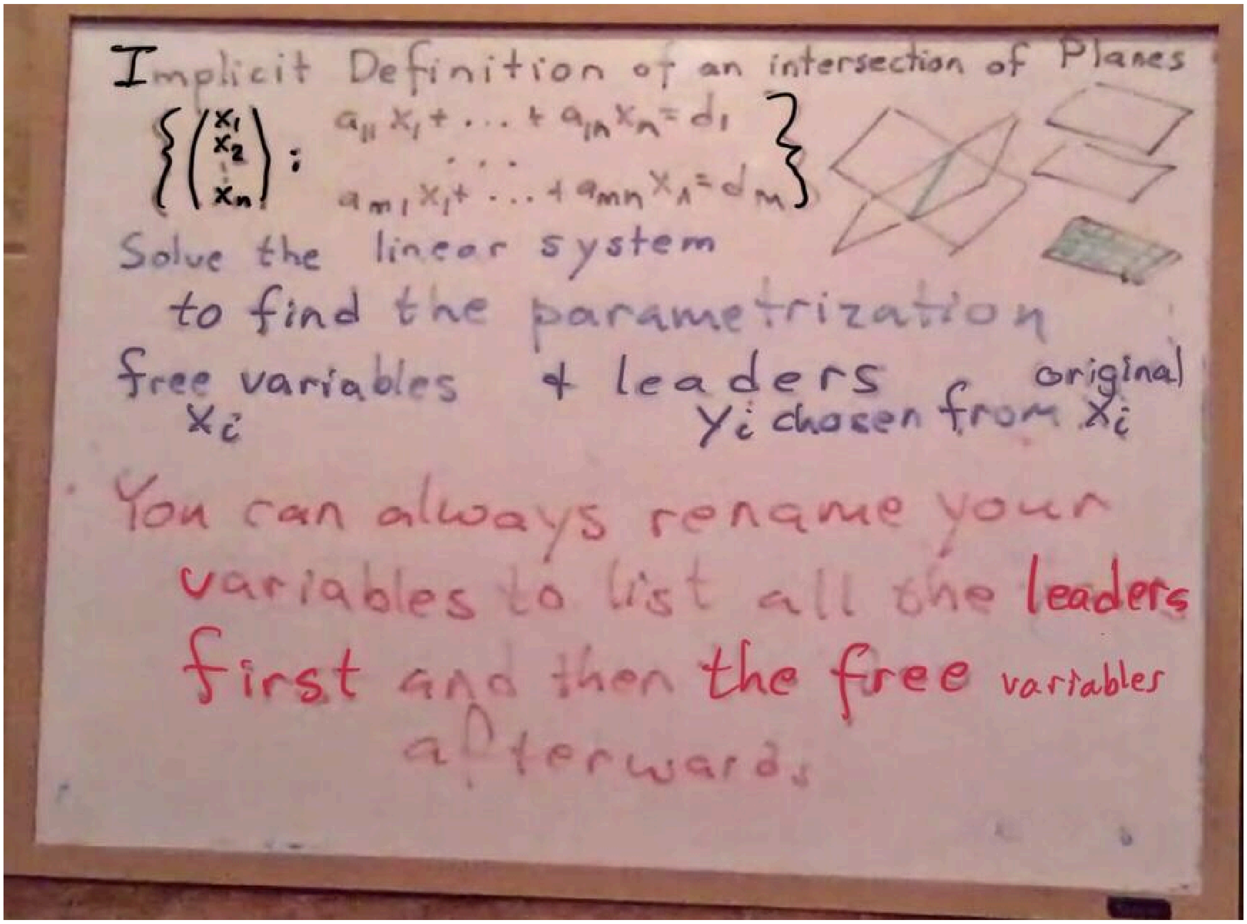


Solve the system

$$\left(\begin{array}{ccc|c} 0&1&0&0\\1&0&0&0\end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} \boxed{1}&0&0&0\\0&\boxed{1}&0&0\end{array}\right)$$

free $z=z$
leaders $x=0$
$y=0$

parametric set $\left\{\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0\\0\\z\end{pmatrix} = z\begin{pmatrix}0\\0\\1\end{pmatrix} \quad z\in\mathbb{R}\right\}$ a line z axis



We next review a theorem from linear algebra which states that a full rank system of linear equations can always be written in parametrized form and introduce the idea of reordering the variables so that the leaders are first and renamed as y variables and the free variables are second and renamed as x variables. See [Video Implicit Part 2](#).

Implicit Definition of an intersection of Planes

$$\hookrightarrow \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} : \begin{array}{l} a_{11} x_1 + \dots + a_{1m} x_n = d_1 \\ \quad \vdots \\ a_{m1} x_1 + \dots + a_{mn} x_n = d_m \end{array} \right\}$$

Solve system to find the parametrization

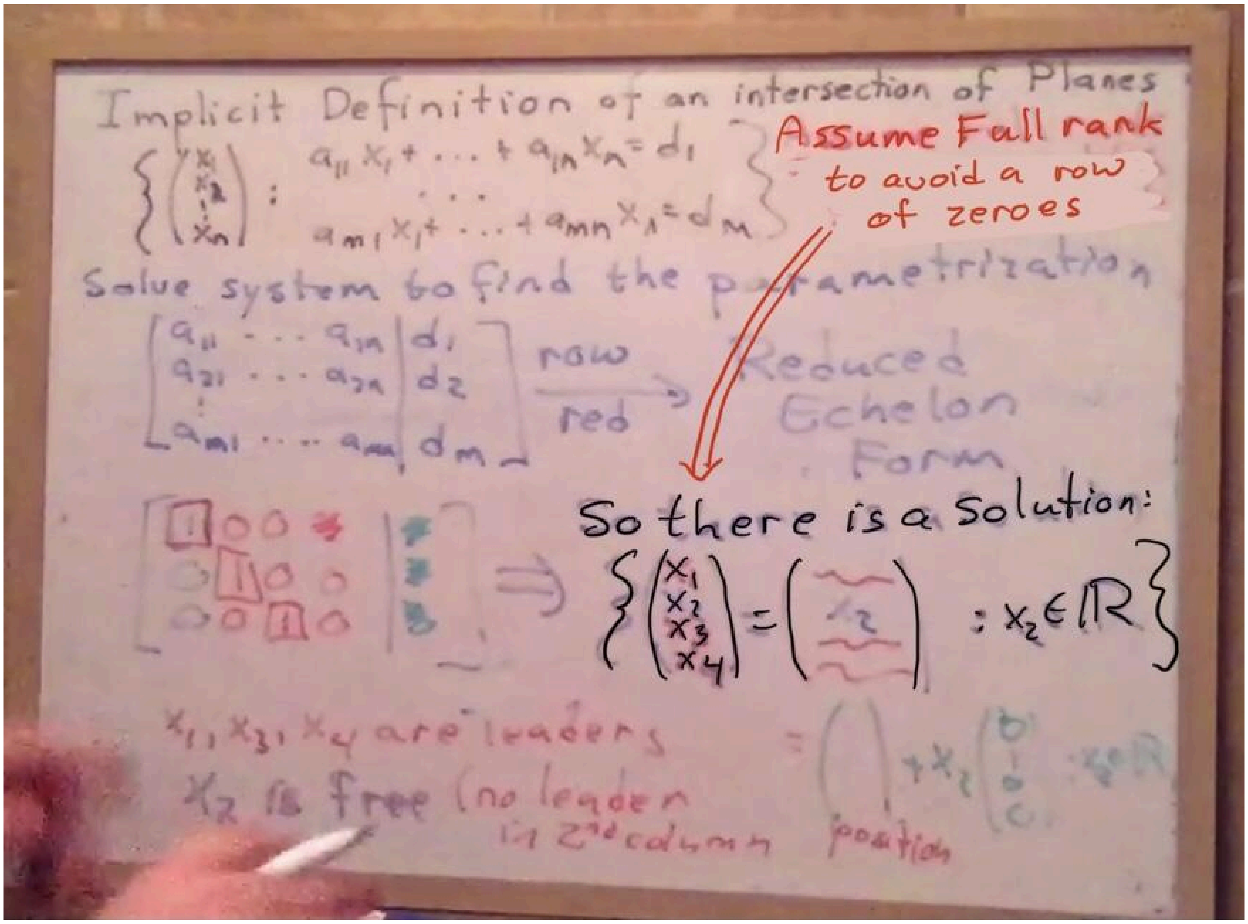
$$\left[\begin{array}{ccc|c} a_{11} & \dots & a_{1m} & d_1 \\ a_{21} & \dots & a_{2m} & d_2 \\ \vdots & & & \\ a_{n1} & \dots & a_{nm} & d_m \end{array}\right] \xrightarrow[\text{red}]{\text{row}} \text{Reduced Echelon Form}$$

$$\left[\begin{array}{cccc|c} \boxed{1} & * & 0 & 0 & * \\ 0 & 0 & \boxed{1} & 0 & * \\ 0 & 0 & 0 & \boxed{1} & * \\ 0 & 0 & 0 & 0 & * \end{array}\right]$$

we must avoid row of zeroes! This could lead to no solution!

x_1, x_3, x_4 are leaders
x_2 is free (no leader in 2nd column

We prefer every row was a leader (Full rank)


Implicit Definition of an intersection of Planes
$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} : \begin{matrix} a_{11}x_1 + \dots + a_{1n}x_n = d_1 \\ \dots \\ a_{m1}x_1 + \dots + a_{mn}x_n = d_m \end{matrix} \right\}$
Assume Full rank
to avoid a row of zeroes
Solve system to find the parametrization
$\left[\begin{array}{ccc|c} a_{11} & \dots & a_{1n} & d_1 \\ a_{21} & \dots & a_{2n} & d_2 \\ \vdots & & & \\ a_{m1} & \dots & a_{mn} & d_m \end{array}\right] \xrightarrow[\text{red}]{\text{row}}$ Reduced Echelon Form
$\left[\begin{array}{cccc|c} \boxed{1} & 0 & 0 & * & * \\ 0 & \boxed{1} & 0 & 0 & * \\ 0 & 0 & \boxed{1} & 0 & * \end{array}\right] \Rightarrow$
So there is a solution:
$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \sim \\ x_2 \\ \sim \\ \sim \end{pmatrix} : x_2 \in \mathbb{R} \right\}$
x_1, x_3, x_4 are leaders
x_2 is free (no leader in 2nd column
$= \begin{pmatrix} \\ \\ \\ \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} : x_2 \in \mathbb{R}$
position

Implicit Definition of an intersection of Planes

$$\left\{ \begin{pmatrix} x_1\\ x_2\\ \vdots\\ x_n\end{pmatrix} : \begin{matrix} a_{11}x_1+\dots+a_{1n}x_n=d_1\\ \dots \\ a_{m1}x_1+\dots+a_{mn}x_n=d_m\end{matrix}\right\}$$

Assume Full rank

Reorder variables so leaders are first

Solve system to find the parametrization

$$\left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & d_1\\ a_{21} & \cdots & a_{2n} & d_2\\ \vdots & & & \\ a_{m1} & \cdots & a_{mn} & d_m \end{array}\right] \xrightarrow[\text{red}]{\text{row}} \text{Reduced Echelon Form}$$

$$\left[\begin{array}{cccc|c} \boxed{1} & 0 & 0 & * & *\\ 0 & \boxed{1} & 0 & 0 & *\\ 0 & 0 & \boxed{1} & 0 & * \end{array}\right]$$

first columns have leaders

So there is a solution

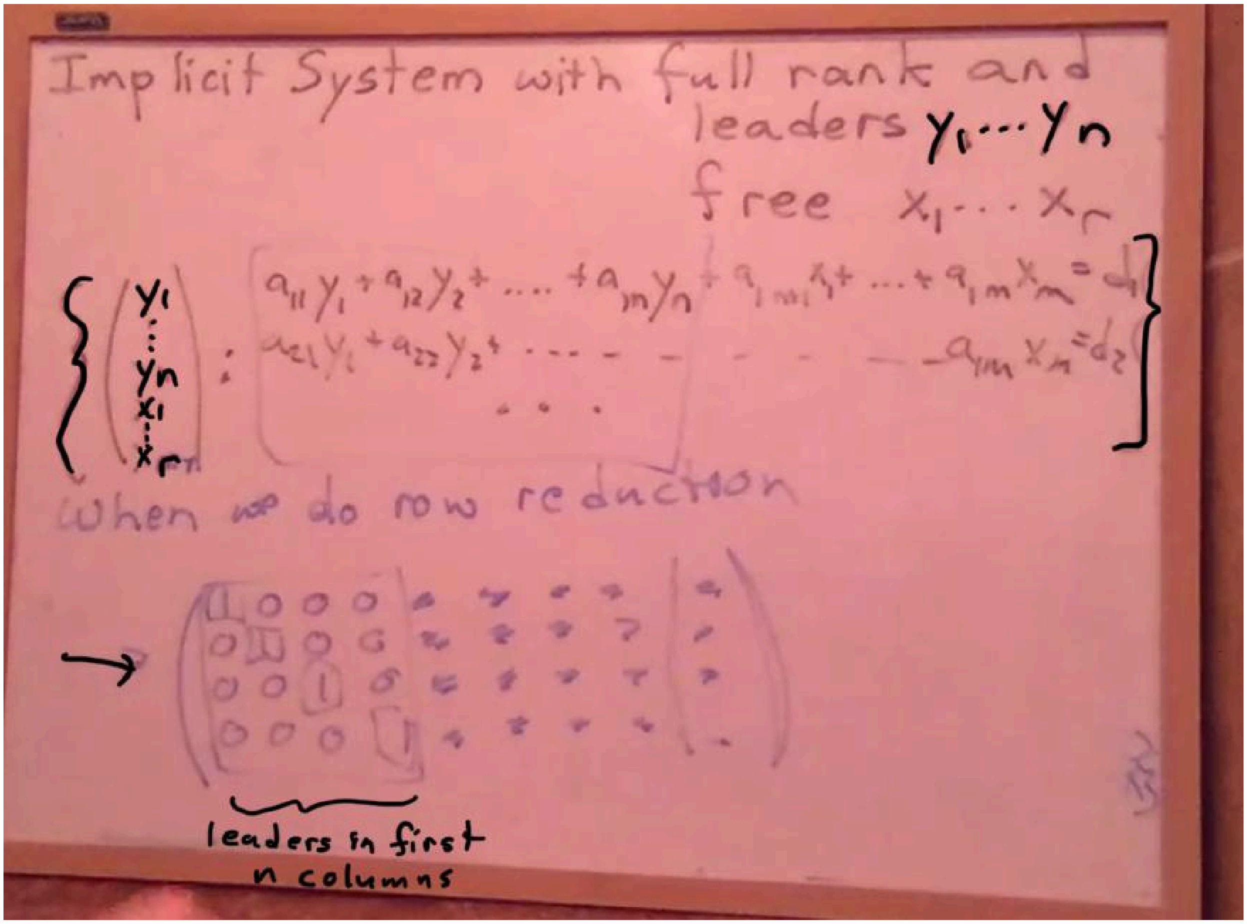
$$\Rightarrow \left\{\begin{pmatrix} x_1\\ x_2\\ x_3\\ x_4\end{pmatrix} = \begin{pmatrix} * + * x_4\\ *\\ *\\ x_4\end{pmatrix} : x_2 \in \mathbb{R}\right\}$$

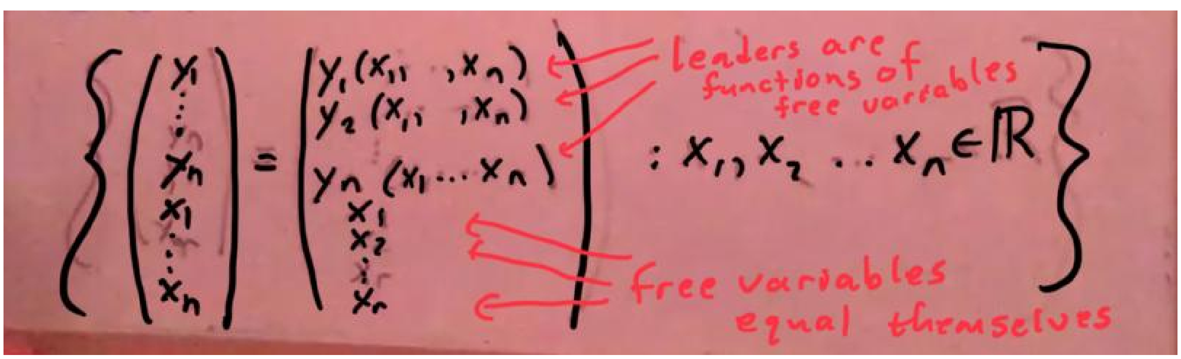
x_1 x_2 x_3 are leaders

$$= \begin{pmatrix} \\ \\ \\ \end{pmatrix} + x_4 \begin{pmatrix} *\\ 0\\ 0\\ 1\end{pmatrix} : x_4 \in \mathbb{R}$$

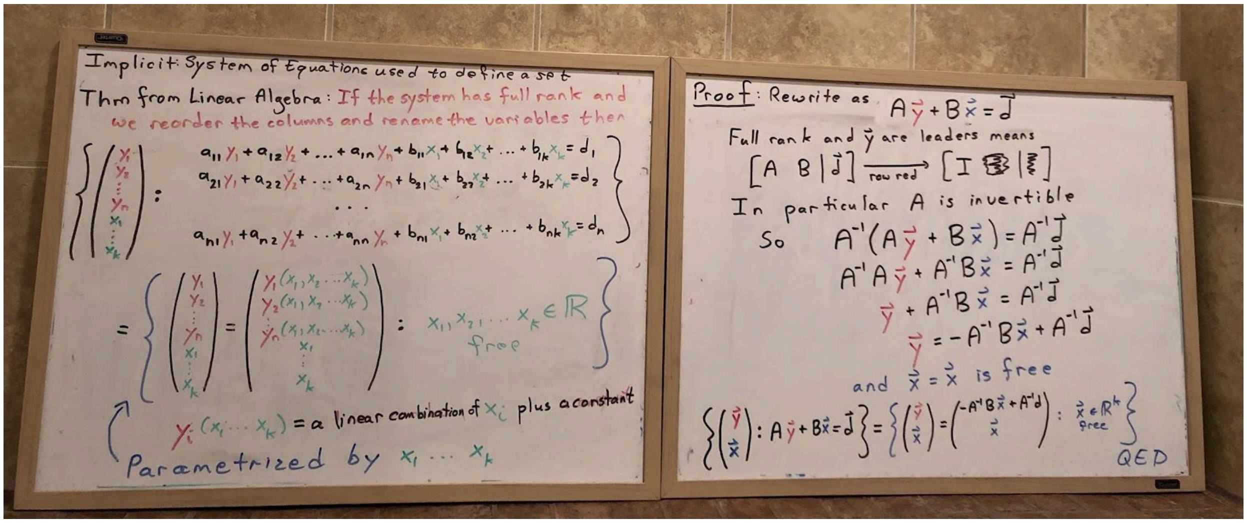
position

x_4 is free (no leader in 2nd column

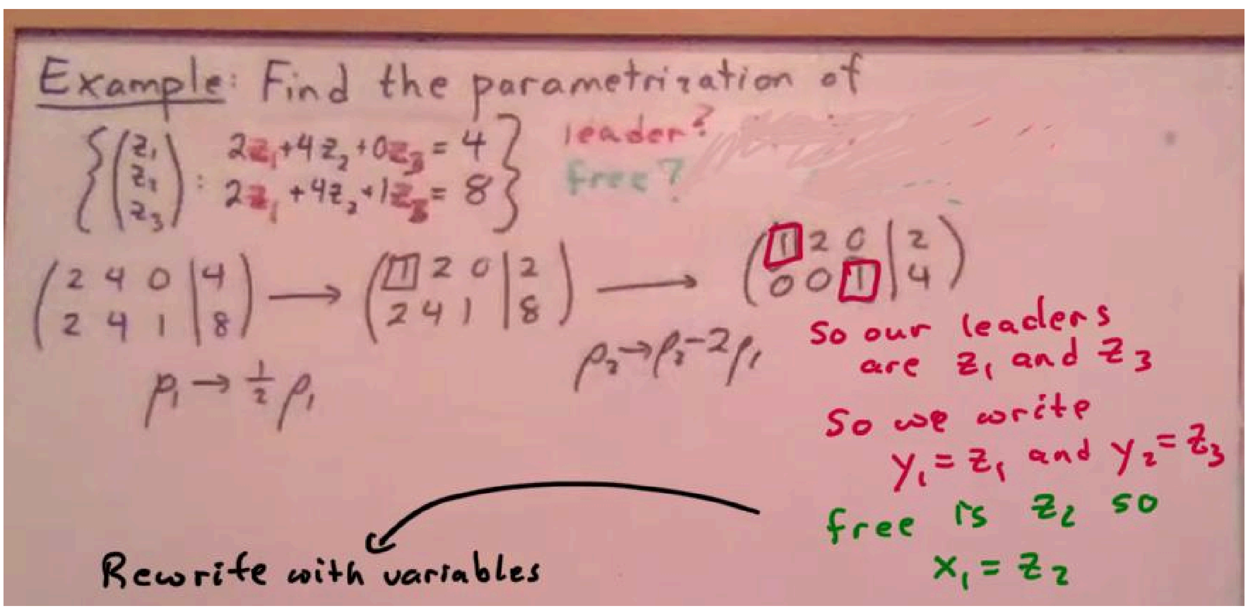




We may now view this as an Implicit Function Theorem for systems of linear equations as in the photo. In the photo we have already reordered the system with the leading variables y first and the free variables second so that the parametrization has formulas for the leading variables depending on the free variables. We provide a constructive proof as well. The constructive proof shows exactly how to find the solution. See [Video Implicit Part 3]:



We work out an example from Linear Algebra explicitly identifying the leaders and free variables and reordering and renaming them and then solving them imitating the constructive proof in the photo above. See [Video Implicit Part 4]()



Example: Find the parametrization of

$$\left\{ \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} : \begin{matrix} 2z_1 + 4z_2 + 0z_3 = 4 \\ 2z_1 + 4z_2 + 1z_3 = 8 \end{matrix} \right\}$$

leader? $z_1 = y_1$ $z_3 = y_2$ $\vec{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

free? $z_2 = x_2$ $\vec{x} = x_1$

$$= \left\{ \begin{pmatrix} y_1 \\ y_2 \\ x_1 \end{pmatrix} : \begin{matrix} 2y_1 + 0y_2 + 4x_1 = 4 \\ 2y_1 + 1y_2 + 4x_1 = 8 \end{matrix} \right\} = \left\{ \begin{pmatrix} \vec{y} \\ \vec{x} \end{pmatrix} : \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix} \vec{y} + \begin{pmatrix} 4 \\ 4 \end{pmatrix} \vec{x} = \begin{pmatrix} 4 \\ 8 \end{pmatrix} \right\}$$

Parametrize this set by solving the system

$$= \left\{ \begin{pmatrix} \vec{y} \\ \vec{x} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix}^{-1} \left(-\begin{pmatrix} 4 \\ 4 \end{pmatrix} \vec{x} + \begin{pmatrix} 4 \\ 8 \end{pmatrix} \right) \\ \vec{x} \end{pmatrix} : \vec{x} \in \mathbb{R}^1 \right\}$$

$$\begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix}^{-1} \left(-\begin{pmatrix} 4x_1 \\ 4x_1 \end{pmatrix} + \begin{pmatrix} 4 \\ 8 \end{pmatrix} \right) = \begin{pmatrix} 1/2 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -4x_1 + 4 \\ -4x_1 + 8 \end{pmatrix} = \begin{pmatrix} -2x_1 + 2 \\ 4 \end{pmatrix}$$

$$= \left\{ \begin{pmatrix} y_1 \\ y_2 \\ x_1 \end{pmatrix} = \begin{pmatrix} -2x_1 + 2 \\ 4 \\ x_1 \end{pmatrix} : x_1 \in \mathbb{R} \right\}$$

$\leftarrow$ parametrized with leaders first

HW1-HW4 are (a)-(d) below: using the method above.
Try each before looking at its solution.

Classwork: Find the parametrization of the following systems:

ⓐ $\left\{ \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} : \begin{array}{l} z_2 + z_3 = 0 \\ z_1 + z_2 + z_3 = 0 \end{array} \right\}$

ⓑ $\left\{ \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} : \begin{array}{l} z_1 + z_2 + z_3 + z_4 = 0 \\ z_1 + z_2 + 2z_3 + 3z_4 = 0 \\ z_1 + 2z_2 + z_3 + z_4 = 0 \end{array} \right\}$

ⓒ $\left\{ \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} : \begin{array}{l} 2z_1 + 2z_2 + 2z_3 + 2z_4 = 4 \\ z_1 + z_2 + 2z_3 + 3z_4 = 8 \end{array} \right\}$

ⓓ $\left\{ \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} : \begin{array}{l} z_1 + z_2 + z_3 = 2 \\ 2z_1 + 2z_2 + 2z_3 = 5 \end{array} \right\}$

Solutions:

ⓐ $\left(\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 0 \\ 0 & \boxed{1} & 1 & 0 \end{array}\right)$ leaders $y_1 = z_1$ $y_2 = z_2$, free $x_1 = z_3$

$\left\{ \begin{pmatrix} y_1 \\ y_2 \\ x_1 \end{pmatrix} : \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} x_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$

parametrized is $\left\{ \begin{pmatrix} y_1 \\ y_2 \\ x_1 \end{pmatrix} = \begin{pmatrix} y_1(x_1) \\ y_2(x_1) \\ x_1 \end{pmatrix} : x_1 \in \mathbb{R} \right\}$ where $\begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{-1} \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} x_1 \right)$

$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^{-1}$ is found: $\left(\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right)$ check $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\left\{ \begin{pmatrix} y_1 \\ y_2 \\ x_1 \end{pmatrix} = \begin{pmatrix} y_1(x_1) \\ y_2(x_1) \\ x_1 \end{pmatrix} : x_1 \in \mathbb{R} \right\}$ where $\begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix} x_1 \right) = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -x_1 \\ -x_1 \end{pmatrix} = \begin{pmatrix} x_1 - x_1 \\ -x_1 \end{pmatrix} = \begin{pmatrix} 0 \\ -x_1 \end{pmatrix}$

ⓑ $\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 2 & 3 & 0 \\ 1 & 2 & 1 & 1 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{cccc|c} \boxed{1} & 1 & 1 & 1 & 0 \\ 0 & \boxed{1} & 0 & 0 & 0 \\ 0 & 0 & \boxed{1} & 2 & 0 \end{array}\right)$ leaders $y_1 = z_1$ $y_2 = z_2$ $y_3 = z_3$, free $x_1 = z_4$

and now finish as in ⓐ

ⓒ $\left(\begin{array}{cccc|c} 2 & 2 & 2 & 2 & 4 \\ 1 & 1 & 2 & 3 & 8 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 2 & 3 & 8 \end{array}\right) \to \left(\begin{array}{cccc|c} \boxed{1} & 1 & 1 & 1 & 2 \\ 0 & 0 & \boxed{1} & 2 & 6 \end{array}\right)$ leaders $y_1 = z_1$ $y_2 = z_3$, free $x_1 = z_2$ $x_2 = z_4$

$\left\{ \begin{pmatrix} y_1 \\ y_2 \\ x_1 \\ x_2 \end{pmatrix} : \begin{array}{l} 2y_1 + 2x_1 + 2y_2 + 2x_2 = 4 \\ y_1 + x_1 + 2y_2 + 3x_3 = 8 \end{array} \right\} = \left\{ \begin{pmatrix} y_1 \\ y_2 \\ x_1 \\ x_2 \end{pmatrix} : \begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \end{pmatrix} \right\}$

$= \left\{ \begin{pmatrix} y_1 \\ y_2 \\ x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y_1(x_1, x_2) \\ y_2(x_1, x_2) \\ x_1 \\ x_2 \end{pmatrix} : x_1, x_2 \in \mathbb{R} \right\}$ where $\begin{pmatrix} y_1(x_1, x_2) \\ y_2(x_1, x_2) \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix}^{-1} \left(\begin{pmatrix} 4 \\ 8 \end{pmatrix} - \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right)$

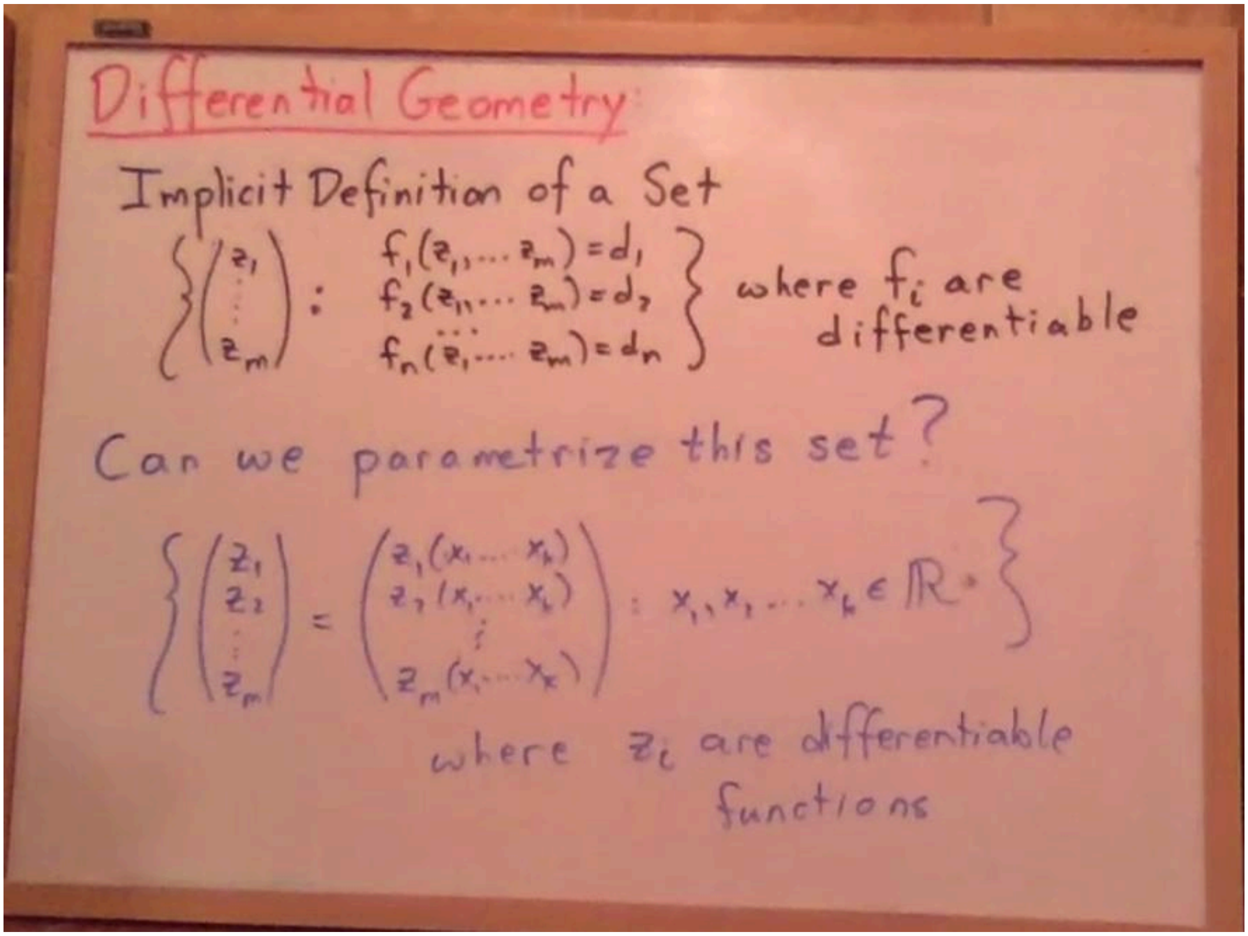
so find the inverse to finish.

ⓓ $\left\{ \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} : \begin{array}{l} z_1 + z_2 + z_3 = 2 \\ 2z_1 + 2z_2 + 2z_3 = 5 \end{array} \right\}$ $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 2 & 2 & 5 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{array}\right)$ row of zeroes ending in one

no solution!

Not full rank

Implicit Systems of Differentiable Equations: **Can they be parametrized too?**
See [Video Implicit Part 5]

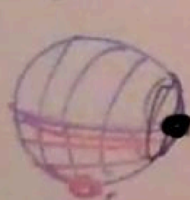


Differential Geometry

Implicit Definition of a Set $\left\{ \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} : z_1^2 + z_2^2 + z_3^2 = 1 \right\}$

We can parametrize locally near $p = (0,1,0)$

$$= \left\{ \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} z_1(z_2, z_3) \\ z_2 \\ z_3 \end{pmatrix} : (z_2, z_3) \in U \right\}$$ where U is an open set $\mathbb{R}^2$ about $(0,0)$



$p=(0,1,0)$

$$z_1^2 + z_2^2 + z_3^2 = 1$$

$$z_1^2 = 1 - z_2^2 - z_3^2$$

$$z_1 = \pm\sqrt{1 - z_2^2 - z_3^2}$$

Not a function near $p=(0,1,0)$

Do not solve for z_1 ← Not a good leader

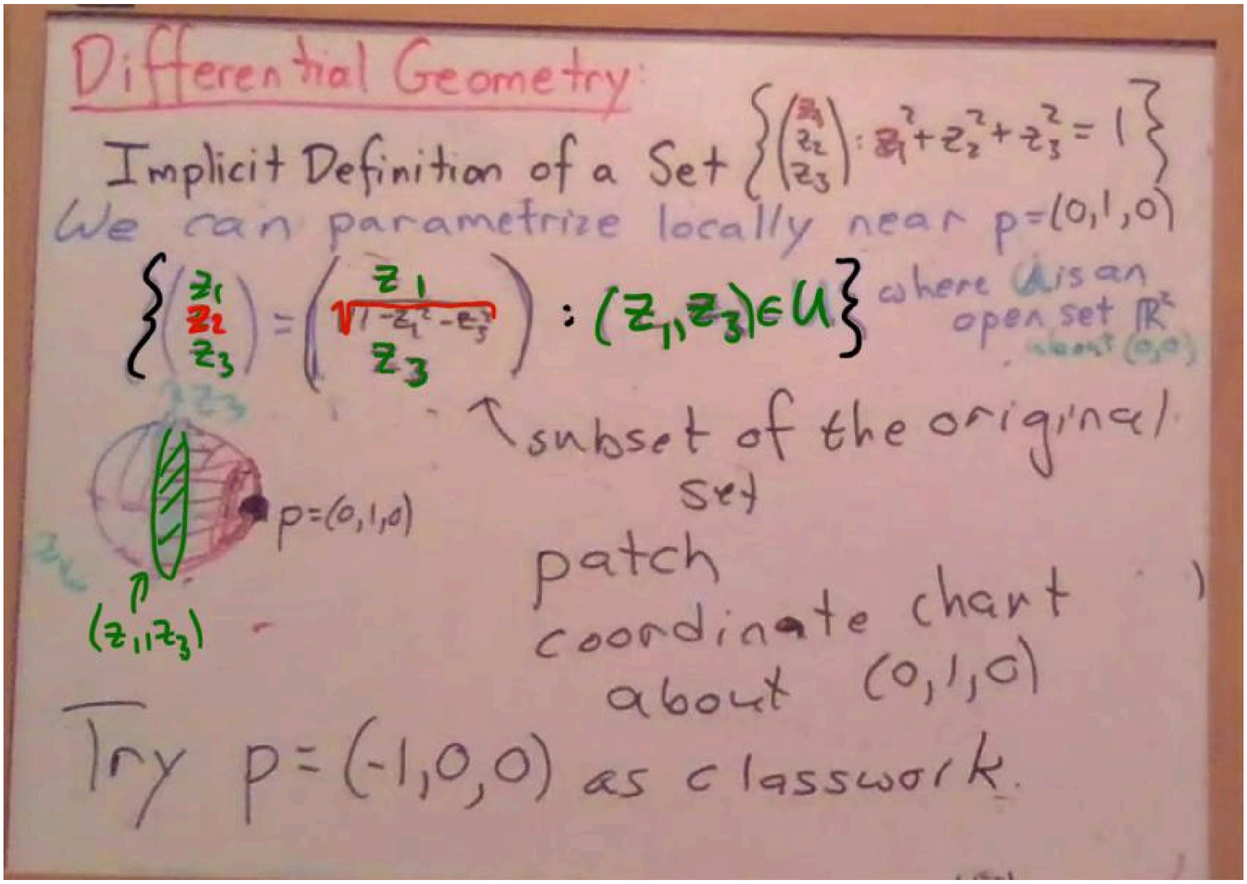
Do not solve for z_1

z_2 is a better leader near

Try $z_2 = +\sqrt{1 - z_1^2 - z_3^2}$ near $p = (0,1,0)$

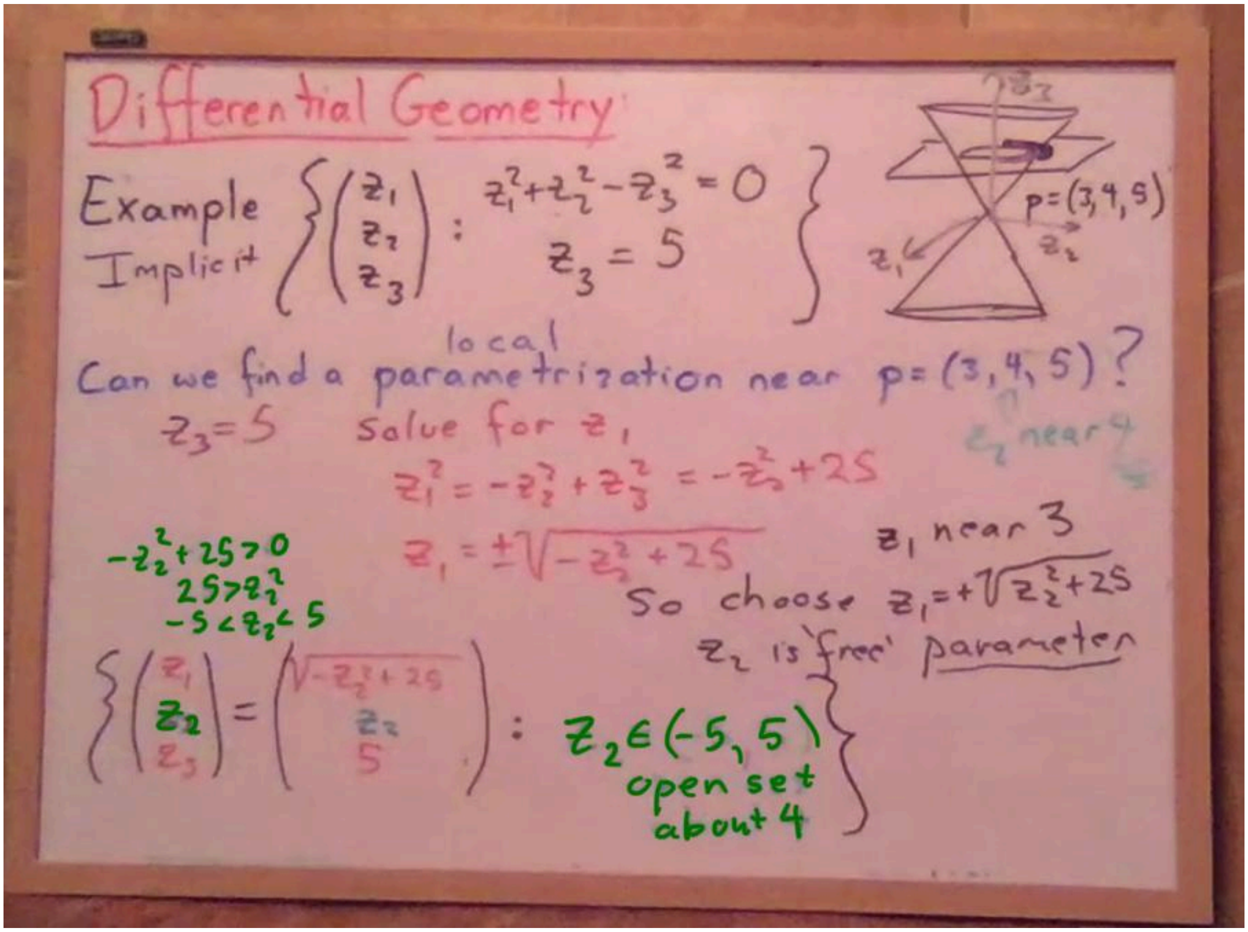
Want $z_2 > 0$ Need $z_1^2 + z_3^2 \leq 1$ So $U = B_{(0,0)}(1)$

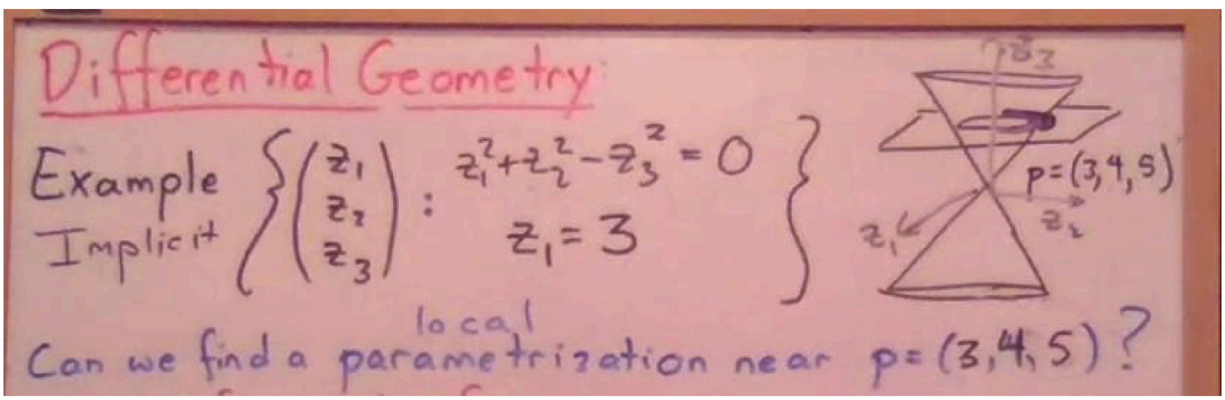
$(z_1, z_3) \in B_{(0,0)}(1)$



HW5: (e) Try the question at the end of the above video finding a local parametrization and use MATLAB [Implicit Plotting](#) to plot the original sphere and use curve plotting to plot the parametrization curve in a different color.

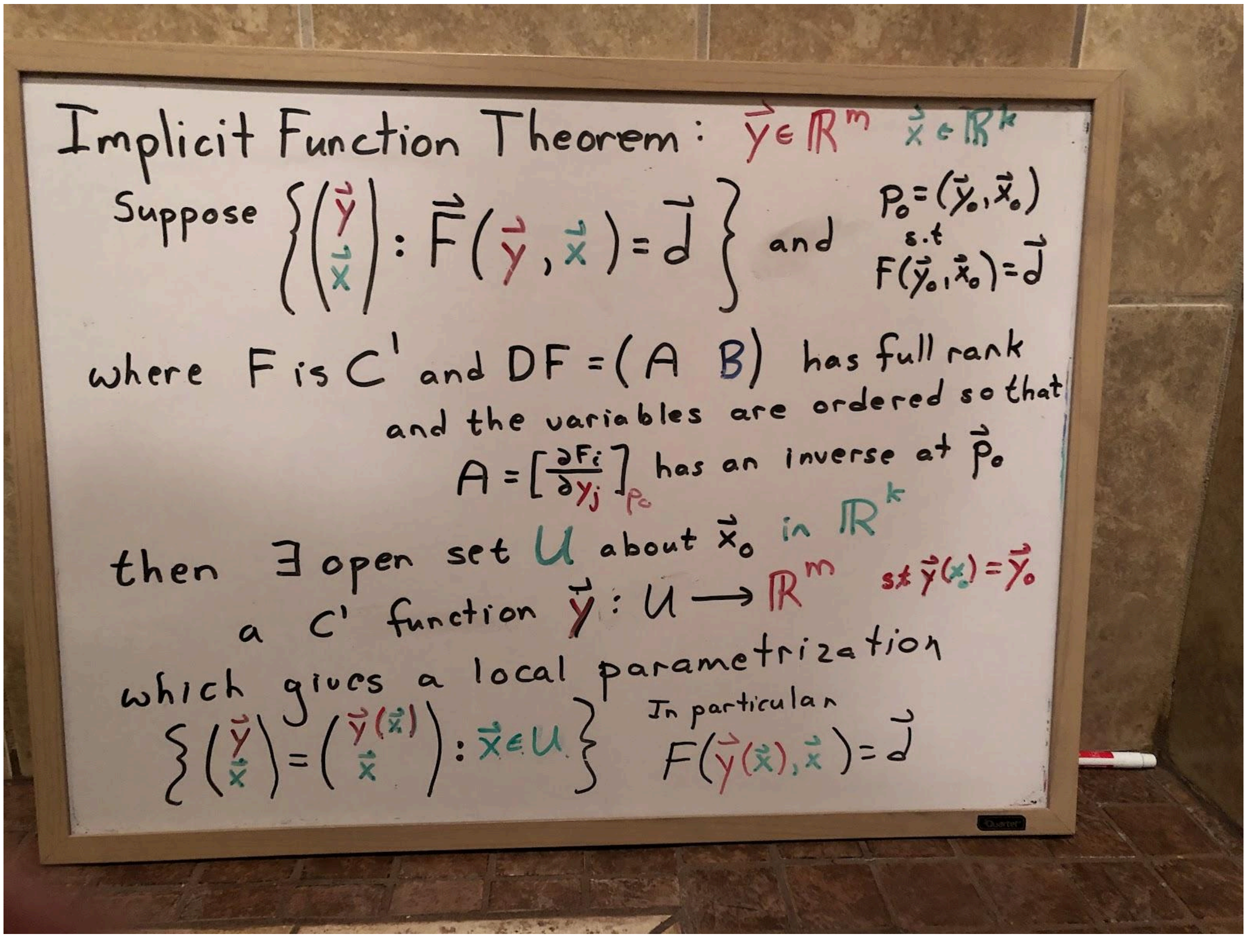
See [Video Implicit Part 6](#) considering a cone intersected with a plane.





HW6: (f) Try the question at the end of the video finding a local parametrization and use MATLAB [Implicit Plotting] to plot the original cone and plane and use curve plotting to plot the parametrization curve in a different color.

Finally we state the implicit function theorem for systems of equations involving differentiable functions:



This above statement is explained and applied with two examples in Video Implicit Part 7

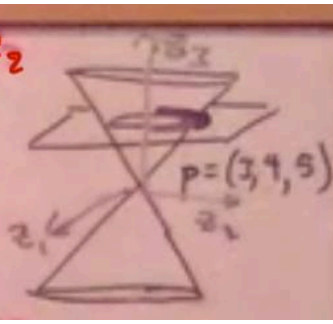
Differential Geometry

$y_1 = z_1 \quad y_2 = z_2$

$x_1 = z_2$

Example Implicit

$$\left\{ \begin{pmatrix} y_1 \\ y_2 \\ x_1 \end{pmatrix} : \begin{matrix} y_1^2 + x_1^2 - y_2^2 = 0 \\ y_2 = 5 \end{matrix} \right\}$$



Differential Geometry

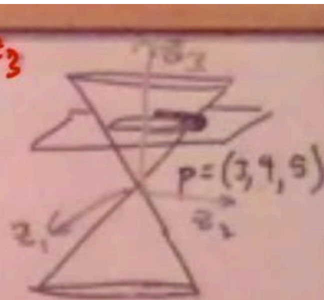
$y_1 = z_1 \quad y_2 = z_3$

$x_1 = z_2$

Reordered:

Example Implicit

$$\left\{ \begin{pmatrix} y_1 \\ y_2 \\ x_1 \end{pmatrix} : \begin{matrix} y_1^2 + x_1^2 - y_2^2 = 0 \\ y_2 = 5 \end{matrix} \right\}$$



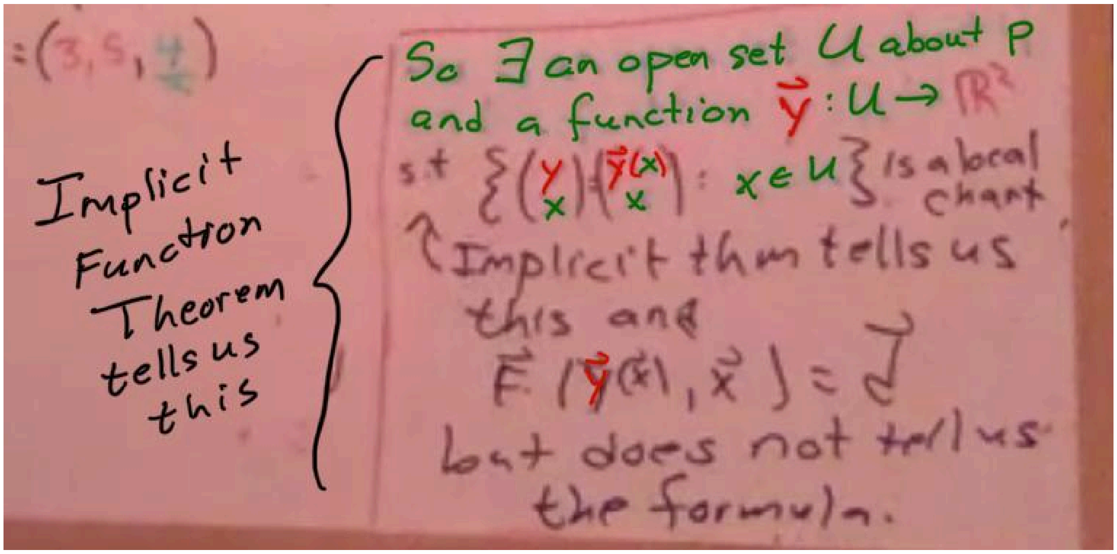
$$F\begin{pmatrix} y_1 \\ y_2 \\ x_1 \end{pmatrix} = \begin{pmatrix} y_1^2 + x_1^2 - y_2^2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$$

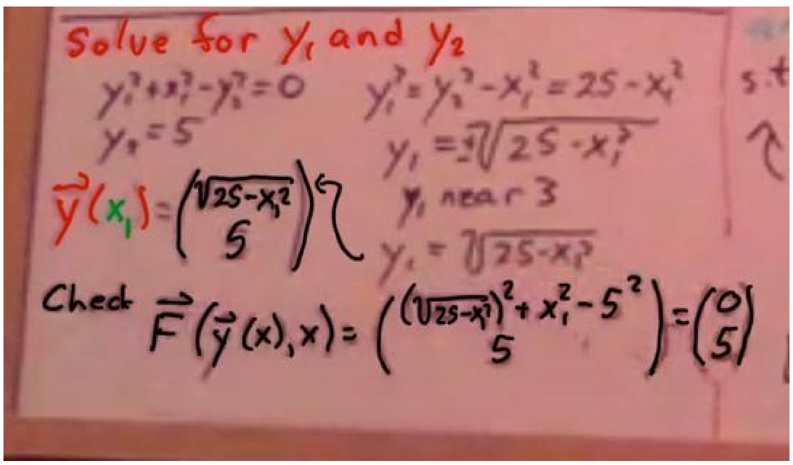
$$\left\{ \vec{z} : \vec{F}(\vec{z}) = \begin{pmatrix} 0 \\ 5 \end{pmatrix} \right\} \qquad \vec{F} : \mathbb{R}^3 \longrightarrow \mathbb{R}$$

$$\underset{(3,5,4)}{DF} = \begin{pmatrix} \underbrace{6 \quad -10}_{} & 8 \\ 0 \quad\ \ 1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 8 \\ 0 \end{pmatrix}$$

$A = \begin{pmatrix} 6 & -10 \\ 0 & 1 \end{pmatrix}$ has an inverse

because $\det(A) = 6 \cdot 1 - (-10) \cdot 0 = 6 \neq 0$





****
Part II: Proof of Inverse and Implicit Function Theorem
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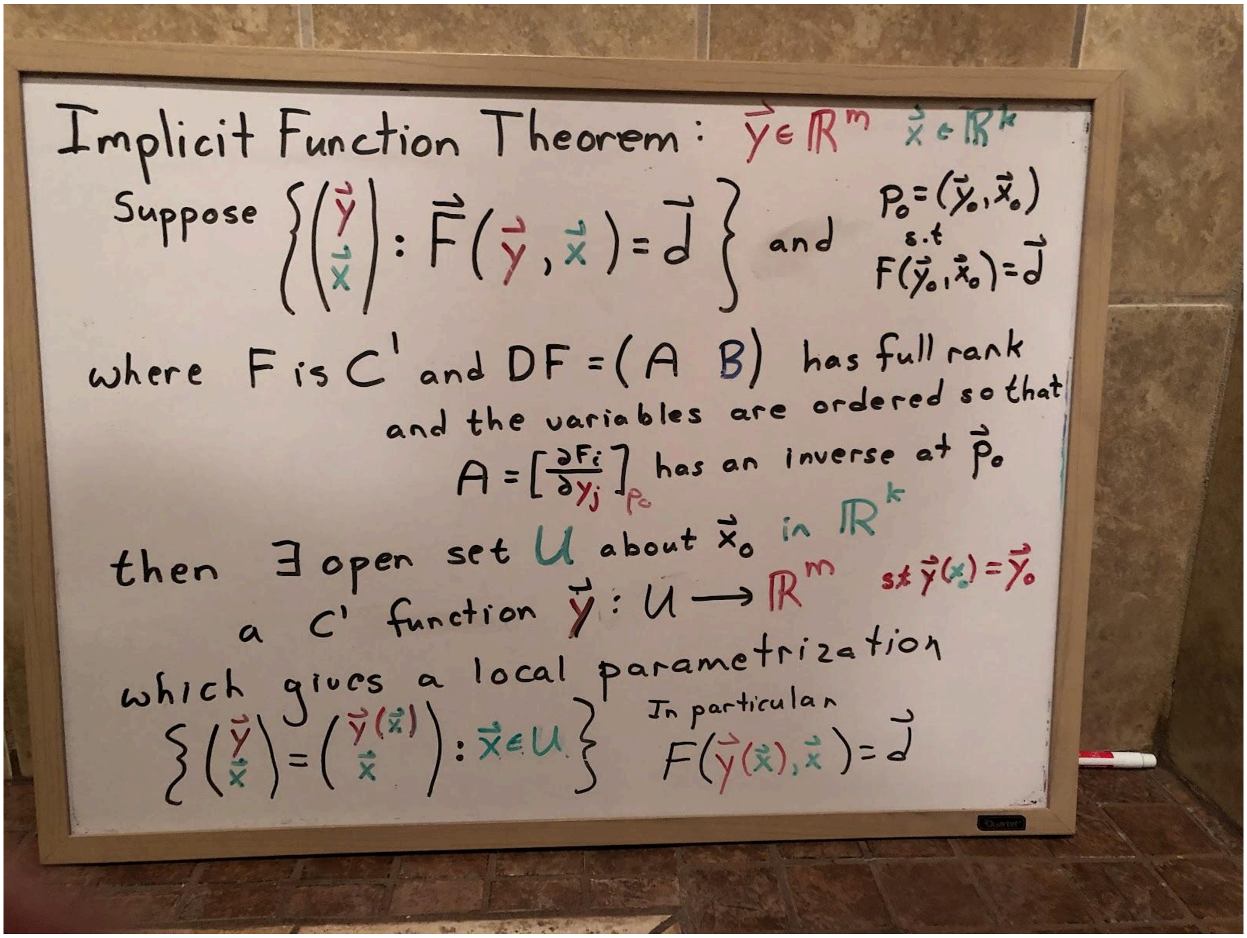
Implicit Function Theorem:
Constructive Proof and Applications

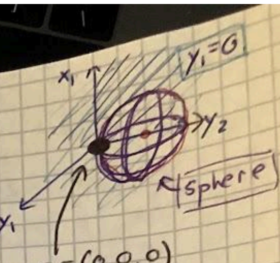
If you wish rewatch [Video Implicit Part 7] for an explanation of this statement.



Next will provide a constructive proof of this theorem and provide applications.

Do the sphere-plane classwork in the photo below where you find the local parametrization of a sphere intersected with a plane. The variables are already ordered correctly. Try each step before looking at the answer:

Consider $\left\{ \begin{pmatrix} y_1 \\ y_2 \\ x_1 \end{pmatrix} : \begin{matrix} y_1 = 0 \\ y_1^2 + (y_2-4)^2 + x_1^2 = 4^2 \end{matrix} \right\}$



$p=(0,0,0)$

$$= \left\{ \begin{pmatrix} y_1 \\ y_2 \\ x_1 \end{pmatrix} : F\begin{pmatrix} y_1 \\ y_2 \\ x_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

where $F\begin{pmatrix} y_1 \\ y_2 \\ x_1 \end{pmatrix} = \big($ Fill in $\big)$ Answer $= \begin{pmatrix} y_1 \\ y_1^2+(y_2-4)^2+x_1^2-4^2 \end{pmatrix}$

Verify $F\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \big($ Fill in $\big)$ Answer $= \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$DF = \big($ Fill in $\big)$ Answer $= \begin{pmatrix} 1 & 0 & 0 \\ 2y_1 & 2(y_2-4) & 2x_1 \end{pmatrix}$

$DF_p = \big($ Fill in $\big)$ Answer $= \begin{pmatrix} \boxed{\begin{matrix} 1 & 0 \\ 0 & -8 \end{matrix}} & \begin{matrix} 0 \\ 0 \end{matrix} \end{pmatrix}$

A

Note A^{-1} exists so Full rank
so can apply implicit function thm.

So lets try to solve explicitly for $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

$y_1 = 0$ so plug in $0^2 + (y_2-4)^2 + x_1^2 = 4^2$

$$(y_2-4)^2 = 4^2 - x_1^2$$
$$y_2 - 4 = \pm\sqrt{4^2-x_1^2}$$
$$y_2 = 4 \pm \sqrt{4^2-x_1^2}$$

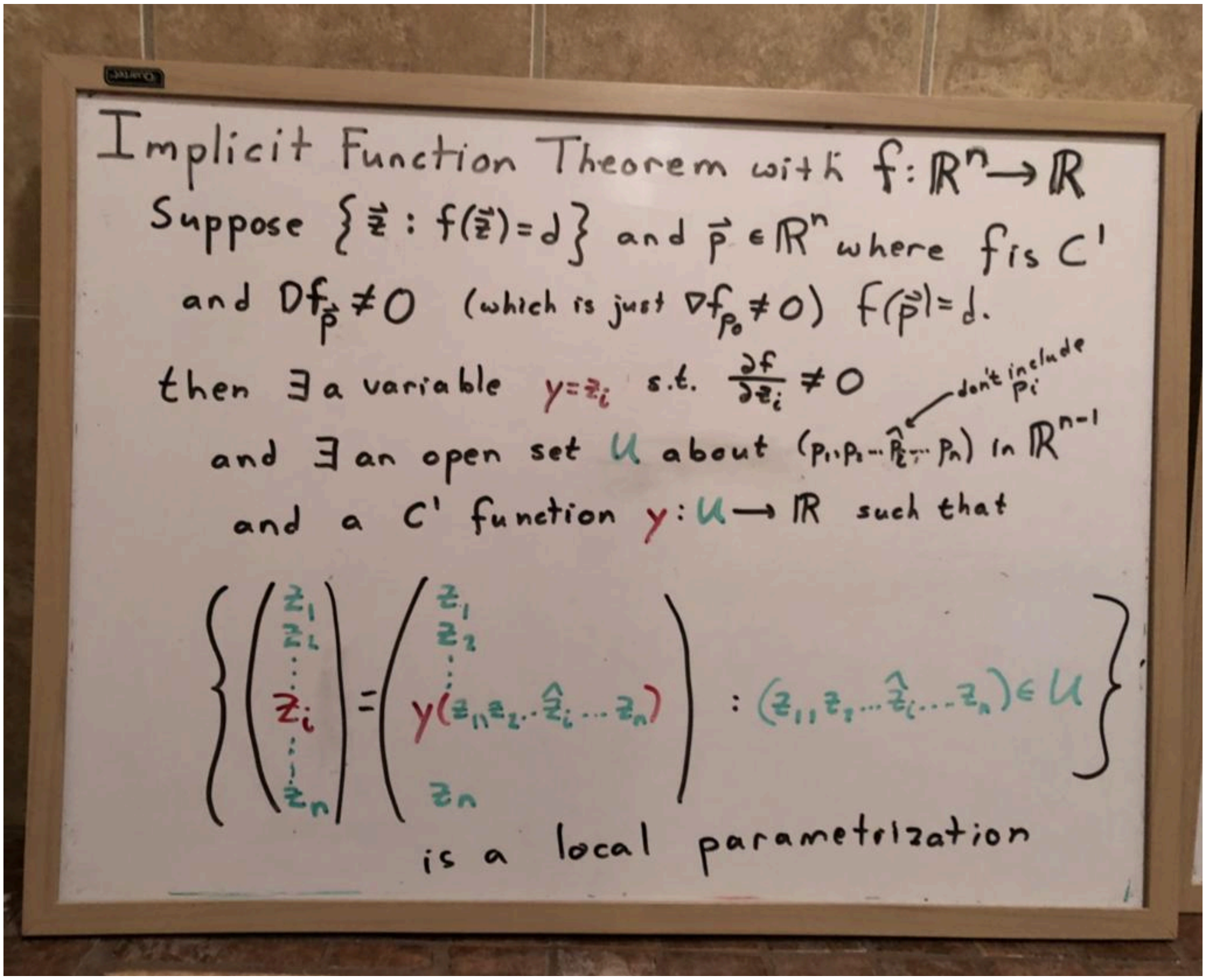
$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 4-\sqrt{4^2-x_1^2} \end{pmatrix}$$

needs $4^2-x_1^2 > 0$
$4^2 > x_1^2$
$-2 < x_1 < 2$
$U = (-2,2)$

Since $p=(0,0,0)$
we choose $\pm$ so that
$0 = 4 \pm \sqrt{4^2-0^2}$
which means we choose $-$

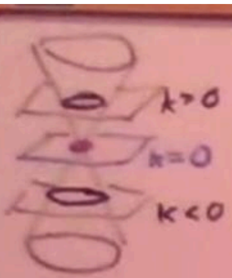
$$\left\{ \begin{pmatrix} y_1 \\ y_2 \\ x_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 4-\sqrt{4^2-x_1^2} \\ x_1 \end{pmatrix} : x_1 \in (-2,2) \right\}$$

Now let us consider the case where there is only one equation not a system. In that case the theorem can be written as in the following photo. See Video Implicit Part 8



In low priority Video Implicit Part 9 we apply the implicit function theorem to a practical problem: the intersection of two cylindrical pipes. This video may be skipped if you are short on time but you may wish to come back to it later.

In high priority Video Implicit Part 10 we consider a problematic situation where the Implicit Function Theorem cannot be applied. This video should be watched by all.



$$\left\{ \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} : \begin{matrix} z_1^2 + z_2^2 - z_3^2 = 0 \\ z_3 = k \end{matrix} \right\}$$

If $\boxed{k=0}$

$p_2 = 0$

so $p_1^2 + p_3^2 = 0$

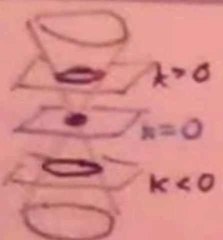
so $p_1 = 0 \quad p_3 = 0$

$$F\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} z_1^2 + z_2^2 - z_3^2 \\ z_3 \end{pmatrix}$$

$$DF_{(0,0,0)} = \begin{pmatrix} 2\cdot 0 & 2\cdot 0 & -2\cdot 0 \\ 0 & 0 & \boxed{1} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{\text{reduce}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$ only rank $1 < 2$

Does not have full rank

Cannot Apply Implicit Function Theorem



$$\left\{ \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} : \begin{matrix} z_1^2 + z_2^2 - z_3^2 = 0 \\ z_3 = k \end{matrix} \right\} \quad k > 0$$

$$F\begin{pmatrix} p_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} z_1^2 + z_2^2 - z_3^2 \\ z_3 \end{pmatrix}$$

$p_3 = k > 0$

$p_1^2 + p_2^2 = k^2$

one of these is not zero

$$DF = \begin{pmatrix} 2p_1 & 2p_2 & -2k \\ 0 & 0 & \boxed{1} \end{pmatrix} \neq \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \boxed{1} \end{pmatrix}$$

RANK

Does have full rank

Can Apply Implicit Function Theorem

Solve with one free variable

$$\left\{ \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} : \begin{matrix} z_1^2 + z_2^2 + z_3^2 = 1 \\ z_3 = k \end{matrix} \right\}$$

$$DF = \begin{pmatrix} 2z_1 & 2z_2 & 2z_3 \\ 0 & 0 & \boxed{1} \end{pmatrix}$$

If $k > 1$ or $k < -1$ there is nothing in the set and no p_0.

because $z_1^2 + z_2^2 + k^2 = 1$ has no solution when $k^2 > 1$.

$$\left\{ \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} : \begin{array}{c} z_1^2 + z_2^2 + z_3^2 = 1 \\ z_3 = k \end{array} \right\}$$

$$DF = \begin{pmatrix} 2z_1 & 2z_2 & 2z_3 \\ 0 & 0 & \boxed{1} \end{pmatrix}$$

$k \in (-1, 1)$ $\quad p_0 = (p_1, p_2, k)$

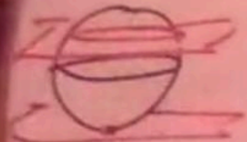
$$p_1^2 + p_2^2 = 1 - k^2 > 0$$

So either $p_1 \neq 0$ or $p_2 \neq 0$

If $p_1 \neq 0$ then $DF_{\vec{p}} = \begin{pmatrix} \boxed{2p_1} & 2p_2 & 2k \\ 0 & 0 & \boxed{1} \end{pmatrix}$ boxing leaders

Full rank Can apply theorem.

$$\left\{ \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} , \begin{matrix} z_1^2 + z_2^2 + z_3^2 = 1 \\ z_3 = k \end{matrix} \right\}$$



$$DF = \begin{pmatrix} 2z_1 & 2z_2 & 2z_3 \\ 0 & 0 & \boxed{1} \end{pmatrix}$$

If $k \in (-1, 1)$ $\quad p_0 = (p_1, p_2, k)$

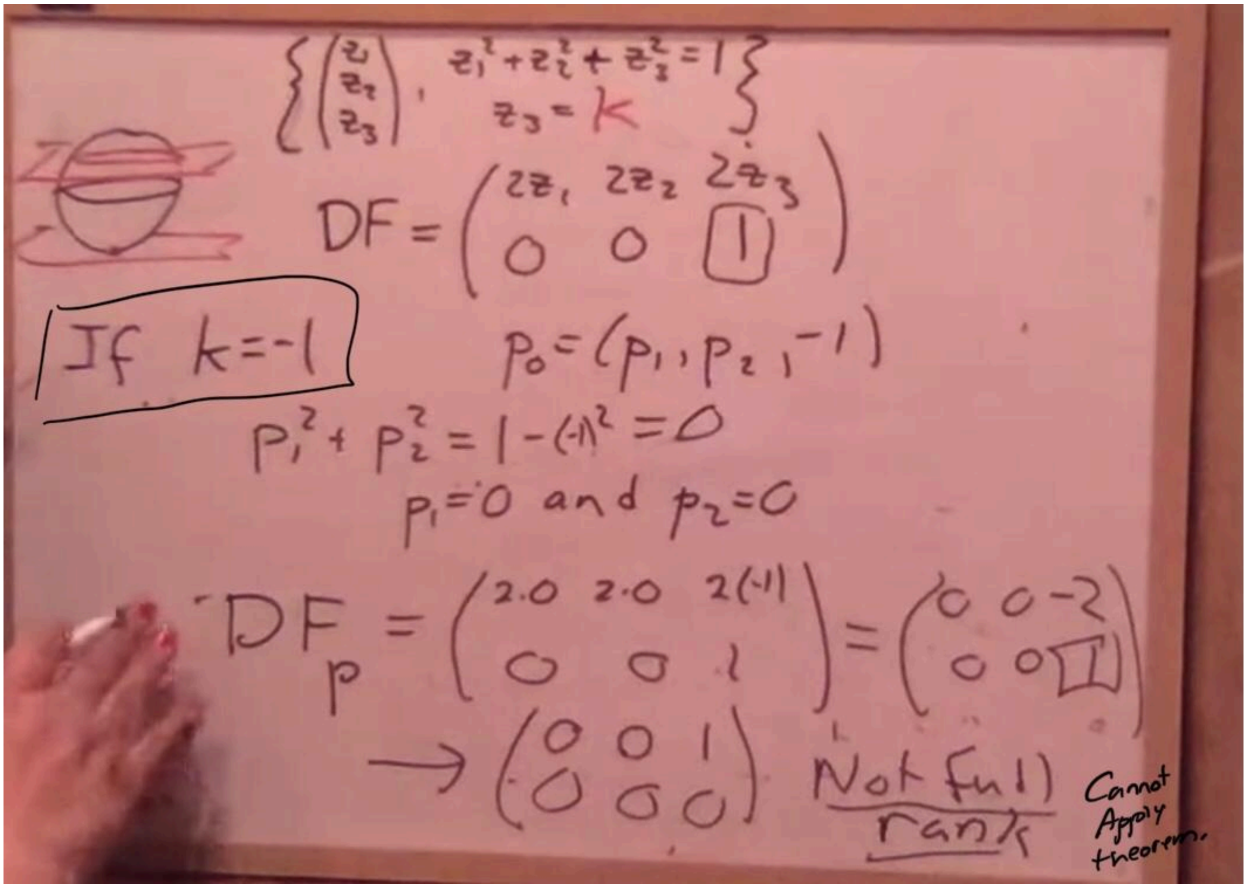
$$p_1^2 + p_2^2 = 1 - k^2 > 0$$

So either $p_1 \neq 0$ or $p_2 \neq 0$

If $p_1 = 0$ then $p_2 \neq 0$

$$DF_{\vec{p}} = \begin{pmatrix} 2p_1 & \boxed{2p_2} & 2k \\ 0 & 0 & \boxed{1} \end{pmatrix}$$

Full rank $\quad$ can apply theorem.



HW7: Check out k=1 case mentioned at the end of the video.

HW8:

Now consider spheres of various radii R about the origin x^2+y^2+z^2=R^2 intersecting with the standard one sheeted hyperboloid z^2= x^2+y^2 - 1

(a) What is F? What is DF?

(b) Consider R=5 and p=(2,3,SQRT(12)) Check p is on the sphere and the hyperboloid Is DF full rank at p for this R? Can you apply the implicit function theorem? If so, which variables are the leaders and find the local parametrization.

(c) Consider R=1 and p=(0,1,0) Check p is on the sphere and the hyperboloid. Is DF full rank at p for this R? Can you apply the implicit function theorem? If so, which variables are the leaders and find the local parametrization.

—- Take a break before continuing —-

Students under stress may skip the rest of this lesson. It is not required to understand subsequent weeks in this course. It includes the proof of the Implicit Function Theorem, one homework problem related to the proof and two extra credit problems. Extra Credit may be done late.

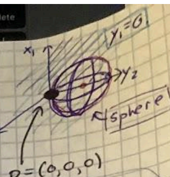
We cannot always find our parametrization explicitly as we have been able to do in our simple examples. If that cannot be done, it is sometimes helpful to at least know the linear approximation of the parametrization. See Video Implicit Part 11

Towards a constructive proof of the Implicit Function Theorem we set up an iterative process to find the parametrization and apply it to an example. See Video Implicit Part 12

Back to our classwork:

Apply this iterative process to the sphere-plane classwork as in the photo below. Try each step before looking at the answer:

Consider $\left\{\begin{pmatrix}y_1\\y_2\\x_1\end{pmatrix} : \begin{array}{l} y_1=0\\ y_1^2+(y_2-4)^2+x_1^2=4^2\end{array}\right\}$



$= \left\{ \begin{pmatrix}y_1\\y_2\\x_1\end{pmatrix} : F\begin{pmatrix}y_1\\y_2\\x_1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}\right\}$

where $F\begin{pmatrix}y_1\\y_2\\x_1\end{pmatrix} = ($ Fill in $)$ Answer $= \begin{pmatrix}y_1\\ y_1^2+(y_2-4)^2+x_1^2-4^2\end{pmatrix}$

Verify $F\begin{pmatrix}0\\0\\0\end{pmatrix} = ($ Fill in $)$ Answer $= \begin{pmatrix}0\\0\end{pmatrix}$

$DF = ($ Fill in $)$ Answer $= \begin{pmatrix}1 & 0 & 0\\ 2y_1 & 2(y_2-4) & 2x_1\end{pmatrix}$

$DF_p = ($ Fill in $)$ Answer $= \begin{pmatrix}\boxed{\begin{matrix}1 & 0\\ 0 & -8\end{matrix}} & \begin{matrix}0\\0\end{matrix}\end{pmatrix}$
A

Find $A^{-1} = ($ Fill in $)$ Answer $A^{-1} = \begin{pmatrix}1 & 0\\ 0 & -1/8\end{pmatrix}$

Let $H(\vec{y}) = -A^{-1}(F(\vec{y},\vec{x})) + \vec{y} = \boxed{\quad}$

Answer $H(\vec{y}) = -\begin{pmatrix}1 & 0\\ 0 & -1/8\end{pmatrix}\begin{pmatrix}y_1\\ y_1^2+(y_2-4)^2+x_1^2-4^2\end{pmatrix} + \begin{pmatrix}y_1\\y_2\end{pmatrix}$

$= \begin{pmatrix}-y_1\\ \frac{1}{8}(y_1^2+(y_2-4)^2+x_1^2-4^2)\end{pmatrix} + \begin{pmatrix}y_1\\y_2\end{pmatrix}$

$H(\vec{y}) = \begin{pmatrix}0\\ \frac{1}{8}y_1^2+\frac{1}{8}(y_2-4)^2+\frac{1}{8}x_1^2-2+y_2\end{pmatrix}$

Take $\vec{y}_1(x) = \vec{0}$ Find $\vec{y}_2(x) = H(\vec{y}_1(x))$

Answer: $\vec{y}_2(x) = \begin{pmatrix}0\\ \frac{1}{8}0^2+\frac{1}{8}(0-4)^2+\frac{1}{8}x_1^2-2+0\end{pmatrix} = \begin{pmatrix}0\\ \cancel{2}+\frac{1}{8}x_1^2-\cancel{2}\end{pmatrix}$

Find $\vec{y}_3(x) = H(\vec{y}_2(x)) = H\begin{pmatrix}0\\ \frac{1}{8}x_1^2\end{pmatrix}$

16
×16

Answer: $= \begin{pmatrix}0\\ \frac{1}{8}0^2+\frac{1}{8}((\frac{1}{8}x_1^2)-4)^2+\frac{1}{8}x_1^2-2+(\frac{1}{8}x_1^2)\end{pmatrix}$

$= \begin{pmatrix}0\\ \frac{1}{8}\frac{1}{64}x_1^4-\frac{1}{8}\frac{\cancel{8}}{\cancel{8}}x_1^2\cdot\cancel{4}+\frac{1}{\cancel{8}}\cancel{16}+\frac{1}{8}x_1^2-\cancel{2}+\frac{1}{8}x_1^2\end{pmatrix} = \begin{pmatrix}0\\ \frac{1}{8}x_1^2+\frac{1}{512}x_1^4\end{pmatrix}$

Our sequence does not repeat but note a Taylor Series!

Notice the sequence is not repeating and in fact never will. We will always get higher and higher powers with each iteration. Contrast this with a Taylor series approximation of the actual solution found in the first classwork problem in the next photo:

But earlier we saw $\vec{y}(\vec{x}) = \begin{pmatrix} 0 \\ 4-\sqrt{16-x_1^2} \end{pmatrix}$

Remember the Taylor Series for $\sqrt{u}$ about $u=16$

$$u^{1/2}\Big|_{u=16} = 4 \qquad \frac{1}{2}u^{-1/2}\Big|_{u=16} = \frac{1}{2}\cdot\frac{1}{4} = \frac{1}{8} \qquad \frac{1}{2}\left(\frac{-1}{2}\right)u^{-3/2}\Big|_{u=16} = \frac{-1}{4}\cdot\frac{1}{(16)^{3/2}} = \frac{-1}{256}$$

So $\sqrt{u} \approx 4 + \frac{1}{8}(u-16) - \frac{1}{256}\frac{(u-16)^2}{2}$ — converges for u near 16

So $\sqrt{16-x^2} \approx 4 + \frac{1}{8}(\cancel{16}-x^2-\cancel{16}) - \frac{1}{256}\frac{(\cancel{16}-x^2-\cancel{16})^2}{2}$

$\approx 4 - \frac{x^2}{8} - \frac{x^4}{512}$

$4-\sqrt{16-x^2} \approx \underbrace{\frac{x^2}{8} + \frac{x^4}{512}}_{y_3(\vec{x})}$ — Converges for x near 0

Notice $\vec{y}_2(\vec{x})$ — in fact $\vec{y}_j(x) \to$ limit of Taylor Series!

So at least for the first few steps the iteration is matching the Taylor series. So like a Taylor series we can hope to have convergence.

Extra Credit: What is the radius of convergence for this Taylor series and how is it related to the open set U?

So now let's return to the proof of the Implicit Function Theorem with our final video: Implicit Part 13.

HW9:

(a) **Submit all parts of Sphere-Plane Classwork**
(b) **Set up the iteration process for the Sphere Plane Classwork**
(c) **Then, at the good point where the implicit function theorem applies, do three iterations. Does it stop at a solution or give a sequence of functions that must converge to a limit?**

Solutions are here but numbered differently. You can consult them as needed after trying.

Extra Credit: Sphere Cylinder

(a) **Plot the sphere of radius 5 about the origin intersected with the cylinder x^2+y^2=9 and mark the point p=(3,0,4) using MATLAB implicit plot.**

(b) **Repeat all the steps of the Sphere-Plane Classwork with this sphere of radius 5 about the origin intersected with the cylinder x^2+y^2=9 at p=(3,0,4).**