Linear Algebra MAT313 Fall 2022
Professor Sormani
Review for the Final

A short video to see what this review is about.

The Final has two 25 minute parts.
It is very challenging
but is only 20% of your course grade
so do not worry too much.

Part I is about Linear Maps
(Lessons 27-28) 60%

Part II is about Vector Spaces and Diagonalization
(Lessons 24 and 26) 40%

A short video to see what this review is about.

You may glance over this sample before reviewing:

MAT313 Final Exam Part I Prof Sormani

60% for Part I

Let $F: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be the linear map

SAMPLE defined by $F\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5x_1 - x_2 + 4x_3 \\ x_2 + 5x_3 \end{pmatrix}$

(1) Find 5% (a) $F\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} =$ 5% (b) $F\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} =$ 5% (c) $F\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} =$

(2) 5% (a) Find $Null(F) = \{ \vec{x} \mid F(\vec{x}) = \vec{0} \}$

5% (b) Find a basis for $Null(F)$.

5% (c) Is F one-to-one? Hint: check if $Null(F) = \{\vec{0}\}$

(3) 5% (a) Find Image of $F = \{ F(\vec{x}) \mid \vec{x} \in \mathbb{R}^3 \}$

5% (b) Find a basis for the Image.

5% (c) Is F onto? Hint is Image of $F = \mathbb{R}^2$

(4) 5% (a) Find $F(\vec{v} + \vec{w}) = F\left(\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}\right) =$

5% (b) Find $F(\vec{v}) + F(\vec{w}) = F\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + F\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} =$

5% (c) Does F preserve addition?

40% for Part II

MAT313 **SAMPLE** Final Part 2 Prof Sormani

① Let $V = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \;\middle|\; x_1 = 3x_2 \right\}$

Show V is closed under scalar multi: 5%

Given $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in V$ we have $\boxed{}$

Given $k \in \mathbb{R}$ $k\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \boxed{} \in V$ 5%

because $\boxed{}$ 10%

② Let $P = \begin{pmatrix} \cos(\frac{\pi}{5}) & -\sin(\frac{\pi}{5}) & 0 \\ \sin(\frac{\pi}{5}) & \cos(\frac{\pi}{5}) & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

ⓐ Check $P^T P = I$ so $P^T = P^{-1}$ 5%

ⓑ If $B = PDP^T$ what are the eigenvalues of B? 5%

ⓒ What are the eigenspaces for these eigenvalues? 5%

ⓓ Describe the transformation PDP^T. 5%

Later you can practice two sample Part I and two sample Part II and read the solutions or watch videos explaining the solutions.

First some key pages from key lessons for our review:

Review Lesson 3 and 5: Row Reduction to Reduced Echelon Form

You will do this for a matrix which is very easy to reduce if you follow the rules of row reduction reviewed here.

Solve the System

Step 1: Convert to Augmented Matrix
Step 2: Row Reductions to Echelon Form
Step 3: Row Reductions to Reduced Echelon Form
Step 4: Convert back to System and Solve for Leaders
Step 5: Write Solution set in Position/Direction Form

$$\boxed{\begin{aligned} x + y + z + w &= 4 \\ 2x + y + 2z + w &= 7 \\ 3x + 3y + 3z + 3w &= 12 \\ 3x + 2y + 3z + 3w &= 11 \\ 4x + 4y + 4z + 4w &= 16 \end{aligned}}$$

Try each step while watching the videos before just reading the solution:
Playlist 313F22-3-extra-1to5

$$\begin{aligned} 1x + 1y + 1z + 1w &= 4 \\ 2x + 1y + 2z + 1w &= 7 \\ 3x + 3y + 3z + 3w &= 12 \\ 3x + 2y + 3z + 3w &= 11 \\ 4x + 4y + 4z + 4w &= 16 \end{aligned}$$

Step 1 Convert to Augmented Matrix

$$\left[\begin{array}{cccc|c} \boxed{1} & 1 & 1 & 1 & 4 \\ 2 & 1 & 2 & 1 & 7 \\ 3 & 3 & 3 & 3 & 12 \\ 3 & 2 & 3 & 3 & 11 \\ 4 & 4 & 4 & 4 & 16 \end{array}\right]$$

skew by row 1

$\rho_2 \to \rho_2 - 2\rho_1$
$\rho_3 \to \rho_3 - 3\rho_1$
$\rho_4 \to \rho_4 - 3\rho_1$
$\rho_5 \to \rho_5 - 4\rho_1$

Step 2: Row Reduction to Echelon Form:

look for leader in the upper left, box it,
is it a 1? Yes, so do not have to scale
get zeroes under the leader using
skew by the leader's row.

$$\left[\begin{array}{cccc|c} \boxed{1} & 1 & 1 & 1 & 4 \\ 0 & -1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

look for 2nd leader
in 2nd column
we see it is -1

scale

$\rho_2 \to -\rho_2$

$$\left[\begin{array}{cccc|c} \boxed{1} & 1 & 1 & 1 & 4 \\ 0 & \boxed{1} & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

get zeroes under the leader
using skew by the leader's row 2

$\rho_4 \to \rho_4 + \rho_2$

$$\left[\begin{array}{cccc|c} \boxed{1} & 1 & 1 & 1 & 4 \\ 0 & \boxed{1} & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Look for third leader
no leader in third column
so go to fourth column
move that leader to the third row
because the third leader must be
in the third row

$\rho_4 \leftrightarrow \rho_3$

Now all leaders are 1s
and only zeroes under all of them

Echelon Form

$$\left[\begin{array}{cccc|c} \boxed{1} & 1 & 1 & 1 & 4 \\ 0 & \boxed{1} & 0 & 1 & 1 \\ 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Step 3 Row Reductions to Reduced Echelon Form

Must be in Echelon Form First ✓

Now get zeroes above each leader.

Start with last leader (in row 3)

skew by last leaders row

$$\rho_1 \to \rho_1 - \rho_3 \qquad \left[\begin{array}{cccc|c} \boxed{1} & 1 & 1 & 0 & 4 \\ 0 & \boxed{1} & 0 & 0 & 1 \\ 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\rho_2 \to \rho_2 - \rho_3$$

Go to 2nd last leader in row 2

skew by his row 2 to get zeroes above him

$$\rho_1 \to \rho_1 - \rho_2$$

We have zeroes above all leaders and so this is Reduced Echelon Form

$$\left[\begin{array}{cccc|c} \boxed{1} & 0 & 1 & 0 & 3 \\ 0 & \boxed{1} & 0 & 0 & 1 \\ 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Step 4 Convert back to a system and solve for the leaders.

We have zeroes above all leaders so this is $\boxed{\text{Reduced Echelon Form}}$

$$\left[\begin{array}{cccc|c} \boxed{1} & 0 & 1 & 0 & 3 \\ 0 & \boxed{1} & 0 & 0 & 1 \\ 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Step 4: Convert back to a system and solve for leaders (box leaders)

$$\boxed{\begin{array}{l} \boxed{1x} + 0y + 1z + 0w = 3 \\ 0x + \boxed{1y} + 0z + 0w = 1 \\ 0x + 0y + 0z + \boxed{1w} = 0 \\ \text{ok} \left\{ \begin{array}{l} 0 = 0 \\ 0 = 0 \end{array} \right. \end{array}}$$

box system

$x = 3 - 1z$ using basic algebra

$y = 1$

$w = 0$

free: $z = z$

Step 5: Write the solution set in Position Direction Form

$$\left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 3 - 1z \\ 1 \\ z \\ 0 \end{pmatrix} \;\middle|\; z \in \mathbb{R} \right\}$$

Next do this

$$= \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} \\ \\ \\ \\ \end{pmatrix} + z \begin{pmatrix} \\ \\ \\ \\ \end{pmatrix} \;\middle|\; z \in \mathbb{R} \right\}$$

position direction

Step 5: Write the solution set in Position Direction Form

$$\left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 3 - 1z \\ 1 + 0z \\ 0 + 1z \\ 0 + 0z \end{pmatrix} \;\middle|\; z \in \mathbb{R} \right\}$$

Next do this

part not multiplied by free variable

we keep track of free variable carefully to find the direction

$$= \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + z \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \;\middle|\; z \in \mathbb{R} \right\}$$

position direction

$$\begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + z \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -z \\ 0 \\ z \\ 0 \end{pmatrix}$$ by scalar multiplication

$$= \begin{pmatrix} 3 - z \\ 1 + 0 \\ 0 + z \\ 0 + 0 \end{pmatrix} = \begin{pmatrix} 3 - z \\ 1 \\ z \\ 0 \end{pmatrix}$$ by vector subtraction ✓

Classwork ①: Convert to an Augmented Matrix
Do Row Reduction to Reduced Echelon Form
Solve for leader and write Solution Set
If there is one free variable write the solution set as a line with position and direction

Six Variables! $x_1\ x_2\ x_3\ x_4\ x_5\ x_6$

$$\boxed{\begin{aligned} 2x_1 + 4x_2 + 6x_6 &= 0 \\ 1x_1 + 2x_2 + 1x_4 &= 0 \\ 4x_1 + 8x_2 + 10x_6 &= 0 \\ 2x_1 + 4x_2 + 2x_5 &= 0 \\ 2x_4 + x_5 &= 0 \\ x_4 + x_5 + 2x_6 &= 0 \end{aligned}}$$

$2x_1 + 4x_2 + 0x_3 + 0x_4 + 0x_5 + 6x_6 = 0$

don't forget missing variables in row

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \\ \end{array}$$

$$\left[\begin{array}{cccccc|c} \boxed{2} & 4 & 0 & 0 & 0 & 6 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 & 0 \\ 4 & 8 & 0 & 0 & 0 & 10 & 0 \\ 2 & 4 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 2 & 0 \end{array}\right]$$

← Our variables

pause + try

← final column is all 0 in a homogeneous system

Box the first leader
turn it into a 1

$\xrightarrow{\rho_1 \to \frac{1}{2}\rho_1}$

$$\left[\begin{array}{cccccc|c} \boxed{1} & 2 & 0 & 0 & 0 & 3 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 & 0 \\ 4 & 8 & 0 & 0 & 0 & 10 & 0 \\ 2 & 4 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 2 & 0 \end{array}\right]$$

$10 - 4 \cdot 3 = 10 - 12 = -2$

$10 - 2 \cdot 3 = 10 - 6 = 4$

must put zeroes under the leader
using skew by leader's row $\rho_i \to \rho_i + k\rho_1$

$\rho_2 \to \rho_2 - \rho_1$
$\rho_3 \to \rho_3 - 4\rho_1$
$\rho_4 \to \rho_4 - 2\rho_1$
last two are 0

$$\longrightarrow \left[\begin{array}{cccccc|c} \boxed{1} & 2 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 2 & 0 \end{array}\right]$$

Find 2nd leader: none in the second column
(2nd variable is free!)
none in third column
(3rd variable is free!)
the leader in the fourth column is a 1
so we do not need to scale
and already in row 2 so no switch needed

Next make 0's under the leader
skew by leaders row ρ_2

$\rho_5 \to \rho_5 - 2\rho_2$
$\rho_6 \to \rho_6 - \rho_2$

$$\longrightarrow \left[\begin{array}{cccccc|c} \boxed{1} & 2 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & 6 & 0 \\ 0 & 0 & 0 & 0 & 1 & 5 & 0 \end{array}\right]$$

Find next leader
switch $\rho_5 \leftrightarrow \rho_3$

$$\xrightarrow{\rho_5 \leftrightarrow \rho_3} \left[\begin{array}{cccccc|c} \boxed{1} & 2 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 & 5 & 0 \end{array}\right]$$

Must make all below the leader to a zero

$$\xrightarrow{\rho_6 \to \rho_6 - \rho_3} \left[\begin{array}{cccccc|c} \boxed{1} & 2 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & \boxed{-2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{array}\right]$$

Find next leader turn it into a 1 scaling it!

$$\xrightarrow{\rho_4 \to \frac{-1}{2}\rho_4} \left[\begin{array}{cccccc|c} \boxed{1} & 2 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{array}\right]$$

change to 0 skew by ρ_4

$$\xrightarrow[\rho_6 \to \rho_5 + \rho_4]{\rho_5 \to \rho_5 - 4\rho_4} \left[\begin{array}{cccccc|c} \boxed{1} & 2 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Echelon Form!

Next want Reduced Echelon Form with zeroes above the leaders starting with bottom leader

skew by leaders row ρ_4

$$\begin{array}{l} \rho_1 \to \rho_1 - 3\rho_4 \\ \rho_2 \to \rho_2 + 3\rho_4 \\ \rho_3 \to \rho_3 - 6\rho_4 \end{array} \longrightarrow \left[\begin{array}{cccccc|c} \boxed{1} & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Check third leader ✓

second leader ✓

Reduced Echelon Form.

Write as a Linear System

$\boxed{1x_1} + 2x_2 = 0$

$\boxed{1x_4} = 0$

$\boxed{1x_5} = 0$

$\boxed{1x_6} = 0$

solve for leaders

$x_1 = -2x_2$

$x_4 = 0$

$x_5 = 0$

$x_6 = 0$

free variables $x_2 = x_2$

$x_3 = x_3$

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} -2x_2 \\ x_2 \\ x_3 \\ 0 \\ 0 \\ 0 \end{pmatrix} : x_2, x_3 \in \mathbb{R} \right\}$$

two free variables!

cannot write in line form!

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 0 - 2x_2 + 0x_3 \\ 0 + 1x_2 + 0x_3 \\ 0 + 0x_2 + 1x_3 \\ 0 + 0x_2 + 0x_3 \\ 0 + 0x_2 + 0x_3 \\ 0 + 0x_2 + 0x_3 \end{pmatrix} : x_2, x_3 \in \mathbb{R} \right\}$$

position with two directions

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} ; x_2, x_3 \in \mathbb{R} \right\}$$

Our Algorithm:

Step 1: Write Augmented Matrix ← Lesson 3

Step 2: Row Actions to Echelon Form ← Lessons 1-2

- Make upper left leader into 1 using scaling (or switch if 0)
- Make zeroes beneath leader using skew by leader's row
- Move down to the next row and make sure next column has a leader which is 1 by repeating red step. Then repeat blue step.

Move down again until Echelon Form. Each leader in red box is a 1. Below each leader are 0's.

reversible row action
- scale
- skew
- switch

Step 3: Row actions to Reduced Echelon Form ← Lesson 3 Reduced Echelon Form

- start with bottom leader. Make sure 0's above it using skew by the leader's row. Repeat with next to last leader until 0's above all leaders

Step 4: Write as equations and solve for leaders and set free variables equal to themselves ← No need to sub up

Write as $\left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} \equiv \end{pmatrix} ; \text{free} \in \mathbb{R} \right\}$

be careful "$0 + 0 \cdots + 0 =$ not 0" no solution

Step 5: write in position directions form ← Lesson 5

$$\left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = \underset{\text{position}}{\begin{pmatrix} \; \end{pmatrix}} + \underset{\text{free variable}}{x_i} \underset{\text{direction for that free variable}}{\begin{pmatrix} \; \end{pmatrix}} + \cdots \;\middle|\; \text{free} \in \mathbb{R} \right\}$$

When the system is homogeneous the position vector is $\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

If a homogeneous system has 1 free variable, t

$\{ \vec{x} = \vec{0} + t\vec{a} \mid t \in \mathbb{R} \}$ a line through the origin.

If it has two free variables, s, t

$\{ \vec{x} = \vec{0} + \underset{\text{direction of t}}{t\vec{a}} + \underset{\text{direction of s}}{s\vec{b}} \mid s, t \in \mathbb{R} \}$

notice that when $s, t = 0$ then $\vec{x} = 0$

So $x_1 = 0$ $x_2 = 0$ ... $x_m = 0$ is one of the solutions to system.

If there are no free variables the only solution is $\{ \vec{x} = \vec{0} \}$

It is never an $\emptyset$.

****

Lessons 7, 9, 10, 14, and 16: Matrices

****

Lesson 7 a matrix times a vector:

n×m matrix (without the column of d_i)

$$\begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,m} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,m} \\ \vdots & & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,m} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix}$$

n×m matrix, n rows and m columns; $\vec{x}\in\mathbb{R}^m$, $\vec{d}\in\mathbb{R}^n$

$$= \begin{pmatrix} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,m}x_m \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,m}x_m \\ \vdots \\ a_{n,1}x_1 + a_{n,2}x_2 + \cdots + a_{n,m}x_m \end{pmatrix}$$

Review:

$$\boxed{\begin{matrix} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,m}x_m = d_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,m}x_m = d_2 \\ \cdots \\ a_{n,1}x_1 + a_{n,2}x_2 + \cdots + a_{n,m}x_m = d_n \end{matrix}}$$

which can be written as

$$\sum_{j=1}^{m} a_{i,j}x_j = d_i \quad \text{for } i = 1 \text{ to } n \text{ (rows)}$$

or as an augmented matrix

← first entry of the answer is the first row of the matrix $\cdot\ \vec{x}$ (dot product)

← second entry is the dot product of the second row with $\vec{x}$

n×m matrix (without the column of d_i)

$$\begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,m} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,m} \\ \vdots & & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,m} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix}$$

n×m matrix; $\vec{x}$, $\vec{d}$

$$= \begin{pmatrix} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,m}x_m \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,m}x_m \\ \vdots \\ a_{n,1}x_1 + a_{n,2}x_2 + \cdots + a_{n,m}x_m \end{pmatrix}$$

$$A\vec{x} = \begin{pmatrix} (\text{row 1 of } A)\cdot\vec{x} \\ (\text{row 2 of } A)\cdot\vec{x} \\ \vdots \\ (\text{row } n \text{ of } A)\cdot\vec{x} \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{m} a_{1,j}x_j \\ \sum_{j=1}^{m} a_{2,j}x_j \\ \vdots \\ \sum_{j=1}^{m} a_{n,j}x_j \end{pmatrix}$$

i^{th} component

$$[A\vec{x}]_i = d_i = \sum_{j=1}^{m} a_{i,j}x_j$$

Example

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 8 & 9 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1x + 2y + 3z \\ 4x + 8y + 9z \end{pmatrix}$$

Example

$$\begin{pmatrix} 1 & 4 \\ 5 & 8 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1\cdot 2 + 4\cdot 3 \\ 5\cdot 2 + 8\cdot 3 \end{pmatrix} = \begin{pmatrix} 2+12 \\ 10+24 \end{pmatrix} = \begin{pmatrix} 14 \\ 34 \end{pmatrix}$$

Classwork

$$\begin{pmatrix} 1 & 3 \\ 5 & 7 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1\cdot 2 + 3\cdot 0 \\ 5\cdot 2 + 7\cdot 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 10 \end{pmatrix}$$

m columns, n rows; $\vec{v}$ is in $\mathbb{R}^m$; answer is in $\mathbb{R}^n$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1x + 0y + 0z \\ 0x + 1y + 0z \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

3 columns, 2 rows; in $\mathbb{R}^3$; ← answer is in $\mathbb{R}^2$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1a + 0b + 0c \\ 0a + 1b + 0c \\ 0a + 0b + 1c \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Null Space of a Matrix

Part II

The Null Space of a Matrix

Defn The null space of a matrix A is the set of vectors

$$\{\vec{x} \mid A\vec{x} = \vec{0}\}$$

$$= \left\{ \vec{x} \;\middle|\; \begin{matrix} a_{11}x_1 + \cdots a_{1,m}x_m = 0 \\ a_{21}x_1 + \cdots + a_{2,m}x_m = 0 \\ \cdots \\ a_{n,1}x_1 + \cdots + a_{n,m}x_m = 0 \end{matrix} \right\}$$

Solution of the homogeneous system that comes from A

what is the null space of $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$?

Solve $\left(\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right)$ homogeneous

$\xrightarrow{\rho_1 \leftrightarrow \rho_3} \left(\begin{array}{ccc|c} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \end{array}\right)$ Reduced Echelon form

$x = 0$, $y = 0$, $z = 0$ all leaders

Null Space $= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$

$= \{\vec{0}\}$

Null Space of $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Solve $\left(\begin{array}{ccc|c} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ already reduced Echelon!

x,y leaders, z is free

$x = 0$
$y = 0$
$z = z$ (free)

$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} : z \in \mathbb{R} \right\}$

$= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = z\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \;\middle|\; z \in \mathbb{R} \right\}$

Null space of $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$

$\left(\begin{array}{ccc|c} \boxed{1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right) \xrightarrow[\text{row}_3 \to \text{row}_3 - \text{row}_1]{\rho_3 \to \rho_3 - \rho_1} \left(\begin{array}{ccc|c} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right)$

$\xrightarrow[\text{row}_3 \to \text{row}_3 - \text{row}_2]{\rho_3 \to \rho_3 - \rho_2} \left(\begin{array}{ccc|c} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

Null space as the ← last example because it has the same reduced Ech form.

Null space of $\begin{pmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix}$ is

$\left(\begin{array}{ccc|c} \boxed{1} & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{array}\right) \xrightarrow[\rho_3 \to \rho_3 - 3\rho_1]{\rho_2 \to \rho_2 - 2\rho_1} \left(\begin{array}{ccc|c} \boxed{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

row$_3$

leader is only x
y and z are free

$x = 0$
$y = y$
$z = z$

so $\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ y \\ z \end{pmatrix} \;\middle|\; y, z \in \mathbb{R} \right\}$

$= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = y\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} ; y, z \in \mathbb{R} \right\}$

Null space of $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

$\left(\begin{array}{ccc|c} \boxed{1} & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) \xrightarrow{\rho_3 \to \rho_3 - \rho_1} \left(\begin{array}{ccc|c} \boxed{1} & 0 & 1 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ Reduced Echelon

leaders are x and y
$z = z$ (free)

$x + z = 0$
$y = 0$
$x = -z$

$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -z \\ 0 \\ z \end{pmatrix} : z \in \mathbb{R} \right\}$

$= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = z\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} : z \in \mathbb{R} \right\}$

Review Lesson 9: proofs with matrix multiplication

Lesson 9 Proof with Matrices

Part I 2×2 and 3×3 Matrices

Part II n×n and n×m matrices

Thm: $\overset{LHS}{A(\vec{v}+\vec{w})} = \overset{RHS}{A\vec{v}+A\vec{w}}$

$\forall$ matrices $A \in M_{n\times m}$ and $\vec{v},\vec{w} \in \mathbb{R}^m$

Proof: (Easy case where $A \in M_{2\times 2}$)

① $\overset{LHS=}{A(\vec{v}+\vec{w})} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\left(\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\right)$ by defn of $M_{2\times 2}$ + $\mathbb{R}^2$

② $= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} v_1+w_1 \\ v_2+w_2 \end{pmatrix}$ by defn of addition on $\mathbb{R}^2$

③ $= \begin{pmatrix} a_{11}(v_1+w_1) + a_{12}(v_2+w_2) \\ a_{21}(v_1+w_1)+a_{22}(v_2+w_2) \end{pmatrix}$ by defn of matrix mult

④ $= \begin{pmatrix} a_{11}v_1 + a_{11}w_1 + a_{12}v_2 + a_{12}w_2 \\ a_{21}v_1 + a_{21}w_1 + a_{22}v_2 + a_{22}w_2 \end{pmatrix}$ by distr of mult over add of reals

$a(v+w) = av + aw$ for $a,v,w \in \mathbb{R}$

⑤ $\overset{RHS}{A\vec{v}+A\vec{w}} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ by defn of $M_{2\times 2}$ and $\mathbb{R}^2$

⑥ $= \begin{pmatrix} a_{11}v_1 + a_{12}v_2 \\ a_{21}v_1 + a_{22}v_2 \end{pmatrix} + \begin{pmatrix} a_{11}w_1 + a_{12}w_2 \\ a_{21}w_1 + a_{22}w_2 \end{pmatrix}$ by defn Matrix mult

⑦ $= \begin{pmatrix} a_{11}v_1 + a_{12}v_2 + a_{11}w_1 + a_{12}w_2 \\ a_{21}v_1 + a_{22}v_2 + a_{21}w_1 + a_{22}w_2 \end{pmatrix}$ by add of vectors

Steps 4 and 7 match So QED.

Thm: $A(k\vec{v}) = k(A\vec{v})$

for all matrices $A \in M_{n\times m}$

all real numbers $k \in \mathbb{R}$

and all vectors $\vec{v} \in \mathbb{R}^m$

Prove for $M_{2\times 3}$ and $\vec{v} \in \mathbb{R}^3$

Proof: 2rows 3columns

① RHS

$$A(k\vec{v}) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}\left(k\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}\right)$$

by defn of $M_{2\times 3}$, $\mathbb{R}$, $\mathbb{R}^3$

②

$$= \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}\begin{pmatrix} kv_1 \\ kv_2 \\ kv_3 \end{pmatrix}$$

by scalar multiplication

③

$$= \begin{pmatrix} a_{11}kv_1 + a_{12}kv_2 + a_{13}kv_3 \\ a_{21}kv_1 + a_{22}kv_2 + a_{23}kv_3 \end{pmatrix}$$

by matrix mult by a vector

$\boxed{HW2}$ complete the LHS

④ LHS

$$k(A\vec{v}) =$$

Do what is in parentheses first

Justify each step.

12:33 AM Mon Sep 14 — Linear Algebra

Thm I $A(\vec{v}+\vec{w}) = A\vec{v} + A\vec{w}$ $\quad \forall A \in M_{n\times m} \quad \forall \vec{v}, \vec{w} \in \mathbb{R}^m$

Thm II $A(k\vec{v}) = k(A\vec{v})$ $\quad \forall A \in M_{n\times m} \quad \forall k \in \mathbb{R} \quad \forall \vec{v} \in \mathbb{R}^m$

Apply these theorems:

Cor I If $\vec{v}, \vec{w} \in Null(A)$ Then $\vec{v}+\vec{w} \in Null(A)$

Cor II (State later)

A corollary is a consequence of a theorem.

12:39 AM Mon Sep 14 — Linear Algebra

Thm I $A(\vec{v}+\vec{w}) = A\vec{v} + A\vec{w}$ $\quad \forall A \in M_{n\times m} \quad \forall \vec{v}, \vec{w} \in \mathbb{R}^m$

Thm II $A(k\vec{v}) = k(A\vec{v})$ $\quad \forall A \in M_{n\times m} \quad \forall k \in \mathbb{R} \quad \forall \vec{v} \in \mathbb{R}^m$

Apply these theorems: Given

Cor I If $\vec{v}, \vec{w} \in Null(A)$ Then $\vec{v}+\vec{w} \in Null(A)$ Show

Cor II (State later)

Proof of Corollary I Use Thm I

Given ① $\vec{v}, \vec{w} \in Null(A)$ ① by given

② $\vec{v}, \vec{w} \in \{\vec{x} \mid A\vec{x} = \vec{0}\}$ ② by defn of Null space

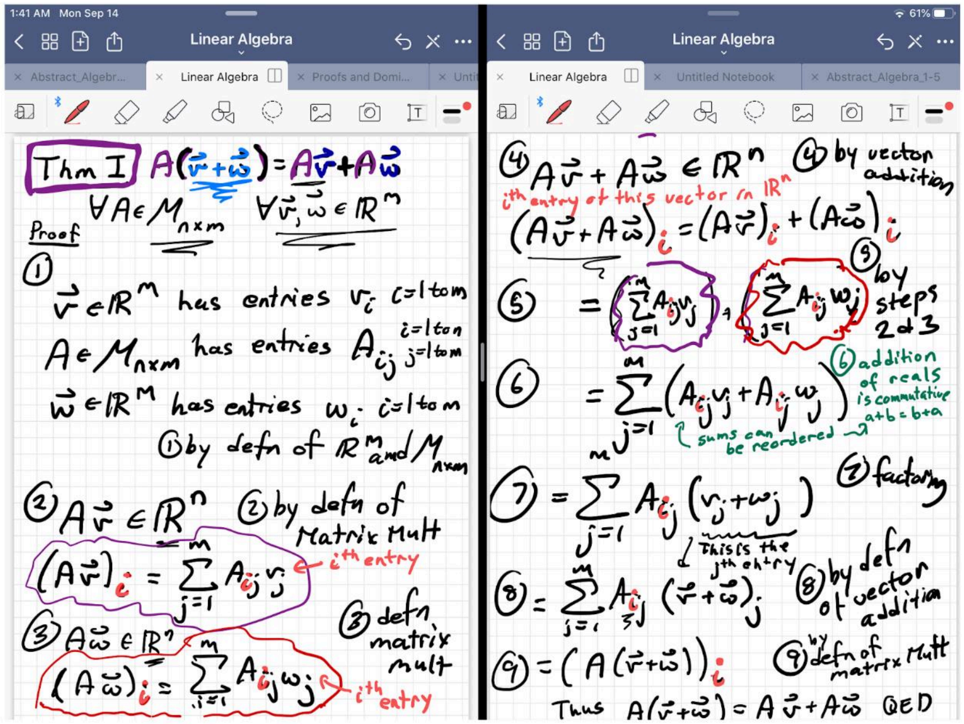
③ $A\vec{v} = \vec{0}$ and $A\vec{w} = \vec{0}$ ③ by defn of the set

{we want to show $\vec{v}+\vec{w} \in Null(A)$}

Must do work to get there

final $\vec{v}+\vec{w} \in Null(A)$ SHOW

Part II Proofs with Sums


Linear Algebra

Thm I $A(\vec{v}+\vec{w}) = A\vec{v} + A\vec{w}$

$\forall A \in M_{n\times m}$ $\forall \vec{v}, \vec{w} \in \mathbb{R}^m$

Proof

① $\vec{v} \in \mathbb{R}^m$ has entries v_i $i=1$ to m

$A \in M_{n\times m}$ has entries A_{ij} $i=1$ to n, $j=1$ to m

$\vec{w} \in \mathbb{R}^m$ has entries w_i $i=1$ to m

① by defn of $\mathbb{R}^m$ and $M_{n\times m}$

② $A\vec{v} \in \mathbb{R}^n$ ② by defn of Matrix Mult

$(A\vec{v})_i = \sum_{j=1}^{m} A_{ij} v_j$ ← i^{th} entry

③ $A\vec{w} \in \mathbb{R}^n$ ③ defn matrix mult

$(A\vec{w})_i = \sum_{j=1}^{m} A_{ij} w_j$ ← i^{th} entry

④ $A\vec{v} + A\vec{w} \in \mathbb{R}^n$ ④ by vector addition

i^{th} entry of this vector in $\mathbb{R}^n$

$(A\vec{v}+A\vec{w})_i = (A\vec{v})_i + (A\vec{w})_i$

⑤ $= \sum_{j=1}^{m} A_{ij} v_j + \sum_{j=1}^{m} A_{ij} w_j$ ⑤ by steps 2 & 3

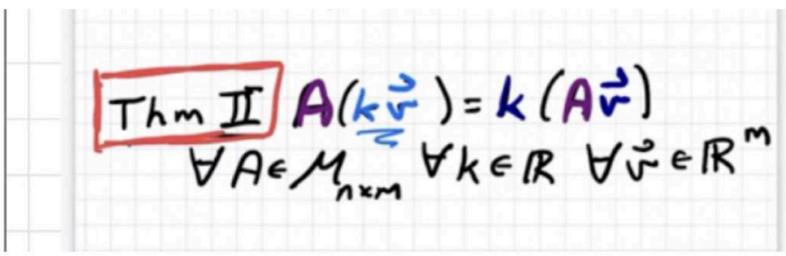
⑥ $= \sum_{j=1}^{m} (A_{ij} v_j + A_{ij} w_j)$ ⑥ addition of reals is commutative $a+b=b+a$ sums can be reordered

⑦ $= \sum_{j=1}^{m} A_{ij} (v_j + w_j)$ ⑦ factoring this is the j^{th} entry

⑧ $= \sum_{j=1}^{m} A_{ij} (\vec{v}+\vec{w})_j$ ⑧ by defn of vector addition

⑨ $= (A(\vec{v}+\vec{w}))_i$ ⑨ by defn of matrix mult

Thus $A(\vec{v}+\vec{w}) = A\vec{v} + A\vec{w}$ QED


Thm II $A(k\vec{v}) = k(A\vec{v})$

$\forall A \in M_{n\times m}$ $\forall k \in \mathbb{R}$ $\forall \vec{v} \in \mathbb{R}^m$

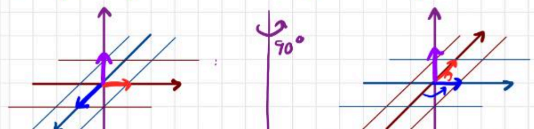
Thm I and Thm II above might be useful on your final. You now know this implies matrix multiplication defines a linear map.

Review Lesson 10: Linear Transformations

You now know Linear Transformations are Linear Maps

Classwork (5): Find a 3D Linear Transformation which rotates about the z axis by 90°

Very Important for Computer Graphics and Robotics



input $\begin{pmatrix}1\\0\\0\end{pmatrix} \longrightarrow$ output $\begin{pmatrix}0\\1\\0\end{pmatrix}$ input $\begin{pmatrix}0\\1\\0\end{pmatrix} \longrightarrow$ output $\begin{pmatrix}-1\\0\\0\end{pmatrix}$ input $\begin{pmatrix}0\\0\\1\end{pmatrix} \longrightarrow$ output $\begin{pmatrix}0\\0\\1\end{pmatrix}$

Find a matrix such that

$$\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}0\\1\\0\end{pmatrix} \text{ and } \begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}=\begin{pmatrix}-1\\0\\0\end{pmatrix} \text{ and } \begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}0\\0\\1\end{pmatrix}$$

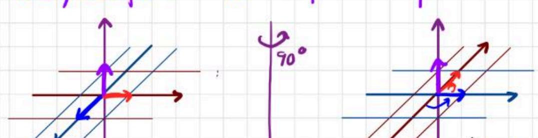
$\Downarrow$

$$\begin{pmatrix}a_{11}\cdot 1+a_{12}\cdot 0+a_{13}\cdot 0\\a_{21}\cdot 1+a_{22}\cdot 0+a_{23}\cdot 0\\a_{31}\cdot 1+a_{32}\cdot 0+a_{33}\cdot 0\end{pmatrix}=\begin{pmatrix}0\\1\\0\end{pmatrix}$$

$\Downarrow$

$$\underbrace{\begin{pmatrix}a_{11}\\a_{21}\\a_{31}\end{pmatrix}}_{1^{st}\text{ column}}=\underbrace{\begin{pmatrix}0\\1\\0\end{pmatrix}}_{1^{st}\text{ output}}$$

Very Important for Computer Graphics and Robotics



1st input $\begin{pmatrix}1\\0\\0\end{pmatrix} \longrightarrow$ 1st output $\begin{pmatrix}0\\1\\0\end{pmatrix}$ 2nd input $\begin{pmatrix}0\\1\\0\end{pmatrix} \longrightarrow$ 2nd output $\begin{pmatrix}-1\\0\\0\end{pmatrix}$ 3rd input $\begin{pmatrix}0\\0\\1\end{pmatrix} \longrightarrow$ 3rd output $\begin{pmatrix}0\\0\\1\end{pmatrix}$

Find a matrix such that

$$\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}0\\1\\0\end{pmatrix} \text{ and } \begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}=\begin{pmatrix}-1\\0\\0\end{pmatrix} \text{ and } \begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$\Downarrow$

$$\begin{pmatrix}a_{11}\cdot 1+a_{12}\cdot 0+a_{13}\cdot 0\\a_{21}\cdot 1+a_{22}\cdot 0+a_{23}\cdot 0\\a_{31}\cdot 1+a_{32}\cdot 0+a_{33}\cdot 0\end{pmatrix}=\begin{pmatrix}0\\1\\0\end{pmatrix} \quad \begin{pmatrix}a_{11}0+a_{12}1+a_{13}0\\a_{21}0+a_{22}1+a_{23}0\\a_{31}0+a_{32}1+a_{33}0\end{pmatrix}=\begin{pmatrix}-1\\0\\0\end{pmatrix} \quad \begin{pmatrix}a_{11}0+a_{12}0+a_{13}1\\a_{21}0+a_{22}0+a_{23}1\\a_{31}0+a_{32}0+a_{33}1\end{pmatrix}=\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$\Downarrow$

$$\underbrace{\begin{pmatrix}a_{11}\\a_{21}\\a_{31}\end{pmatrix}}_{1^{st}\text{ column}}=\underbrace{\begin{pmatrix}0\\1\\0\end{pmatrix}}_{1^{st}\text{ output}} \quad \underbrace{\begin{pmatrix}a_{12}\\a_{22}\\a_{32}\end{pmatrix}}_{2^{nd}\text{ column}}=\underbrace{\begin{pmatrix}-1\\0\\0\end{pmatrix}}_{2^{nd}\text{ output}} \quad \underbrace{\begin{pmatrix}a_{13}\\a_{23}\\a_{33}\end{pmatrix}}_{3^{rd}\text{ column}}=\underbrace{\begin{pmatrix}0\\0\\1\end{pmatrix}}_{3^{rd}\text{ output}}$$

So the matrix is $\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}=\begin{pmatrix}0&-1&0\\1&0&0\\0&0&1\end{pmatrix}$

Review Lesson 14: Multiplying Matrices

Linear Algebra

Multiplying Matrices

Review

$A \in M_{n\times m}$ — m columns, n rows

$[A\vec{v}]_i = \sum_{j=1}^{m} a_{ij} v_j$

$\vec{v} \in \mathbb{R}^m$ $\quad A\vec{v} \in \mathbb{R}^n$

$$A\vec{v} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{pmatrix} = \begin{pmatrix} \overrightarrow{\text{row } 1} \cdot \vec{v} \\ \overrightarrow{\text{row } 2} \cdot \vec{v} \\ \vdots \\ \overrightarrow{\text{row } n} \cdot \vec{v} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}v_1 + a_{12}v_2 + \cdots + a_{1m}v_m \\ a_{21}v_1 + a_{22}v_2 + \cdots + a_{2m}v_m \\ \vdots \\ a_{n1}v_1 + a_{n2}v_2 + \cdots + a_{nm}v_m \end{pmatrix} \in \mathbb{R}^n$$

Linear Algebra

Multiplying Matrices

Review

$A \in M_{n\times m}$ — m columns, n rows

$[A \times B]_{ik} = \sum_{j=1}^{m} a_{ij} b_{jk}$

$B \in M_{m\times \ell}$ $\quad A\times B \in M_{n\times \ell}$

$$A\times B = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1\ell} \\ b_{21} & b_{22} & \cdots & & \vdots \\ \vdots & & & & \\ b_{m1} & & & & b_{m\ell} \end{pmatrix} = ?$$

Consider $B = (\vec{B}_1\ \vec{B}_2 \cdots \vec{B}_\ell) \in M_{m\times \ell}$ $\quad \vec{B}_1, \ldots, \vec{B}_\ell \in \mathbb{R}^m$

$A \times B = (A\vec{B}_1\ A\vec{B}_2\ \cdots\ A\vec{B}_\ell) \in M_{n\times \ell}$ $\quad A\vec{B}_1, \ldots, A\vec{B}_\ell \in \mathbb{R}^n$

n rows

Linear Algebra

$$A\times B = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1\ell} \\ b_{21} & b_{22} & b_{23} & & \vdots \\ \vdots & & \vdots & & \\ b_{m1} & & b_{m3} & & b_{m\ell} \end{pmatrix} = ?$$

j counter goes across → the i^{th} row of A

↓ j counter goes down in the k^{th} column of B

$[A\times B]_{ik}$ = dot product of i^{th} row of A, k^{th} column of B

Example

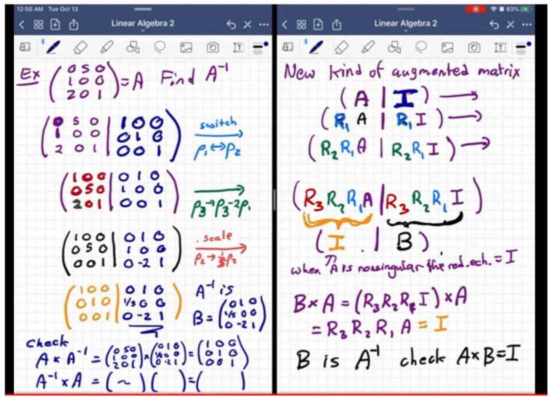
$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 2 & 3 & 4 \end{pmatrix} = \begin{array}{cc} 1^{st}\text{ row } 1^{st}\text{ column} & 1^{st}\text{ row } 2^{nd}\text{ column} \\ \end{array}$$

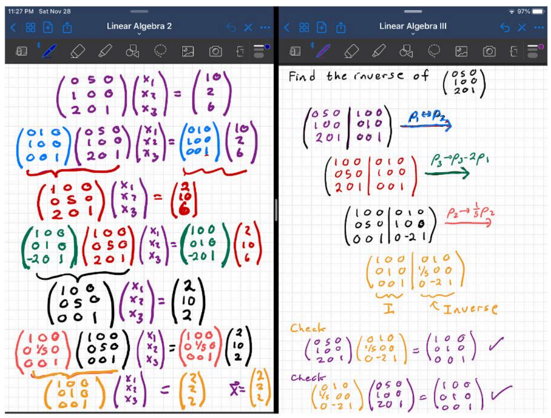
$$\begin{pmatrix} 1\cdot 0 + 2\cdot 1 + 3\cdot 1 & 1\cdot 1 + 2\cdot 0 + 3\cdot 2 & \ & \ \\ \ & \ & \ & \ \end{pmatrix}$$

$M_{2\times 3}$ $\quad M_{3\times 4}$ $\quad M_{2\times 4}$

↑ fill in + watch next video

Review Lesson 16: Inverses of Matrices





How to find an inverse and why it works

for nonsingular matrices whose Reduced Echelon Form = I.

For singular matrices, there are no inverses!

****

Review from Lesson 20 and 21: Basis Span Subspaces Null spaces

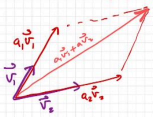
****

New Defn: Given a collection of vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3 \dots \vec{v}_k \in \mathbb{R}^m$

a linear combination

$$\sum_{i=1}^{k} a_i \vec{v}_i = a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_k\vec{v}_k$$

where $a_1, a_2 \dots a_k \in \mathbb{R}$



Recall our solution sets of homogeneous systems

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} = x_2 \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{pmatrix} + x_3 \begin{pmatrix} w_1 \\ \vdots \\ w_m \end{pmatrix} + x_5 \begin{pmatrix} z_1 \\ \vdots \\ z_m \end{pmatrix} : \underbrace{x_2, x_3, x_5 \in \mathbb{R}}_{\text{free}} \right\}$$

summed up

free variables times directions

Linear Combo of the Directions.

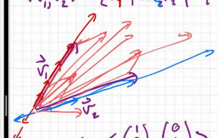
New Defn Given a collection of vectors $\vec{v}_1\ \vec{v}_2 \dots \vec{v}_k \in \mathbb{R}^m$ the span $\langle \vec{v}_1, \vec{v}_2, \dots, \vec{v}_k \rangle$ is the set of all linear combinations:

$$= \left\{ a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_k\vec{v}_k : \underbrace{a_i \in \mathbb{R}}_{\text{free}} \right\}$$

$$= \left\{ \sum_{i=1}^{k} a_i \vec{v}_i : a_i \in \mathbb{R} \right\}$$

run free through $\mathbb{R}$

$$\langle \vec{v}_1, \vec{v}_2 \rangle = \{ a_1\vec{v}_1 + a_2\vec{v}_2 \mid a_i \in \mathbb{R} \}$$

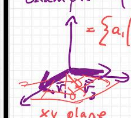


Example $\left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle$

$$= \left\{ a_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \middle| a_i \in \mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix} a_1 \\ a_2 \\ 0 \end{pmatrix} \middle| a_i \in \mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \middle| x, y \in \mathbb{R} \right\}$$

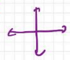


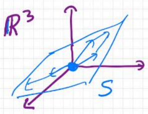
Lesson 20 Part 2
Subspaces

Recall $\vec{v} \in \mathbb{R}^m$ vectors

$\mathbb{R}^m$ Euclidean Space of dimension m

$\mathbb{R}^1$ = real line (dim = 1)

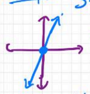
$\mathbb{R}^2$  Euclidean plane (dim = 2)

$\mathbb{R}^3$ 

Euclidean Space of dim 3

A <u>subspace</u> $S \subset \mathbb{R}^m$ is a collection of vectors that includes $\vec{0}$ and is "closed under addition" $(\forall \vec{v}, \vec{w} \in S \quad \vec{v}+\vec{w} \in S)$ and "closed under scalar mult" $(\forall R \in \mathbb{R} \; \vec{v} \in S, \; R\vec{v} \in S)$

<u>Example</u> $S = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ mx \end{pmatrix} : x \in \mathbb{R} \right\} \subset \mathbb{R}^2$



Check this is a subspace

$\vec{0} \in S$? <u>yes</u> at $x=0$ $\begin{pmatrix} 0 \\ m \cdot 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ ✓

Closed under addition?

$\vec{v}, \vec{w} \in S \quad \vec{v} = \begin{pmatrix} x_1 \\ mx_1 \end{pmatrix} \quad \vec{w} = \begin{pmatrix} x_2 \\ mx_2 \end{pmatrix}$

$\vec{v}+\vec{w} = \begin{pmatrix} x_1 \\ mx_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ mx_2 \end{pmatrix} = \begin{pmatrix} x_1+x_2 \\ mx_1+mx_2 \end{pmatrix}$

$= \begin{pmatrix} x_1+x_2 \\ m(x_1+x_2) \end{pmatrix} \in S$

<u>Yes</u> because $x_1+x_2 \in \mathbb{R}$

Closed under scalar?

$R \in \mathbb{R} \quad \vec{v} \in S \quad \vec{v} = \begin{pmatrix} x \\ mx \end{pmatrix}$

$R\vec{v} = R \begin{pmatrix} x \\ mx \end{pmatrix} = \begin{pmatrix} Rx \\ Rmx \end{pmatrix} = \begin{pmatrix} Rx \\ m(Rx) \end{pmatrix}$

$\in S$ because $Rx \in \mathbb{R}$.

<u>Yes</u>

So S is a subspace of $\mathbb{R}^2$.

Watch Video 313F20-10-5a for the definition and then Video 313F20-10-5b for classwork and hw hints:

Linearly Independent Vectors

Defn: A collection of vectors, $\vec{v}_1, \vec{v}_2, \dots \vec{v}_k$, is linearly independent if

$$\underbrace{\sum_{j=1}^{k} t_j \vec{v}_j}_{\text{linear combination}} = \vec{0} \iff \text{all the } t_j = 0 \text{ for } j=1,2 \dots k$$

$$\underbrace{t_1 \vec{v}_1 + t_2 \vec{v}_2 + \dots + t_k \vec{v}_k}_{} = \vec{0}$$

solve for $t_1, t_2 \dots t_k$ to this homogeneous system.

Watch Video 313F20-10-5b for the following classwork and HW9-10 hints:

Classwork:

Are $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}$ and $\begin{pmatrix}2\\3\\0\end{pmatrix}$ lin. indep?

$$t_1 \begin{pmatrix}1\\0\\0\end{pmatrix} + t_2 \begin{pmatrix}0\\1\\0\end{pmatrix} + t_3 \begin{pmatrix}2\\3\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}1t_1 + 0t_2 + 2t_3\\ 0t_1 + 0t_2 + 3t_3\\ 0t_1 + 0t_2 + 0t_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

no leaders in a column so a free variable

$$\left[\begin{array}{ccc|c} \boxed{1} & 0 & 2 & 0\\ 0 & 0 & \boxed{3} & 0\\ 0 & 0 & 0 & 0\end{array}\right] \xrightarrow{\rho_2 \to \frac{1}{3}\rho_2} \left[\begin{array}{ccc|c} \boxed{1} & 0 & 2 & 0\\ 0 & 0 & \boxed{1} & 0\\ 0 & 0 & 0 & 0\end{array}\right]$$

2 leaders and one free variable

The solution set is not just $\{\vec{0}\}$

There are t_1, t_2, t_3 not all zero such that

$$t_1 \begin{pmatrix}1\\0\\0\end{pmatrix} + t_2 \begin{pmatrix}0\\1\\0\end{pmatrix} + t_3 \begin{pmatrix}2\\3\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

Thus $\begin{pmatrix}1\\0\\0\end{pmatrix} \begin{pmatrix}0\\1\\0\end{pmatrix}$ and $\begin{pmatrix}2\\3\\0\end{pmatrix}$ is not lin indep.

Lesson 20 Part IV

Basis of a Subspace

Defn

Suppose $S \subset \mathbb{R}^m$ is a subspace

we say $\vec{v}_1, \vec{v}_2 \ldots \vec{v}_k$ is a "basis" for S if

$$S = \langle \vec{v}_1, \vec{v}_2, \ldots \vec{v}_k \rangle$$

and

$\vec{v}_1 \ldots \vec{v}_k$ are linearly independent

Example:

$$S = \left\{ \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} : x, y \in \mathbb{R} \right\}$$

$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ are a basis.

To see this we check

$$S = \langle \vec{v}_1, \vec{v}_2 \rangle$$

proof:

$$\langle \vec{v}_1, \vec{v}_2 \rangle = \{ t_1 \vec{v}_1 + t_2 \vec{v}_2 \mid t_1, t_2 \in \mathbb{R} \}$$
$$= \left\{ t_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \mid t_1, t_2 \in \mathbb{R} \right\}$$
$$= \left\{ \begin{pmatrix} t_1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ t_2 \\ 0 \end{pmatrix} \mid t_1, t_2 \in \mathbb{R} \right\}$$
$$= \left\{ \begin{pmatrix} t_1 \\ t_2 \\ 0 \end{pmatrix} \mid \underbrace{t_1, t_2 \in \mathbb{R}}_{\substack{\text{free variable} \\ t_1 = x \quad t_2 = y}} \right\} = S$$

Also check linearly indep

$$t_1 v_1 + t_2 v_2 = 0 \iff t_1 = 0 \text{ and } t_2 = 0.$$

Lesson 20 Part IV

Basis of a Subspace

Defn

Suppose $S \subset \mathbb{R}^m$ is a subspace

we say $\vec{v}_1, \vec{v}_2 \ldots \vec{v}_k$ is a "basis" for S if

$$S = \langle \vec{v}_1, \vec{v}_2, \ldots \vec{v}_k \rangle$$

and

$\vec{v}_1 \ldots \vec{v}_k$ are linearly independent

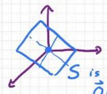
$$t_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} t_1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ t_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} t_1 \\ t_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$t_1 = 0$ and $t_2 = 0$

So they are lin indep

Span + Lin Indep

Thus they are a basis.



$\mathbb{R}^n$ Euclidean Space of dimension n

S is a subspace:

$\vec{0} \in S$

closed under + $\to$ $\forall \vec{v}, \vec{w} \in S$ $\vec{v}+\vec{w} \in S$

closed under scalar $\to$ $\forall R \in \mathbb{R}$ $\forall \vec{v} \in S$ $R\vec{v} \in S$

$\vec{v}_1 \dots \vec{v}_k$ are linearly independent if $\sum_{i=1}^{k} a_i \vec{v}_i = 0 \Leftrightarrow$ all $a_i = 0$, $i = 1 \dots k$

$S = \langle \vec{v}_1, \vec{v}_2 \dots \vec{v}_k \rangle = \left\{ \sum_{i=1}^{k} a_i \vec{v}_i \mid a_i \in \mathbb{R} \right\}$ span

Defn: $\vec{v}_1 \dots \vec{v}_k$ are basis for S if $S = \langle \vec{v}_1 \dots v_k \rangle$ and $\vec{v}_1 \dots \vec{v}_k$ are lin. indep.

Defn The dimension of $S \subset \mathbb{R}^n$ is equal to k if it has basis with k vectors.

$$\mathbb{R}^n = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \middle| \begin{array}{c} x_i \in \mathbb{R} \\ \text{free} \end{array} \right\}$$

$$= \left\{ x_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \dots + x_n \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \middle| x_i \in \mathbb{R} \right\}$$

ndim has n basis vectors

standard basis of $\mathbb{R}^n$

$$\hat{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \hat{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \dots \hat{e}_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

$\boxed{\text{HW1}}$ Verify that $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ are a basis for $S = \mathbb{R}^3$

basis has 3 vectors

3dim

Check: span $\mathbb{R}^3$ (by defn of $\mathbb{R}^3$)

Check: linearly indep.

Linear Algebra

$\mathbb{R}^n$ Euclidean Space of dimension n

S is a subspace:

$\vec{0} \in S$

closed under + $\to$ $\forall \vec{v}, \vec{w} \in S$ $\vec{v}+\vec{w} \in S$

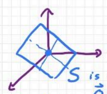
closed under scalar $\to$ $\forall R \in \mathbb{R}$ $\forall \vec{v} \in S$ $R\vec{v} \in S$

$\vec{v}_1 \dots \vec{v}_k$ are linearly independent if $\sum_{i=1}^{k} a_i \vec{v}_i = 0 \Leftrightarrow$ all $a_i = 0$, $i = 1 \dots k$

$S = \langle \vec{v}_1, \vec{v}_2 \dots \vec{v}_k \rangle = \left\{ \sum_{i=1}^{k} a_i \vec{v}_i \mid a_i \in \mathbb{R} \right\}$ span

Defn: $\vec{v}_1 \dots \vec{v}_k$ are basis for S if $S = \langle \vec{v}_1 \dots v_k \rangle$ and $\vec{v}_1 \dots \vec{v}_k$ are lin. indep.

Defn The dimension of $S \subset \mathbb{R}^n$ is equal to k if it has basis with k vectors.



Thm: If $\vec{v}_1, \vec{v}_2 \dots \vec{v}_k$ is a basis for S and $\vec{w}_1, \vec{w}_2 \dots \vec{w}_m$ is a basis for S then $k = m$.

Thus dim is defined.

Proof by Contradiction:

(1) Assume on the contrary that $k \neq m$ — (1) Indirect Hypothesis

Examine the consequences (plan)

Reach a Contradiction

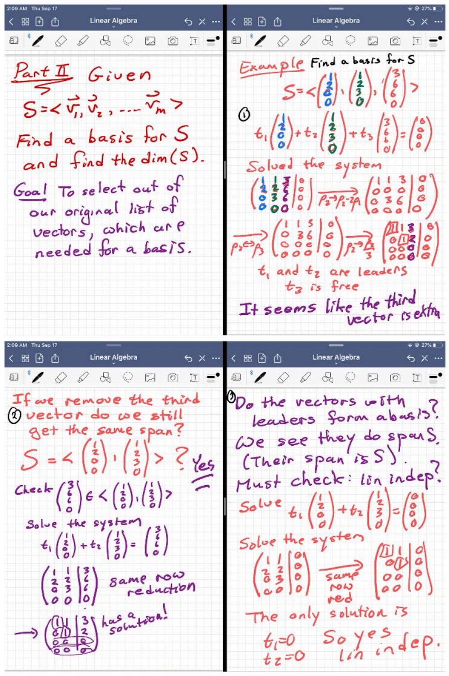
Thus $k = m$ QED

****

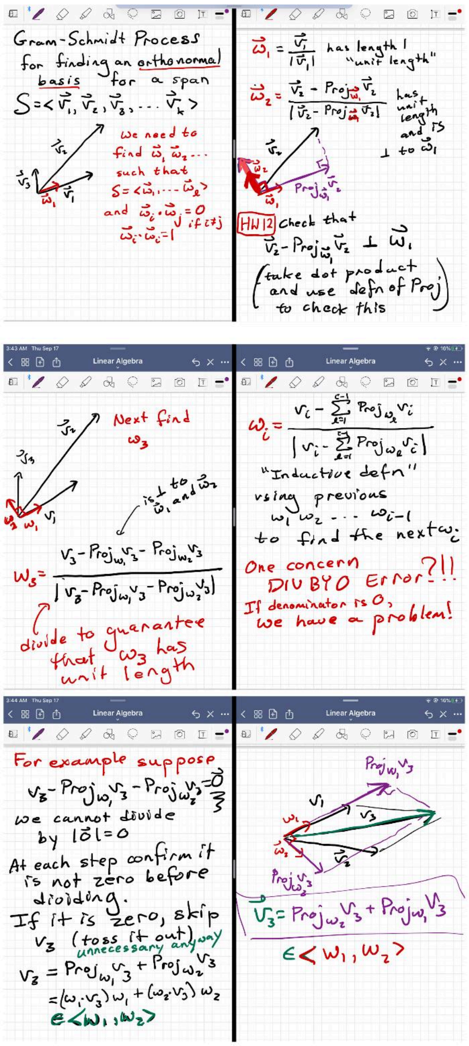
Part II **Finding a basis using pivot columns (required)**

****

Watch [Video 313F20-11-4]



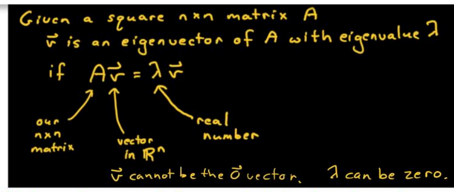
Gram-Schmidt is not on the final but remember it is useful:



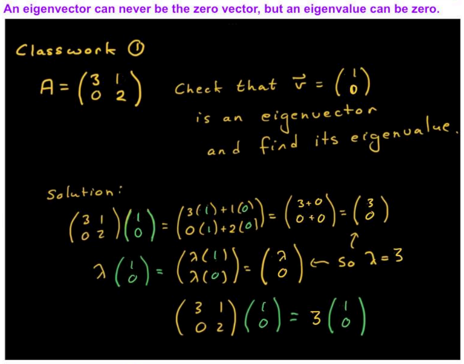
****

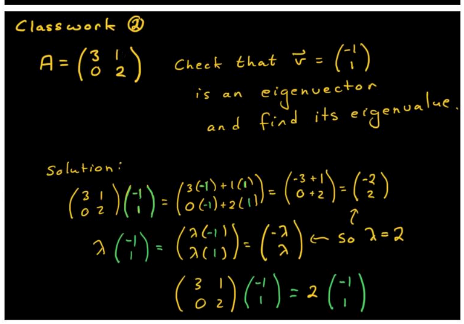
Review Lesson 11: Eigenvalues and Eigenvectors
Review Lesson 23: Eigenspaces

****



An eigenvector can never be the zero vector, but an eigenvalue can be zero.





$\boxed{\text{Theorem}}$ Suppose $A \in \mathbb{R}^{n\times n}$

has $A\vec{v}_i = \lambda_i \vec{v}_i$ for $i=1$ to n

then $$A^k\left(\sum_{i=1}^{n} c_i \vec{v}_i\right) = \sum_{i=1}^{n} c_i \lambda_i^k \vec{v}_i$$

and if you can solve the system

$$c_1\vec{v}_1 + \dots + c_n\vec{v}_n = \vec{\omega}$$

This is the same as saying the matrix whose columns are $\vec{v}_1, \dots, \vec{v}_n$ is a nonsingular matrix

then $$A^k \vec{\omega} = c_1\lambda_1^k \vec{v}_1 + \dots + c_n \lambda_n^k \vec{v}_n$$

$\boxed{\text{Note}}$ The 3×3 matrix in $\boxed{\text{HW1}}$ is especially nice because you can always solve the system

$$c_1\begin{pmatrix}1\\0\\0\end{pmatrix} + c_2\begin{pmatrix}0\\1\\-1\end{pmatrix} + c_3\begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}\omega_1\\\omega_2\\\omega_3\end{pmatrix}$$

$$\begin{pmatrix}5&0&0\\0&2&4\\0&4&2\end{pmatrix}^k\begin{pmatrix}\omega_1\\\omega_2\\\omega_3\end{pmatrix}$$

We know this because

$$\left(\begin{array}{ccc|c}1&0&0&\omega_1\\0&1&1&\omega_2\\0&-1&1&\omega_3\end{array}\right) \xrightarrow{\rho_3\to\rho_3+\rho_2} \left(\begin{array}{ccc|c}1&0&0&\omega_1\\0&1&1&\omega_2\\0&0&2&\omega_2+\omega_3\end{array}\right)$$

$$\xrightarrow{\rho_3\to\frac{1}{2}\rho_3} \left(\begin{array}{ccc|c}1&0&0&\omega_1\\0&1&1&\omega_2\\0&0&1&(\omega_2+\omega_3)/2\end{array}\right)$$

$$\xrightarrow{\rho_2\to\rho_2-\rho_3} \left(\begin{array}{ccc|c}1&0&0&\omega_1\\0&1&0&\omega_2-(\omega_2+\omega_3)/2\\0&0&1&(\omega_2+\omega_3)/2\end{array}\right)$$

always has the solution

$$c_1 = \omega_1$$
$$c_2 = \omega_2 - (\omega_2+\omega_3)/2$$
$$c_3 = (\omega_2+\omega_3)/2$$

So $$\begin{pmatrix}5&0&0\\0&2&4\\0&4&2\end{pmatrix}^k\begin{pmatrix}\omega_1\\\omega_2\\\omega_3\end{pmatrix} = \begin{pmatrix}5&0&0\\0&2&4\\0&4&2\end{pmatrix}^k\left(c_1\begin{pmatrix}1\\0\\0\end{pmatrix} + c_2\begin{pmatrix}0\\1\\-1\end{pmatrix} + c_3\begin{pmatrix}0\\1\\1\end{pmatrix}\right)$$

$$= c_1(5)^k\begin{pmatrix}1\\0\\0\end{pmatrix} + c_2(-2)^k\begin{pmatrix}0\\1\\-1\end{pmatrix} + c_3(6)^k\begin{pmatrix}0\\1\\1\end{pmatrix}$$

can always be found for any $\vec{\omega}$.

HW8 Use the power method to find the largest eigenvalue of $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$ starting with $\vec{x}_0 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Example #1 $A = \begin{bmatrix} 7 & 9 \\ 9 & 7 \end{bmatrix}$ $\lambda_1 = 16$, $\lambda_2 = -2$

① $x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

② $k=0$: $p = Ax_0 = \begin{bmatrix} 7 \\ 9 \end{bmatrix}$, $n_0 = 9$, $x_1 = \begin{bmatrix} 0.77 \\ 1 \end{bmatrix}$

$k=1$: $p = Ax_1 = \begin{bmatrix} 14.39 \\ 13.93 \end{bmatrix}$, $n_1 = 14.39$, $x_2 = \begin{bmatrix} 1.00 \\ 0.97 \end{bmatrix}$

$k=2$: $p = Ax_2 = \begin{bmatrix} 15.73 \\ 15.79 \end{bmatrix}$, $n_2 = 15.79$, $x_3 = \begin{bmatrix} 0.996 \\ 1 \end{bmatrix}$

$k=3$: $p = Ax_3 = \begin{bmatrix} 15.972 \\ 15.964 \end{bmatrix}$, $n_3 = 15.972$, $x_4 = \begin{bmatrix} 1 \\ 0.999 \end{bmatrix}$

③ $n \to 16 = \lambda_1$

A·P

HW8 Use the power method to find the largest eigenvalue of $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$ starting with $x_0 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Hint for HW8 follow this method

Solution for HW8 is below to check your work

① $\vec{x}_0 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

② $k=0$

$A\vec{x}_0 = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3\cdot 1 + 0\cdot 2 \\ 0\cdot 1 + 2\cdot 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

n_0 is the largest number here

so $n_0 = 4$

$\vec{x}_1 = \frac{1}{n_0} A\vec{x}_0 = \frac{1}{4}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 3/4 \\ 1 \end{pmatrix}$

$k=1$

$A\vec{x}_1 = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 3/4 \\ 1 \end{pmatrix} = \begin{pmatrix} 3(3/4) + 0\cdot 1 \\ 0(3/4) + 2\cdot 1 \end{pmatrix} = \begin{pmatrix} 9/4 \\ 2 \end{pmatrix} = \begin{pmatrix} 2.25 \\ 2 \end{pmatrix}$

n_1 is the largest number here

so $n_1 = 2.25$

$\vec{x}_2 = \frac{1}{2.25} A\vec{x}_1 = \frac{1}{2.25}\begin{pmatrix} 2.25 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 8/9 \end{pmatrix}$

$k=2$ $A\vec{x}_2 = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 8/9 \end{pmatrix} = \begin{pmatrix} 3 \\ 16/9 \end{pmatrix}$

$n_2 = 3$ $\vec{x}_3 = \frac{1}{3} A\vec{x}_2 = \frac{1}{3}\begin{pmatrix} 3 \\ 16/9 \end{pmatrix} = \begin{pmatrix} 1 \\ 16/27 \end{pmatrix}$

$k=3$ $A\vec{x}_3 = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 16/27 \end{pmatrix} = \begin{pmatrix} 3 \\ 32/27 \end{pmatrix}$

$n_3 = 3$ $\vec{x}_4 = \frac{1}{n_3} A\vec{x}_3 = \begin{pmatrix} 1 \\ 32/81 \end{pmatrix}$

same as n_2 so 3 is the eigenvalue

$k=4$ $A\vec{x}_3 = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 32/81 \end{pmatrix} = \begin{pmatrix} 3 \\ 64/81 \end{pmatrix}$

$n_4 = 3$ $\vec{x}_4 = \frac{1}{3} A\vec{x}_3 = \frac{1}{3}\begin{pmatrix} 3 \\ 64/81 \end{pmatrix} \approx \begin{pmatrix} 1 \\ 21/81 \end{pmatrix}$ ← I approximated here

$k=5$ $A\vec{x}_4 = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 21/81 \end{pmatrix} = \begin{pmatrix} 3 \\ 10/81 \end{pmatrix}$

$n_4 = 3$ $\vec{x}_5 = \frac{1}{3} A\vec{x}_4 = \begin{pmatrix} 1 \\ 3/81 \end{pmatrix}$

Notice $\vec{x}_k = \begin{pmatrix} 1 \\ \text{something small} \end{pmatrix} \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ as $k \to \infty$

Check $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ ✓

Why does HW8 work?

these are the eigenvectors

these are the eigenvalues

$$\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\left(1\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = 1\cdot 3\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2\cdot 2\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$n_1 = 4$

$$\text{So } \vec{x}_1 = \frac{1}{4}\left(1\cdot 3\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2\cdot 2\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$$

$$\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\vec{x}_1 = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\frac{1}{4}\left(1\cdot 3\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2\cdot 2\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$$

$$= \frac{1}{4}\left(1\cdot 3^2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2\cdot 2^2\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 9/4 \\ 2 \end{pmatrix}$$

$n_2 = \frac{9}{4}$

$$\text{So } \vec{x}_2 = \left(\frac{\cancel{4}}{9}\right)\left(\frac{1}{\cancel{4}}\right)\left(1\cdot 3^2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2\cdot 2^2\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$$

$$\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}\vec{x}_2 = \frac{1}{9}\left(1\cdot 3^3\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2\cdot 2^3\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 3 \\ 2^4/9 \end{pmatrix}$$

$n_3 = 3$

$$\text{So } \vec{x}_3 = \frac{1}{3}\frac{1}{9}\left(1\cdot 3^3\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2\cdot 2^3\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$$

...

$$\text{So } \vec{x}_k = \frac{1}{3^k}\left(1\cdot 3^k\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2\cdot 2^k\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{2\cdot 2^k}{3^k}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

as $k \to \infty$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 0\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

first eigenvector dominates

Part I Finding eigenvalues using the characteristic polynomial

Watch Playlist 313F22-23-1-1to9

Today's Topics

Eigenvalues, Eigenvectors

Characteristic Polynomials

Eigenspaces

Brief Overview and Review

Defn: Given a square matrix A we say that λ (lambda) is an eigenvalue of A with eigenvector $\vec{v}$ if

$$A\vec{v} = \lambda\vec{v}$$

where $\vec{v} \neq 0$ but $\lambda \in \mathbb{R}$ (even $\mathbb{C}$)
real (or complex)

Notice: $A\vec{v} - \lambda\vec{v} = \vec{0}$

$A\vec{v} - \lambda I\vec{v} = \vec{0}$ because $I\vec{v} = \vec{v}$

$(A - \lambda I)\vec{v} = \vec{0}$

Theorem: λ is an eigenvalue for A with eigenvector $\vec{v} \neq \vec{0}$ if $(A - \lambda I)\vec{v} = \vec{0}$

Given an eigenvalue λ then find the eigenvectors by solving the homogeneous system

$$\left([A - \lambda I] \middle| \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \right)$$

for null space $\{\vec{v} = \dots \mid \dots\}$

The solution set is called the eigenspace for λ.

All the vectors in the eigenspace except for $\vec{v} = \vec{0}$ are eigenvectors for λ.

What do we do if we do not know the eigenvalues for A?

By the theorem above:

λ is an eigenvalue for A iff $\left([A - \lambda I] \middle| \vec{0} \right)$ has a solution set with nonzero vectors.

This happens when $A - \lambda I$ is singular.

Today's Topics

Eigenvalues, Eigenvectors
Characteristic Polynomials
Eigenspaces

Brief Overview and Review

Defn: Given a square matrix A we say that λ (lambda) is an eigenvalue of A with eigenvector $\vec{v}$ if

$$A\vec{v} = \lambda\vec{v}$$

where $\vec{v} \neq 0$ but $\lambda \in \mathbb{R}$ (even $\mathbb{C}$)
real (or complex)

Notice: $A\vec{v} - \lambda\vec{v} = \vec{0}$
$A\vec{v} - \lambda I\vec{v} = \vec{0}$ because $I\vec{v} = \vec{v}$
$(A - \lambda I)\vec{v} = \vec{0}$

Theorem: λ is an eigenvalue for A with eigenvector $\vec{v} \neq \vec{0}$ if $(A - \lambda I)\vec{v} = \vec{0}$

Given an eigenvalue λ then find the eigenvectors by solving the homogeneous system $\left([A - \lambda I] \middle| \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}\right)$

for null space $\{\vec{v} = \ldots \mid \ldots\}$

The solution set is called the eigenspace for λ.
All the vectors in the eigenspace except for $\vec{v} = \vec{0}$ are eigenvectors for λ.

What do we do if we do not know the eigenvalues for A?

By the theorem above:
λ is an eigenvalue for A iff $\left([A - \lambda I] \middle| \vec{0}\right)$ has a solution set with nonzero vectors.

This happens when $A - \lambda I$ is singular.

This happens when

$$\det(A - \lambda I) = 0$$

This is actually a polynomial with λ as the variable
Called the characteristic polynomial.

Classwork:
Use $\det(A-\lambda I)=0$ to find the eigenvalues of A where:

$$A=\begin{pmatrix}1&0&1&1\\0&1&1&1\\1&1&1&0\\1&1&0&1\end{pmatrix}$$

4x4 matrix

To find det of an nxn matrix
Method of Minors

$$\det\begin{pmatrix}1&0&1&1\\0&1&1&1\\1&1&1&0\\1&1&0&1\end{pmatrix}$$

choose a row with some zeroes

$$\begin{matrix}+&-&+&-\\-&+&-&+\\+&-&+&-\\-&+&-&+\end{matrix}$$

← we chose first row

$$= +a_{11}\det A_{11} - a_{12}\det A_{12} + a_{13}\det A_{13} - a_{14}\det A_{14}$$

where A_{ij} is the minor for a_{ij}: found by crossing out row i column j

$$\det(A-\lambda I) = \det\left(\begin{pmatrix}1&0&1&1\\0&1&1&1\\1&1&1&0\\1&1&0&1\end{pmatrix}-\lambda\begin{pmatrix}&&&\\&&&\\&&&\\&&&\end{pmatrix}\right)$$

↑ Id matrix

$$\det(A-\lambda I) = \det\left(\begin{pmatrix}1&0&1&1\\0&1&1&1\\1&1&1&0\\1&1&0&1\end{pmatrix}-\lambda\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\right)$$

↑ Id matrix

Identity Matrix 4x4

$$\det(A-\lambda I) = \det\left(\begin{pmatrix}1&0&1&1\\0&1&1&1\\1&1&1&0\\1&1&0&1\end{pmatrix}-\lambda\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\right)$$

↑ Id matrix

Identity Matrix 4x4

$$= \det\begin{pmatrix}(1-\lambda)&0&1&1\\0&(1-\lambda)&1&1\\1&1&(1-\lambda)&0\\1&1&0&(1-\lambda)\end{pmatrix}$$

← notice this is just the matrix A with $-\lambda$ on diagonals

$$\det(A-\lambda I) = \det\left(\begin{pmatrix}1&0&1&1\\0&1&1&1\\1&1&1&0\\1&1&0&1\end{pmatrix}-\lambda\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\right)$$

↑ Id matrix

Identity Matrix 4x4

$$= \det\begin{pmatrix}(1-\lambda)&0&1&1\\0&(1-\lambda)&1&1\\1&1&(1-\lambda)&0\\1&1&0&(1-\lambda)\end{pmatrix}$$

← notice this is just the matrix A with $-\lambda$ on diagonals

Take the det using the method of minors

$$\begin{matrix}+&-&+&-\\-&+&-&+\\+&-&+&-\\-&+&-&+\end{matrix}$$

$$= +(1-\lambda)\det A_{11} - 0\det A_{12} + 1\det A_{13} - 1\det A_{14}$$

How to find the minors:

$$\begin{pmatrix}(1-\lambda)&0&1&1\\0&(1-\lambda)&1&1\\1&1&(1-\lambda)&0\\1&1&0&(1-\lambda)\end{pmatrix}$$

$$= (1-\lambda)\det\begin{pmatrix}(1-\lambda)&1&1\\1&(1-\lambda)&0\\1&0&(1-\lambda)\end{pmatrix}$$

$$- 0\det\begin{pmatrix}0&1&1\\1&(1-\lambda)&0\\1&0&(1-\lambda)\end{pmatrix}$$

$$+ 1\det\begin{pmatrix}0&(1-\lambda)&1\\1&1&0\\1&1&(1-\lambda)\end{pmatrix}$$

$$- 1\det\begin{pmatrix}0&(1-\lambda)&1\\1&1&(1-\lambda)\\1&1&0\end{pmatrix}$$

To find each of these 3x3 det use 3x3 trick

$$= (1-\lambda)\begin{pmatrix} (1-\lambda)(1-\lambda)(1-\lambda) + 1\cdot 0\cdot 1 + 1\cdot 1\cdot 0 \\ -1(1-\lambda)1 - (1-\lambda)\cdot 0\cdot 0 - 1\cdot 1\cdot (1-\lambda) \end{pmatrix} - 0$$

parentheses around the whole det

a real number

$$\left(\begin{array}{ccc|cc} (1-\lambda) & 1 & 1 & (1-\lambda) & 1 \\ 1 & (1-\lambda) & 0 & 1 & (1-\lambda) \\ 1 & 0 & (1-\lambda) & 1 & 0 \end{array}\right)$$

$$+1\begin{pmatrix} 0+0+1 \\ -1-0-(1-\lambda)1(1-\lambda) \end{pmatrix} - 1\begin{pmatrix} 0+(1-\lambda)^2+1 \\ -1-0-0 \end{pmatrix}$$

Simplify the Polynomial:

Simplify the Polynomial:

$$= (1-\lambda)\left((1-\lambda)^3 - 2(1-\lambda)\right) - 0$$
$$+1\left(-(1-\lambda)^2\right) - 1\left((1-\lambda)^2\right)$$
$$= (1-\lambda)^4 - 2(1-\lambda)^2 - (1-\lambda)^2 - (1-\lambda)^2$$
$$= (1-\lambda)^4 - 4(1-\lambda)^2$$
$$= \left((1-\lambda)^2 - 4\right)(1-\lambda)^2$$
$$= (1-2\lambda+\lambda^2-4)(1-\lambda)^2$$
$$= (\lambda^2-2\lambda-3)(1-\lambda)^2$$
$$= (\lambda-3)(\lambda+1)(1-\lambda)^2 = 0$$

be very careful with parentheses!

So $\lambda = 3$ $\lambda = -1$ $\lambda = 1$

are the eigenvalues of $A = \begin{pmatrix} 1&0&1&1 \\ 0&1&1&1 \\ 1&1&1&0 \\ 1&1&0&1 \end{pmatrix}$

Next Part of Lesson

find eigenvectors and eigenspaces.

Hint: If you cannot factor ax^2+bx+c use the following theorem:

THEOREM: "Quadratic Formula" The roots of $ax^2+bx+c=0$

are $$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

Part II Finding eigenspaces and eigenvectors for each eigenvalue

You may wish to do this on a different day.

Watch [Playlist 313F22-23-2-1to3]

Defn: Given an $n \times n$ matrix A

A real or complex number λ is an **eigenvalue** of A if there is a nonzero vector $\vec{v}$ such that

$$A\vec{v} = \lambda\vec{v}$$

Any such vector $\vec{v}$ is an **eigenvector**.

The set of all such vectors is the **eigenspace** for eigenvalue λ.

Theorem: Given an $n \times n$ matrix A

A real or complex number λ is an **eigenvalue** of A if there is a nonzero solution to the homogeneous system:

$$(A - \lambda I)\vec{v} = \vec{0}$$

Any such vector $\vec{v}$ is an **eigenvector**.

The set of all such vectors is the **eigenspace** for eigenvalue λ.

Corollary: λ is an eigenvalue of $A \Leftrightarrow$
$\Leftrightarrow$ $A - \lambda I$ singular $\Longleftrightarrow \det(A - \lambda I) = 0$

Corollary: The eigenspace of λ is the null space of $A - \lambda I$: $\text{Null}(A - \lambda I)$.

$\boxed{\text{Proof}}$ of the Theorem:

$$A v = \lambda \vec{v} \Longleftrightarrow (A - \lambda I)\vec{v} = 0$$

Must prove both directions

$\boxed{\text{Part I}}$ $\underbrace{A\vec{v} = \lambda\vec{v}}_{\text{given}} \Longrightarrow \underbrace{(A - \lambda I)\vec{v} = 0}_{\text{show}}$

① $A\vec{v} = \lambda\vec{v}$	① Given
② $A\vec{v} = \lambda I\vec{v}$	② $I\vec{v} = \vec{v}$
③ $A\vec{v} - \lambda I\vec{v} = \vec{0}$	③ subtracting $\lambda I\vec{v}$ on both sides
④ $(A - \lambda I)\vec{v} = \vec{0}$	④ $(A - B)\vec{w} = A\vec{w} - B\vec{w}$ where $B = \lambda I$ $\vec{w} = \vec{v}$

Part I done

$\boxed{\text{Part II}}$ $(A - \lambda I)\vec{v} = \vec{0} \Longrightarrow A\vec{v} = \lambda\vec{v}$

① $(A - \lambda I)\vec{v} = \vec{0}$	① Given
② $A\vec{v} - \lambda I\vec{v} = \vec{0}$	② $(A - B)\vec{w} = A\vec{w} - B\vec{w}$
③ $A\vec{v} = \lambda I\vec{v}$	③ add to both sides
④ $A\vec{v} = \lambda\vec{v}$	④ $I\vec{v} = \vec{v}$

Part II done

QED

Classwork

Find the eigenspace for $\lambda=3$ of $A=\begin{pmatrix}4&1&1\\1&4&1\\1&1&4\end{pmatrix}$

$$\text{Null}(A-\lambda I)=\text{Null}\left(\begin{pmatrix}4&1&1\\1&4&1\\1&1&4\end{pmatrix}-3\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\right)$$

$$=\text{Null}\begin{pmatrix}(4-3)&1&1\\1&(4-3)&1\\1&1&(4-3)\end{pmatrix}=\text{Null}\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$$

$$\left(\begin{array}{ccc|c}\boxed{1}&1&1&0\\1&1&1&0\\1&1&1&0\end{array}\right)\xrightarrow[\rho_3\to\rho_3-\rho_1]{\rho_2\to\rho_2-\rho_1}\left(\begin{array}{ccc|c}\boxed{1}&1&1&0\\0&0&0&0\\0&0&0&0\end{array}\right)$$ Reduced Echelon Form

$$\boxed{\begin{array}{r}\boxed{1x_1}+1x_2+1x_3=0\\0=0\\0=0\end{array}}$$

Solve for leaders $x_1=-x_2-x_3$

free $x_2=x_2$, $x_3=x_3$

$$\text{Null}\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}=\left\{\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}-x_2-x_3\\x_2\\x_3\end{pmatrix}\ \middle|\ x_2,x_3\in\mathbb{R}\right\}$$

$$\text{eigenspace}=\left\{\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=x_2\begin{pmatrix}-1\\+1\\0\end{pmatrix}+x_3\begin{pmatrix}-1\\0\\1\end{pmatrix}\ \middle|\ x_2,x_3\in\mathbb{R}\right\}$$

two eigenvectors

many more: for example $x_2=4$, $x_3=5$

$$4\begin{pmatrix}-1\\+1\\0\end{pmatrix}+5\begin{pmatrix}-1\\0\\1\end{pmatrix}=\begin{pmatrix}-4\\4\\0\end{pmatrix}+\begin{pmatrix}-5\\0\\5\end{pmatrix}=\begin{pmatrix}-9\\4\\5\end{pmatrix}$$ is also an eigenvector

$$\begin{pmatrix}4&1&1\\1&4&1\\1&1&4\end{pmatrix}\begin{pmatrix}-1\\+1\\0\end{pmatrix}=\begin{pmatrix}4(-1)+1(1)+1\cdot 0\\1(-1)+4(1)+1(0)\\1(-1)+1(0)+4(0)\end{pmatrix}=\begin{pmatrix}-3\\3\\0\end{pmatrix}=3\begin{pmatrix}-1\\+1\\0\end{pmatrix}\ \checkmark$$

$$\begin{pmatrix}4&1&1\\1&4&1\\1&1&4\end{pmatrix}\begin{pmatrix}-1\\0\\1\end{pmatrix}=$$ You check this!

[redacted]

HW What if you try $\lambda=2$? What happens?

warning not an eigenvalue!

If the λ you use gives $\text{Null}(A-\lambda I)=\{\vec{0}\}$ no free variables

Then you have either a mistake when you decided λ was an evalue or an error finding its eigenspace.

HW5 and HW6 above

HW Find the eigenspaces of $A=\begin{pmatrix}1&0&1&1\\0&1&1&1\\1&1&1&0\\1&1&0&1\end{pmatrix}$

Recall: $\lambda=3$ $\lambda=-1$ $\lambda=1$ are the eigenvalues of $A=\begin{pmatrix}1&0&1&1\\0&1&1&1\\1&1&1&0\\1&1&0&1\end{pmatrix}$

So find the eigenspace for each.

Be sure to check your answers!

Check $A\vec{v}=\lambda\vec{v}$

HW7 above is very long. So it counts as HW7 and HW8. Here's the solution for eigenvalue=1

Solution for $\lambda=1$:

$$\left(\begin{array}{cccc|c}1-1&0&1&1&0\\0&1-1&1&1&0\\1&1&1-1&0&0\\1&1&0&1-1&0\end{array}\right)=\left(\begin{array}{cccc|c}0&0&1&1&0\\0&0&1&1&0\\1&1&0&0&0\\1&1&0&0&0\end{array}\right)\xrightarrow{\rho_1\leftrightarrow\rho_3}\left(\begin{array}{cccc|c}1&1&0&0&0\\0&0&1&1&0\\0&0&1&1&0\\1&1&0&0&0\end{array}\right)\xrightarrow{\rho_4\to\rho_4-\rho_1}\left(\begin{array}{cccc|c}\boxed{1}&1&0&0&0\\0&0&\boxed{1}&1&0\\0&0&1&1&0\\0&0&0&0&0\end{array}\right)$$

subtract λ on diagonal

$$\xrightarrow{\rho_3\to\rho_3-\rho_2}\left(\begin{array}{cccc|c}1&1&0&0&0\\0&0&1&1&0\\0&0&0&0&0\\0&0&0&0&0\end{array}\right)$$

$x_1+x_2=0$
$x_3+x_4=0$

$x_1=-x_2$
$x_2=x_2$ (free)
$x_3=-x_4$
$x_4=x_4$ (free)

$$\left\{\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=x_2\begin{pmatrix}-1\\1\\0\\0\end{pmatrix}+x_4\begin{pmatrix}0\\0\\-1\\1\end{pmatrix}\ \middle|\ x_2,x_4\in\mathbb{R}\right\}$$

both eigenvectors
can check them.

$$\begin{pmatrix}1&0&1&1\\0&1&1&1\\1&1&1&0\\1&1&0&1\end{pmatrix}\begin{pmatrix}-1\\1\\0\\0\end{pmatrix}=\begin{pmatrix}-1+0+0+0\\0+1+0+0\\-1+1+0+0\\-1+1+0+0\end{pmatrix}=\begin{pmatrix}-1\\1\\0\\0\end{pmatrix}=1\begin{pmatrix}-1\\1\\0\\0\end{pmatrix}\ \checkmark$$

$$\begin{pmatrix}1&0&1&1\\0&1&1&1\\1&1&1&0\\1&1&0&1\end{pmatrix}\begin{pmatrix}0\\0\\-1\\1\end{pmatrix}=\text{you show this}=1\begin{pmatrix}0\\0\\-1\\1\end{pmatrix}$$

Do the other eigenvalues too

Note that if an eigenspace has two directions, you can use Gram-Schmidt to find an orthonormal basis to get perpendicular eigenvectors.

← Gram-Schmidt

Theorem: If $A^T = A$ and

$A\vec{v}_1 = \lambda_1 \vec{v}_1$ and $A\vec{v}_2 = \lambda_2 \vec{v}_2$

and $\lambda_1 \neq \lambda_2 \neq 0$

then $\vec{v}_1 \cdot \vec{v}_2 = 0$

HW12: Needs complex numbers!

HW Find eigenvalues and eigenspaces for $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ using the characteristic polynomial and then solving for eigenspaces

Solution Hints

$$\det\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$$

$$= \det\begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} = 0$$

$$(-\lambda)(-\lambda) - (-1)(1) = 0$$

$$\lambda^2 + 1 = 0$$

$$\lambda^2 = -1$$

has no real solutions!

$i = \sqrt{-1}$ $\quad \lambda = \pm i$

$-i = -\sqrt{-1}$

$\lambda = +i$ Find its evector

Solve $\quad -i(i) = -i^2 = -(-1) = +1$

$$\left(\begin{array}{cc|c} -i & -1 & 0 \\ 1 & -i & 0 \end{array}\right) \xrightarrow{\rho_1 \to i\rho_1} \left(\begin{array}{cc|c} 1 & -i & 0 \\ 1 & -i & 0 \end{array}\right)$$

$$\xrightarrow{\rho_2 \to \rho_2 - \rho_1} \left(\begin{array}{cc|c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$x_1 - ix_2 = 0 \quad x_1 = ix_2$

$\qquad x_2 = x_2$ (free)

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} ix_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} i \\ 1 \end{pmatrix} \;\middle|\; x_2 \in \mathbb{C} \right\}$$

So the evector is $\begin{pmatrix} i \\ 1 \end{pmatrix}$ free in $\mathbb{C}$

Check

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} i \\ 1 \end{pmatrix} = \begin{pmatrix} 0i - 1\cdot 1 \\ 1\cdot i + 0\cdot 1 \end{pmatrix} = \begin{pmatrix} -1 \\ i \end{pmatrix} \checkmark$$

$$i\begin{pmatrix} i \\ 1 \end{pmatrix} = \begin{pmatrix} i^2 \\ i \end{pmatrix} = \begin{pmatrix} -1 \\ i \end{pmatrix} \checkmark$$

Now Find evector for $\lambda = -i$ yourself!

Review Lesson 24: Diagonalization and Orthogonal Matrices

<u>Defn</u> An $n \times n$ matrix, B, is diagonalizable if $\exists$ a matrix $P \in M_{n \times n}$ such that

$$B = PDP^{-1}$$

where D is a diagonal matrix.

The first classwork is:

Example:
Diagonalize $B = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

So we need to find a diagonal matrix D and a matrix P such that:

$$B = PDP^{-1}$$

which is the same as
BP=PD

We will show how to diagonalize B before explaining why the method works.

The first step is to find the eigenvalues and eigenvectors:

Last lesson we showed B had eigenvalues 3 and -1 with eigenvectors: $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ for 3 and eigenvector: $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ for -1

The details on how to do this is in the previous lesson.

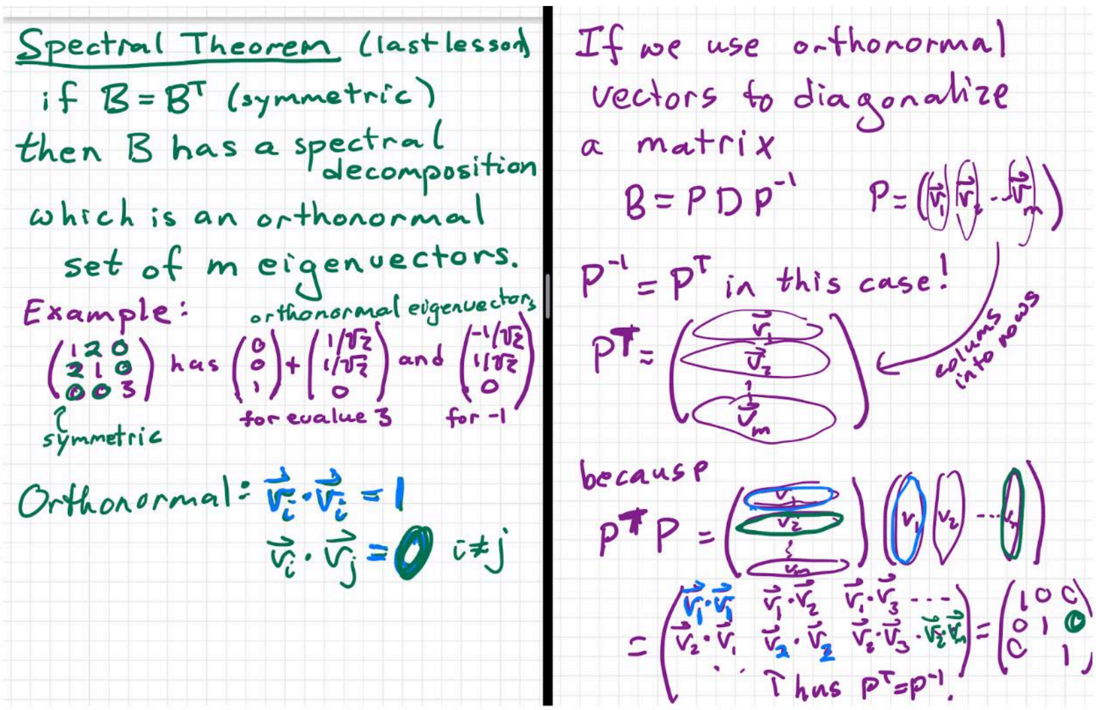
Next the matrix P is made out of the eigenvectors and the diagonal matrix D is made out of the eigenvalues:

$$P = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \text{ and } D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Note the first column of P is the eigenvector in green corresponding to the first eigenvalue in D also in green. The second column of P is the eigenvector in black corresponding to the second eigenvalue in D also in black. The third column of P is the eigenvector in purple corresponding to the third eigenvalue in D also in purple. The corresponding eigenvalues are put on the diagonal of D in the same order as their eigenvectors. This is explained in this video.

In particular, if you have a diagonalized matrix, you know its eigenvalues from D and the eigenspaces can be found using the columns of P that correspond to each eigenvalue. The span of the corresponding columns will be the eigenspace.

Finally we have the Spectral Theorem which involves special orthonormal eigenvectors to create an orthogonal matrix P. A matrix P is Orthogonal if its inverse is equal to its transpose which happens when the columns are orthonormal.



****

Review Lesson 26: Vector Spaces

Review Lesson 27: Linear Maps

Review Lesson 28: Basis and Dimension, one to one, onto,

****

Defn: A Vector Space V is a set with addition and scalar multiplication that has the following ten properties:

Properties of Vector Addition

- Closed under Vector Addition: $\boxed{\forall \vec{v}, \vec{w} \in V \quad \vec{v} + \vec{w} \in V}$ *
- Associativity Property: $\forall \vec{v}, \vec{w}, \vec{u} \in V \quad (\vec{v} + \vec{w}) + \vec{u} = \vec{v} + (\vec{u} + \vec{w})$ ✓
- Commutativity Property: $\forall \vec{v}, \vec{w} \in V \quad \vec{v} + \vec{w} = \vec{w} + \vec{v}$ ✓
- Additive Identity Property: $\exists \vec{0} \in V$ such that $\forall \vec{v} \in V \quad \vec{0} + \vec{v} = \vec{v} = \vec{v} + \vec{0}$
- Additive Inverses Property: $\forall \vec{v} \in V \quad \exists -\vec{v} \in V$ s.t $\vec{v} + (-\vec{v}) = (-\vec{v}) + \vec{v} = \vec{0}$

Properties of Scalar Multiplication

- Closed under Scalar Multiplication: $\boxed{\forall t \in \mathbb{R} \quad \vec{v} \in V \quad t\vec{v} \in V}$ *
- Compatibility Property: $\forall s, t \in \mathbb{R} \quad \forall \vec{v} \in V \quad (st)\vec{v} = s(t\vec{v})$ ✓
- Scalar Identity Property: $\exists 1 \in \mathbb{R}$ s.t. $\forall \vec{v} \in V \quad 1\vec{v} = \vec{v}$ ✓
- Distribution over Vector Addition Property: $\forall t \in \mathbb{R} \quad \forall \vec{v}, \vec{w} \in V \quad t(\vec{v} + \vec{w}) = t\vec{v} + t\vec{w}$ ✓
- Distribution over Scalar Addition Property: $\forall s, t \in \mathbb{R} \quad \forall \vec{v} \in V \quad (s+t)\vec{v} = s\vec{v} + t\vec{v}$ ✓

$\boxed{\text{Thm}}$ For a Vector Subspace only need to check

* closed under addition

and

* closed under scalar mult

Defn: A Linear Map $F: V \to W$

- Preserves Addition: $\forall \vec{v}, \vec{u} \in V \quad F(v+u) = F(v) + F(u)$
- Preserves Scalar Mult: $\forall v \in V \ \forall t \in \mathbb{R} \quad F(tv) = tF(v)$

Vector Subspace Thm:

If V is a nonempty subset of W which is

* Closed Under Addition
* Closed Under Scalar Mult.

then V is a vector subspace of W

Defn: Given a linear map $F: V \to W$ we have

$$Null(F) = \{\vec{v} \in V \mid F(\vec{v}) = \vec{0}\}$$

Defn F is one to one means

$F(\vec{v}) = F(\vec{\omega}) \Rightarrow \vec{v} = \vec{\omega}$

Thm: If $F: V \to W$ is a linear map between vector spaces V and W and if $Null(F) = \{\vec{0}\}$ then F is one-to-one.

Defn: Given $\vec{v}_1 \dots \vec{v}_k \in V$ the span $<\vec{v}_1, \vec{v}_2 \dots \vec{v}_k> =$

$$= \{t_1\vec{v}_1 + \dots + t_k\vec{v}_k \mid t_i \in \mathbb{R}\}$$

$$= \left\{\sum_{i=1}^{k} t_i \vec{v}_i \mid t_i \in \mathbb{R}\right\}$$

Defn: The Image of a map $F: V \to W$ is $F(V) = \{F(\vec{v}) \mid \vec{v} \in V\}$

Thm: If $V = <\vec{v}_1, \dots, \vec{v}_k>$ then $F(V) = <F(\vec{v}_1), \dots, F(\vec{v}_k)>$

So if $F: \mathbb{R}^3 \to W$ then

$$F(\mathbb{R}^3) = < F\begin{pmatrix}1\\0\\0\end{pmatrix}, F\begin{pmatrix}0\\1\\0\end{pmatrix}, F\begin{pmatrix}0\\0\\1\end{pmatrix} >$$

Defn: $F: V \to W$ is **onto** $\Leftrightarrow \forall \vec{w} \in W \ \exists \vec{v} \in V$ s.t $F(\vec{v}) = \vec{w}$

Thm: Onto $\Leftrightarrow F(V) = W$

****

SAMPLE FINALS

****

Two samples are included for each part for you to practice after reviewing:

Sample Part I

MAT313

Final Exam Part I

Prof Sormani

Let $F: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be the linear map

SAMPLE

defined by $F\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5x_1 - x_2 + 4x_3 \\ x_2 + 5x_3 \end{pmatrix}$

(1) Find (a) $F\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} =$ (b) $F\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} =$ (c) $F\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} =$

(2) (a) Find $Null(F) = \{ \vec{x} \mid F(\vec{x}) = \vec{0} \}$

(b) Find a basis for $Null(F)$.

(c) Is F one-to-one? Hint: check if $Null(F) = \{\vec{0}\}$

(3) (a) Find Image of $F = \{ F(\vec{x}) \mid \vec{x} \in \mathbb{R}^3 \}$

(b) Find a basis for the Image.

(c) Is F onto? Hint is Image of $F = \mathbb{R}^2$

(4) (a) Find $F(\vec{v} + \vec{w}) = F\left(\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}\right) =$

(b) Find $F(\vec{v}) + F(\vec{w}) = F\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + F\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} =$

(c) Does F preserve addition?

Try it before looking at the solution Playlist 313F22-FP1-S1:

MAT313 SAMPLE Final Exam Part I Prof Sormani

Let $F: \mathbb{R}^3 \to \mathbb{R}^2$ be the linear map defined by $\boxed{F\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}5x_1 - x_2 + 4x_3\\ x_2 + 5x_3\end{pmatrix}}$

1. Find (a) $F\begin{pmatrix}1\\0\\0\end{pmatrix} =$ (b) $F\begin{pmatrix}0\\1\\0\end{pmatrix} =$ (c) $F\begin{pmatrix}0\\0\\1\end{pmatrix} =$

2. (a) Find $Null(F) = \{\vec{x} \mid F(\vec{x}) = \vec{0}\}$
   (b) Find a basis for $Null(F)$.
   (c) Is F one-to-one? Hint: check if $Null(F) = \{\vec{0}\}$

3. (a) Find Image of $F = \{F(\vec{x}) \mid \vec{x} \in \mathbb{R}^3\}$
   (b) Find a basis for the Image.
   (c) Is F onto? Hint is Image of $F = \mathbb{R}^2$

4. (a) Find $F(\vec{v} + \vec{w}) = F\left(\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix} + \begin{pmatrix}w_1\\w_2\\w_3\end{pmatrix}\right) =$
   (b) Find $F(\vec{v}) + F(\vec{w}) = F\begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix} + F\begin{pmatrix}w_1\\w_2\\w_3\end{pmatrix} =$
   (c) Does F preserve addition?

Solutions ← Show work!

1. $F\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}5\cdot 1 - 0 + 4\cdot 0\\ 0 + 5\cdot 0\end{pmatrix} = \begin{pmatrix}5\\0\end{pmatrix}$ ✓

$F\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}5\cdot 0 - 1 + 4\cdot 0\\ 1 + 5\cdot 0\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix}$ ✓

$F\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}5\cdot 0 - 0 + 4\cdot 1\\ 0 + 5\cdot 1\end{pmatrix} = \begin{pmatrix}4\\5\end{pmatrix}$ ✓

Matrix for F is $\begin{pmatrix}5 & -1 & 4\\ 0 & 1 & 5\end{pmatrix}$

Check

$\begin{pmatrix}5 & -1 & 4\\ 0 & 1 & 5\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}5\\0\end{pmatrix}$ ✓

$\begin{pmatrix}5 & -1 & 4\\ 0 & 1 & 5\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix}$ ✓

$\begin{pmatrix}5 & -1 & 4\\ 0 & 1 & 5\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}4\\5\end{pmatrix}$ ✓

2. (a) Null (F) is found by solving $F(\vec{x}) = \vec{0}$

$$\boxed{\begin{aligned}5x_1 - 1x_2 + 4x_3 &= 0\\ 0x_1 + 1x_2 + 5x_3 &= 0\end{aligned}}$$

$$\left(\begin{array}{ccc|c}5 & -1 & 4 & 0\\ 0 & 1 & 5 & 0\end{array}\right) \xrightarrow{\rho_1 \to \frac{1}{5}\rho_1} \left(\begin{array}{ccc|c}1 & -\frac{1}{5} & \frac{4}{5} & 0\\ 0 & 1 & 5 & 0\end{array}\right)$$

$$\xrightarrow{\rho_1 \to \rho_1 + \frac{1}{5}\rho_2} \left(\begin{array}{ccc|c}1 & 0 & \frac{9}{5} & 0\\ 0 & 1 & 5 & 0\end{array}\right)$$ Red Ech Form

$$\boxed{\begin{aligned}\boxed{x_1} + \tfrac{9}{5}x_3 &= 0\\ \boxed{x_2} + 5x_3 &= 0\end{aligned}}$$

$\left.\begin{aligned}x_1 &= -\tfrac{9}{5}x_3\\ x_2 &= -5x_3\end{aligned}\right\}$ leaders

$x_3 = x_3$ (free)

(a) $Null(F) = \left\{\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = x_3\begin{pmatrix}-9/5\\-5\\1\end{pmatrix} \;\middle|\; x_3 \in \mathbb{R}\right\}$

(b) basis for Null (F) are the directions $\begin{pmatrix}-9/5\\-5\\1\end{pmatrix}$

(c) Is F one-to-one No

because $F\begin{pmatrix}-9/5\\-5\\1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$ and $F\begin{pmatrix}0\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$.

If we found $Null(F) = \{\vec{0}\}$ then F is one-to-one by Thm proven Lesson 28.

MAT313 Final Exam Part I Prof Sormani

Let $F: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be the linear map

defined by $F\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5x_1 - x_2 + 4x_3 \\ x_2 + 5x_3 \end{pmatrix}$

(1) Find (a) $F\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} =$ (b) $F\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} =$ (c) $F\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} =$

(2) (a) Find $Null(F) = \{\vec{x} \mid F(\vec{x}) = \vec{0}\}$

(b) Find a basis for Null (F).

(c) Is F one-to-one? Hint: check if $Null(F) = \{\vec{0}\}$

(3) (a) Find Image of $F = \{F(\vec{x}) \mid \vec{x} \in \mathbb{R}^3\}$

(b) Find a basis for the Image.

(c) Is F onto? Hint is Image of $F = \mathbb{R}^2$

(4) (a) Find $F(\vec{v}+\vec{w}) = F\left(\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}\right) =$

(b) Find $F(\vec{v}) + F(\vec{w}) = F\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + F\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} =$

(c) Does F preserve addition?

(3)(a) Find Image of F

Image of F = Span $F\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} F\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} F\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$= \left\langle F\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, F\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, F\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle$

$= \left\langle \begin{pmatrix} 5 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \end{pmatrix} \right\rangle$

(3b) Find a basis

Must find pivot columns of

$\begin{pmatrix} 5 & -1 & 4 \\ 0 & 1 & 5 \end{pmatrix}$ we saw already 1st & 2nd columns are pivot columns

So our basis is $\begin{pmatrix} 5 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

(3c) Is F onto?

Is Image of $F = \mathbb{R}^2$?

2 basis elements so

2 dimensional so $= \mathbb{R}^2$

A 2D subspace of $\mathbb{R}^2$ is $\mathbb{R}^2$

Yes it is onto.

(4)(a) $F(\vec{v}+\vec{w}) = F\left(\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}\right) = F\begin{pmatrix} v_1+w_1 \\ v_2+w_2 \\ v_3+w_3 \end{pmatrix} = \begin{pmatrix} 5(v_1+w_1) - (v_2+w_2) + 4(v_3+w_3) \\ (v_2+w_2) + 5(v_3+w_3) \end{pmatrix}$

$= \begin{pmatrix} 5v_1 + 5w_1 - v_2 - w_2 + 4v_3 + 4w_3 \\ v_2 + w_2 + 5v_3 + 5w_3 \end{pmatrix} = \begin{pmatrix} 5v_1 - v_2 + 4v_3 + 5w_1 - w_2 + 4w_3 \\ v_2 + 5v_3 + w_2 + 5w_3 \end{pmatrix}$

(b) $F(\vec{v}) + F(\vec{w}) = F\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + F\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 5v_1 - v_2 + 4v_3 \\ v_2 + 5v_3 \end{pmatrix} + \begin{pmatrix} 5w_1 - w_2 + 4w_3 \\ w_2 + 5w_3 \end{pmatrix}$

$= \begin{pmatrix} 5v_1 - v_2 + 4v_3 + 5w_1 - w_2 + 4w_3 \\ v_2 + 5v_3 + w_2 + 5w_3 \end{pmatrix}$ Matches

(c) Yes F preserves addition becaus $F(\vec{v}+\vec{w}) = F(\vec{v}) + F(\vec{w})$

Second sample Part I:

MAT313 | Final Exam Part I | Prof Sormani

Let $F: \mathbb{R}^3 \longrightarrow \mathbb{R}^4$ be the linear map

SAMPLE

defined by $F\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ x_1 + x_2 \\ x_3 \\ x_3 \end{pmatrix}$

① Find ⓐ $F\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} =$ ⓑ $F\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} =$ ⓒ $F\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} =$

② ⓐ Find $Null(F) = \{\vec{x} \mid F(\vec{x}) = \vec{0}\}$

ⓑ Find a basis for $Null(F)$.

ⓒ Is F one-to-one? Hint: check if $Null(F) = \{\vec{0}\}$

③ ⓐ Find Image of $F = \{F(\vec{x}) \mid \vec{x} \in \mathbb{R}^3\}$

ⓑ Find a basis for the Image.

ⓒ Is F onto? Hint is Image of $F = \mathbb{R}^4$

④ ⓐ Find $F(\vec{v} + \vec{w}) = F\left(\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}\right) =$

ⓑ Find $F(\vec{v}) + F(\vec{w}) = F\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + F\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} =$

ⓒ Does F preserve addition?

Try it before looking at the solutions Playlist 313F22-FP1-S2 :

MAT313 Final Exam Part I Prof Sormani

Let $F: \mathbb{R}^3 \to \mathbb{R}^4$ be the linear map

SAMPLE defined by $F\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ x_1 + x_2 \\ x_3 \\ x_3 \end{pmatrix}$

① Find ⓐ $F\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} =$ ⓑ $F\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} =$ ⓒ $F\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} =$

② ⓐ Find $Null(F) = \{\vec{x} \mid F(\vec{x}) = \vec{0}\}$

ⓑ Find a basis for $Null(F)$.

ⓒ Is F one-to-one? Hint: check if $Null(F) = \{\vec{0}\}$

③ ⓐ Find Image of $F = \{F(\vec{x}) \mid \vec{x} \in \mathbb{R}^3\}$

ⓑ Find a basis for the Image.

ⓒ Is F onto? Hint is Image of $F = \mathbb{R}^2$

④ ⓐ Find $F(\vec{v} + \vec{w}) = F\left(\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}\right) =$

ⓑ Find $F(\vec{v}) + F(\vec{w}) = F\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + F\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} =$

ⓒ Does F preserve addition?

Solutions

① ⓐ $F\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1-0 \\ 1+0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ ⓑ $F\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0-1 \\ 0+1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ ⓒ $F\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0-0 \\ 0+0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$

Matrix is $\begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

② $Null(F) = \{\vec{x} \mid F(\vec{x}) = \vec{0}\} = \left\{\vec{x} \;\middle|\; \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}\vec{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right\}$

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \xrightarrow{\rho_2 \to \rho_2 - \rho_1} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \xrightarrow{\rho_2 \to \frac{\rho_2}{2}} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{\rho_4 \to \rho_4 - \rho_3} \underbrace{\left(\begin{array}{ccc|c} \boxed{1} & -1 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)}_{\text{Echelon}} \xrightarrow{\rho_1 \to \rho_1 + \rho_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$Null(F) = \left\{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right\}$

ⓑ No Basis because $Null(F)$ is 0 dim'l because $Null(F) = \{\vec{0}\}$

ⓒ Yes F is one-to-one because $Null(F) = \vec{0}$ by Lesson 28 Thm.

MAT313 Final Exam Part I Prof Sormani

Let $F: \mathbb{R}^3 \to \mathbb{R}^4$ be the linear map

SAMPLE defined by $F\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ x_1 + x_2 \\ x_3 \\ x_3 \end{pmatrix}$

① Find ⓐ $F\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} =$ ⓑ $F\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} =$ ⓒ $F\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} =$

② ⓐ Find $Null(F) = \{\vec{x} \mid F(\vec{x}) = \vec{0}\}$

ⓑ Find a basis for $Null(F)$.

ⓒ Is F one-to-one? Hint: check if $Null(F) = \{\vec{0}\}$

③ ⓐ Find Image of $F = \{F(\vec{x}) \mid \vec{x} \in \mathbb{R}^3\}$

ⓑ Find a basis for the Image.

ⓒ Is F onto? Hint is Image of $F = \mathbb{R}^4$

④ ⓐ Find $F(\vec{v} + \vec{w}) = F\left(\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}\right) =$

ⓑ Find $F(\vec{v}) + F(\vec{w}) = F\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + F\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} =$

ⓒ Does F preserve addition?

③ ⓐ Image of F

$= \langle F\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, F\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, F\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \rangle$ span

$= \langle \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \rangle \subset \mathbb{R}^4$

ⓑ Basis pivot columns: all three were pivot columns so all columns are linearly indep.

basis is $\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ $\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$.

ⓒ Is F onto? Is it onto $\mathbb{R}^4$

No, Image is only 3 dimensional and $\mathbb{R}^4$ is 4 dimensional

In fact $\begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \notin$ Image of F.

④ ⓐ Find $F(\vec{v} + \vec{w}) = F\left(\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}\right) = F\begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \\ v_3 + w_3 \end{pmatrix}$ where $F\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ x_1 + x_2 \\ x_3 \\ x_3 \end{pmatrix}$

$$= \begin{pmatrix} (v_1 + w_1) - (v_2 + w_2) \\ (v_1 + w_1) + (v_2 + w_2) \\ v_3 + w_3 \\ v_3 + w_3 \end{pmatrix} = \begin{pmatrix} v_1 + w_1 - v_2 - w_2 \\ v_1 + w_1 + v_2 + w_2 \\ v_3 + w_3 \\ v_3 + w_3 \end{pmatrix} = \begin{pmatrix} v_1 - v_2 + w_1 - w_2 \\ v_1 + v_2 + w_1 + w_2 \\ v_3 + w_3 \\ v_3 + w_3 \end{pmatrix}$$ ← Match!

ⓑ Find $F(\vec{v}) + F(\vec{w}) = F\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} + F\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} v_1 - v_2 \\ v_1 + v_2 \\ v_3 \\ v_3 \end{pmatrix} + \begin{pmatrix} w_1 - w_2 \\ w_1 + w_2 \\ w_3 \\ w_3 \end{pmatrix} = \begin{pmatrix} v_1 - v_2 + w_1 - w_2 \\ v_1 + v_2 + w_1 + w_2 \\ v_3 + w_3 \\ v_3 + w_3 \end{pmatrix}$

ⓒ Does F preserve addition? Yes

Sample Part II

MAT313 SAMPLE

Final Part 2

Prof Sormani

① Let $V = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \;\middle|\; x_1 = 3x_2 \right\}$

Show V is closed under scalar multi

Given $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in V$ we have $\boxed{}$

Given $k \in \mathbb{R}$ $k\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \boxed{} \in V$

because $\boxed{}$

② Let $P = \begin{pmatrix} \cos(\frac{\pi}{5}) & -\sin(\frac{\pi}{5}) & 0 \\ \sin(\frac{\pi}{5}) & \cos(\frac{\pi}{5}) & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

(a) Check $P^T P = I$ so $P^T = P^{-1}$

(b) If $B = PDP^T$ what are the eigenvalues of B?

(c) What are the eigenspaces for these eigenvalues?

(d) Describe the transformation PDP^T.

Try it before looking at these solutions **Video 313F22-FP2-S1**

MAT313 Prof Sormani

SAMPLE Final Part 2

① Let $V = \{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mid x_1 = 3x_2 \}$

Show V is closed under scalar multi:

Given $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in V$ we have $\boxed{}$

Given $k \in \mathbb{R}$ $k\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \boxed{} \in V$

because $\boxed{}$

② Let $P = \begin{pmatrix} \cos(\frac{\pi}{5}) & -\sin(\frac{\pi}{5}) & 0 \\ \sin(\frac{\pi}{5}) & \cos(\frac{\pi}{5}) & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

ⓐ Check $P^T P = I$ so $P^T = P^{-1}$

ⓑ If $B = PDP^T$ what are the eigenvalues of B?

ⓒ What are the eigenspaces for these eigenvalues?

ⓓ Describe the transformation PDP^T.

Solution

① Let $V = \{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mid x_1 = 3x_2 \}$

Show V is closed under scalar mult.

Given $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in V$ we have $\boxed{v_1 = 3v_2}$

Given $k \in \mathbb{R}$ $k\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \boxed{\begin{pmatrix} kv_1 \\ kv_2 \end{pmatrix}} \in V$

because $kv_1 \stackrel{?}{=} 3kv_2$

$\boxed{kv_1 = k(3v_2) = 3kv_2}$

② $P = \begin{pmatrix} \cos(\frac{\pi}{5}) & -\sin(\frac{\pi}{5}) & 0 \\ \sin(\frac{\pi}{5}) & \cos(\frac{\pi}{5}) & 0 \\ 0 & 0 & 1 \end{pmatrix}$ Rotation around z axis by $(\frac{\pi}{5})$

$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ diagonal matrix

ⓐ $P^T P = \begin{pmatrix} \cos(\frac{\pi}{5}) & \sin(\frac{\pi}{5}) & 0 \\ -\sin(\frac{\pi}{5}) & \cos(\frac{\pi}{5}) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\frac{\pi}{5}) & -\sin(\frac{\pi}{5}) & 0 \\ \sin(\frac{\pi}{5}) & \cos(\frac{\pi}{5}) & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$= \begin{pmatrix} \cos^2(\frac{\pi}{5}) + \sin^2(\frac{\pi}{5}) + 0^2 & \cos(\frac{\pi}{5})(-\sin(\frac{\pi}{5})) + \sin(\frac{\pi}{5})\cos(\frac{\pi}{5}) + 0^2 & 0+0+0 \\ -\sin(\frac{\pi}{5})\cos(\frac{\pi}{5}) + \cos(\frac{\pi}{5})\sin(\frac{\pi}{5}) + 0^2 & \sin^2(\frac{\pi}{5}) + \cos^2(\frac{\pi}{5}) + 0^2 & 0+0+0 \\ 0+0+0 & 0+0+0 & 0+0+1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

TRIG

$\cos^2\theta + \sin^2\theta = 1$

Thus $P^T = P^{-1}$ Orthogonal Matrices

$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

ⓑ If $B = PDP^T = PDP^{-1}$ Diagonalization

what are the eigenvalues? $\lambda = 2, \lambda = 3, \lambda = 4$

ⓒ For $\lambda = 2$ evector is $\begin{pmatrix} \cos(\frac{\pi}{5}) \\ \sin(\frac{\pi}{5}) \\ 0 \end{pmatrix}$ only 1^{st} column of P

The eigenspace is $\{ t\begin{pmatrix} \cos(\frac{\pi}{5}) \\ \sin(\frac{\pi}{5}) \\ 0 \end{pmatrix} \mid t \in \mathbb{R} \}$

For $\lambda = 3$ 2^{nd} column of P $= \{ t\begin{pmatrix} -\sin(\frac{\pi}{5}) \\ \cos(\frac{\pi}{5}) \\ 0 \end{pmatrix} \mid t \in \mathbb{R} \}$

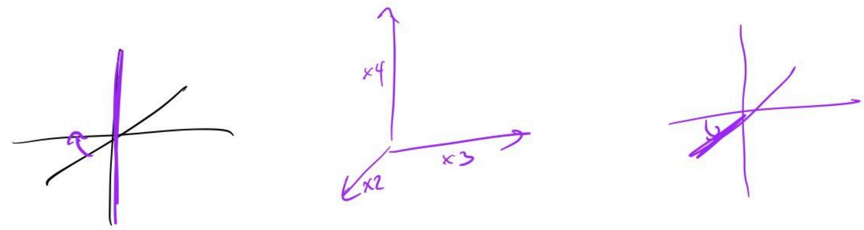
For $\lambda = 4$ 3^{rd} column of P $= \{ t\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \mid t \in \mathbb{R} \}$

(d) Describe the transformation PDP^T

First do P^T which rotates around
z axis by $-\pi/5$

Then do D which stretches x by 2
y by 3
z by 4

Then do P which rotates around
z axis by $\pi/5$



Second sample Part II

MAT313 Final Part 2 Prof Sormani

SAMPLE

① Let $V = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \;\middle|\; \begin{matrix} x_1 + x_2 = 0 \\ x_2 + x_3 = 0 \end{matrix} \right\}$

Show V is closed under scalar multi

Given $\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \in V$ we have $\boxed{}$

Given $k \in \mathbb{R}$ $k \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \boxed{} \in V$

because $\boxed{}$

② Let $P = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ $D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

ⓐ Check $P^T P = I$ so $P^T = P^{-1}$

ⓑ If $B = P D P^T$ what are the eigenvalues of B?

ⓒ What are the eigenspaces for these eigenvalues?

ⓓ Describe the transformation $P D P^T$.

Try it before looking at these solutions [Video 313F22-FP2-S2]

MAT313 SAMPLE Final Part 2 Prof Sormani

① Let $V = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \middle| \begin{array}{l} x_1+x_2=0 \\ x_2+x_3=0 \end{array} \right\}$

Show V is closed under scalar mult:

Given $\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \in V$ we have $\boxed{}$

Given $k \in \mathbb{R}$ $k \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \boxed{} \in V$

because $\boxed{}$

② Let $P = \begin{pmatrix} 0&0&1 \\ 1&0&0 \\ 0&1&0 \end{pmatrix}$ $D = \begin{pmatrix} 0&0&0 \\ 0&2&0 \\ 0&0&2 \end{pmatrix}$.

ⓐ Check $P^T P = I$ so $P^T = P^{-1}$

ⓑ If $B = PDP^T$ what are the eigenvalues of B?

ⓒ What are the eigenspaces for these eigenvalues?

ⓓ Describe the transformation PDP^T.

Solutions $V = \left\{ \begin{pmatrix} x_1 \\ x_1 \\ x_3 \end{pmatrix} \middle| \begin{array}{l} x_1+x_2=0 \\ x_2+x_3=0 \end{array} \right\}$ Rules

①

Show V is closed under scalar mult.

Given $\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \in V$ we have $\boxed{\begin{array}{l} v_1+v_2=0 \\ v_2+v_3=0 \end{array}}$

Given $k \in \mathbb{R}$ $k \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \boxed{\begin{pmatrix} kv_1 \\ kv_2 \\ kv_3 \end{pmatrix}} \in V$

because $\boxed{\begin{array}{l} kv_1 + kv_2 = k(v_1+v_2) = k\cdot 0 = 0 \ ✓ \\ kv_2 + kv_3 = k(v_2+v_3) = k\cdot 0 = 0 \ ✓ \end{array}}$

② $P = \begin{pmatrix} 0&0&1 \\ 1&0&0 \\ 0&1&0 \end{pmatrix}$ Permutation Matrix $D = \begin{pmatrix} 0&0&0 \\ 0&2&0 \\ 0&0&2 \end{pmatrix}$ diagonal matrix

ⓐ $P^T P = \begin{pmatrix} 0&0&1 \\ 1&0&0 \\ 0&1&0 \end{pmatrix}^T \begin{pmatrix} 0&0&1 \\ 1&0&0 \\ 0&1&0 \end{pmatrix} = \begin{pmatrix} 0&1&0 \\ 0&0&1 \\ 1&0&0 \end{pmatrix} \begin{pmatrix} 0&0&1 \\ 1&0&0 \\ 0&1&0 \end{pmatrix} = \begin{pmatrix} 0^2+1^2+0^2 & 0^2+0+0 & 0+0+0 \\ 0+0+0 & 0^2+0^2+1^2 & 0+0+0 \\ 0+0+0 & 0+0+0 & 1^2+0+0 \end{pmatrix}$

So $P^T = P^{-1}$ $= \begin{pmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&1 \end{pmatrix} = I$ ✓

ⓑ $B = PDP^T = PDP^{-1}$ is diagonalized so its eigenvalues are on the diagonal of D: $\lambda=0$, $\lambda=2$, $\lambda=2$

ⓒ Eigenspaces

for $\lambda=2$ eigenspace $= \left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle = \left\{ t_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + t_2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \middle| t_1, t_2 \in \mathbb{R} \right\}$

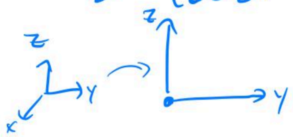
for $\lambda=0$ eigenspace $= \left\langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle = \left\{ t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \middle| t \in \mathbb{R} \right\}$

ⓓ Describe the transformation

$$PDP^T = \begin{pmatrix} 0&0&1 \\ 1&0&0 \\ 0&1&0 \end{pmatrix} \begin{pmatrix} 0&0&0 \\ 0&2&0 \\ 0&0&2 \end{pmatrix} \begin{pmatrix} 0&1&0 \\ 0&0&1 \\ 1&0&0 \end{pmatrix}$$

$P^T = \begin{pmatrix} 0&1&0 \\ 0&0&1 \\ 1&0&0 \end{pmatrix}$ takes x axis to z axis because 1st column $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, y axis to x axis because 2nd column is $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, z axis to y axis because 3rd column is $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad \left.\begin{array}{l} \text{then takes } x \text{ axis to } \underline{0} \\ \underline{\text{stretches}}\ y \text{ axis by } 2 \\ \underline{\text{stretches}}\ z \text{ axis by } 2 \end{array}\right\}$$



$$P = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \text{ takes}$$

x axis back to y axis	y axis back to z axis	z axis back to x axis