2 Binomial

PURPOSE & HIGHLIGHTS

This activity is designed to **elicit students' ideas** about subsets, binomial coefficients, and counting; **prompt alternative ways** to think about a problem; and **layer binomial coefficient** vocabulary and techniques over intuitive experiences.

Highlights: Students see the value of approaching a problem in different ways. Discovering the connection between the visual triangle and the algebraic approach to binomial coefficients creates joy.

LESSON OVERVIEW

Plan for the lesson

Introduction and Teamwork Preparation (7 min): Introduce the binomial triangle. Make sure students understand the pattern used to construct the triangle and establish some way of discussing locations within the triangle. Assign groups and roles.

Core Activity, Thinking about subsets with puppies (24 min): Students count possible combinations of puppies based on the number of leashes. They use their reasoning to understand and explain the standard recurrence for binomial coefficients. Students share.

Extension Activity, Doing the problem two ways (34 min): Students explore the relationships between binomial coefficients and the binomial triangle. They use the values from the Core Activity to link and compare the notations from both concepts. Students share.

Formative Activity (5 min): Students discuss the pros and cons of using the binomial triangle or recurrence as opposed to an algebraic formula (which your students may or may not already know). Students create examples and support/challenge arguments. Students share.

Wrap-up (5 min):
Conclude with student survey or further reflection.

Class Time Estimate:

- Core + Formative Activities: 50 min
- Core + Extension + Formative Act.: 75 min

Key Ideas for Instructors

- Some students may do the Core Activity with the algebraic formula; encourage them to think in terms of sets as well.
- The goal is to make the connections between visual/diagrammatic and formula-based methods and be open to alternate approaches.

Student Prerequisites

The activity assumes the students have seen combinations, in particular:
(n choose k) = the ways to pick k unordered objects from n objects. The activity does **NOT** assume students have used the factorial formula for binomial coefficients:
(n choose k) = n!/(k! (n-k)!)
though it is okay if they have seen or used it.

BINOMIAL RESOURCES

- [Instructor Roadmap]
- [Student Handout]
- [Team Roles Handout]

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Binomial/Number Triangle Lesson

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Purpose

This activity is designed to engage students in **productive struggle** as it

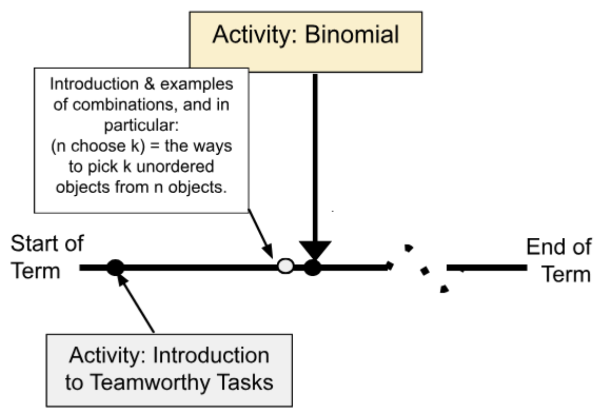
- **elicits** student ideas about subsets, binomial coefficients, and counting;
- **prompts alternative ways** to think about a problem, including visually via Pascal's triangle and seeing certain mathematical properties in multiple ways; and
- **layers binomial coefficient** vocabulary and techniques over intuitive experiences.

The mathematics computation, symbols and vocabulary aim for students to discover that:

- The number triangle[1] and binomial coefficients (interpreted combinatorially) satisfy the same recurrence relation.
- Since they also both start with the same edge values – either the sides of the triangle, or $\binom{n}{0}$ and $\binom{n}{n}$, the triangle is made out of binomial coefficients. In other words, one goal of this activity is for students to discover the combinatorial explanation of the fact that the number triangle is made of binomial coefficients.
- Combinatorial thinking, the visual format of the number triangle, and formulas are all valid methods for solving problems and understanding phenomena like $\binom{n}{r} = \binom{n}{n-r}$.
- Different methods have different strengths.

Lesson Fit and Student Prerequisites

- The activity assumes students understand the meaning of combinatorial wording, particularly: (n choose k) = the ways to pick k unordered objects from a collection of n objects. Students are reminded about this in Activity 1.
- The activity does **NOT** assume students have used the algebraic factorial formula for binomial coefficients: (n choose k)= $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.
- It can be especially useful for student learning if this activity is done BEFORE introducing the formula.



Instructor Preparation

Reminder: see the **User Guide** for suggestions on how to create and assign groups, how to orchestrate active engagement by students in their in-class teams, and other teaching tips.

- **Please read this entire roadmap before planning or doing your class meeting.** This is more important for this lesson than it might be for other lessons, because this lesson contains more connections between formal and informal representations that you need to have in mind so you can facilitate your students discovering it.
- Do (work through) each of the problems in the Student Activity Handout.
- Decide the number of student groups based on class size. Groups of 3 are ideal but groups of 4 can work with more effort on your part to keep people interacting.
- Provide multiple colors of markers/chalk for student use.
- **Handouts:** Use the Team Roles handout and Student Activity Handout with at least one copy visible for each team (on paper or on a display). The activity handout includes the QR code for the end-of-session survey.

[1] The number triangle is often called Pascal's triangle, but was a well-known object centuries earlier in Persia, India, China, Germany, and Italy.

Lesson Overview

The activity flow of this lesson is:

Activity 1: Students discover Pascal's Formula by generalizing an example (and probably also thinking combinatorially). $\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$

Activity 2: Students discover that the number triangle is made of binomial coefficients, and maybe use Pascal's Formula to explain why.

Activity 3 (if you have 75 min): Students use combinatorial thinking, or the triangle, or maybe the factorial formula to explain why $\binom{n}{r} = \binom{n}{n-r}$.

Set-up [pre-class]	Make sure boards are clear and markers in multiple colors are available. Post static cling whiteboards as needed to increase workspace.
Launch ~10 min	1. Quickly introduce or review different notations for the number of ways to choose a subset of *k* things from a collection of *n* things, including: $\binom{n}{k}$ *nCk* and "*n* choose *k*" 2. Review with the whole class the idea of teamwork. 3. **Announce "Roles Rules" for teams** (Team Roles handout) 4. Students count off or otherwise form teams of 3 or 4. 5. Have teams move to their shared board space. Remind teams to document their role assignments (**you might want to collect one handout per team at the end of class**). 6. Distribute the Student Activity Handout (on paper if possible). At least 1 per team. *Suggestion: Have Facilitators come to you for handout and pens/chalk*. 7. In teams, students do the icebreaker activity (first thing on the handout). 8. Before Activity 1, teams make a copy of the number triangle in their shared space *and* they continue the triangle pattern for two more rows. Remind students: **do not erase the triangle <u>and</u> leave room for other work**.
Activity 1 **Dog Walking** ~23 min	[~15 min] Students work in teams. Each team works on the same activity on counting and thinking about subsets with dog walking. The situations are intentionally under-determined as problems, so students must decide on assumptions to make in order to propose a solution. [~8 min] Reporters from selected groups share their group's thinking and solution(s) with the whole class.
Activity 2 ~17 min	[~10 min] The second activity has students understand how Pascal's Formula is related to the number triangle, and then figure out how the numbers in the triangle are all binomial coefficients. [~7 min] Reporters share findings with the class, as with Activity 1.
Activity 3 ~14 min (optional)	[~8 min] Students work on subsets with dog walking again, but this time looking at the symmetry of Pascal's Triangle. [~6 min] Reporters share findings with the class.
Wrap-up ~5 min	Students reflect on the activity and briefly discuss in their groups the pros and cons of understanding Pascal's Formula algebraically versus what was done in the lesson. Students complete the anonymous feedback survey.

Lesson Details (~75 minutes for the lesson)

Launch (~10 minutes)

1. Quickly introduce or review different notations for the number of ways to choose a subset of *k* things from a collection of *n* things, including: *nCk* and "*n* choose *k*" and $\binom{n}{k}$.
2. Review with the whole class the idea of teamwork.
3. Announce "Roles Rules" for teams (Team Roles handout)
4. Students count off or otherwise form teams of 4 or 3.
5. Have teams move to their shared board space. Remind teams to record role assignments on their boards AND the handout (**you will collect one handout per team at the end of class**).
6. Distribute the Student Activity Handout **on paper**. At least 1 per team.
   Suggestion: Have each Facilitator come to you for handout and pens/chalk.
7. In teams, students do the icebreaker activity (first thing on the handout).
8. BEFORE beginning Activity 1, have the students create and work on the triangle as instructed in the handout:
   a. **Copy the number triangle onto their shared workspace** (e.g., whiteboard). Space out the entries so there's room to annotate the triangle later; **the goal is for the triangle to persist as a tool** throughout the activities**.** Suggestion: use no more than half of the team space for the triangle so there is still room for other work.
   b. Continue the triangle pattern for two more rows.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
... ...
... ...

Answers (which students will be able to discover on their own):

- Seventh row is: 1 6 15 20 15 6 1. Eighth row is: 1 7 21 35 35 21 7 1.
- Pattern answer: each number is the sum of the 2 numbers above it.

Note that the triangle part of the lesson has students discover the pattern in the number triangle, whereas Activity 1 has students discover essentially the same pattern in (n choose k). Activity 2 will then have students discover the explanation (really a proof) that the number triangle is made of (n choose k) because it satisfies the same pattern and has the same edge values (one possible explanation/proof).

Activity 1. Subsets with Puppies (23 min, including report-out by some groups)

- Have students work in groups, preferably at different sections of a big board, next best is in clusters of chairs. Circulate among groups and observe/help with the group work.
- As you circulate, remind students of the combinatorial meaning of (n choose k): it is the **number of ways to pick k unordered objects from n objects**.
- After 15 minutes (or better choice of stopping time based on observations of student progress) call on at least two different team's Reporters to present their team's ideas.

***Italics* in this section indicate text in the Student Handout**; plain text is for you as instructor.

Context: *We have five puppies at home: Retriever, Dachshund, Shepherd, Husky, Bulldog. Suppose we currently only have two leashes and can only take two dogs out on a walk.*

1(a) In how many ways can we make pairs to take two dogs for a walk? Use one of the "choose" notations to represent how many ways we can make the pair.

Answer: (5 choose 2) = 10

1(b) Now, look at the Retriever in particular: how many ways can we choose two different dogs to take out on a walk if our Retriever has to be one of the two dogs? (Again, try to use the "choose" notation.)

Answer: (4 choose 1) = 4. Explanation: one leash is for the Retriever and we are just picking 1 out of the remaining 4 for the other leash.

*1(c) How many different two-dog pairs can we take on a walk if we do **not** take the Retriever?*

Answer: (4 choose 2) = 6 (we can assume we could walk the Retriever separately)

*1(d) Notice that the answer to (a) equals the answer to (b) plus the answer to (c). **Why**?*

Answer: (b)+(c)=(a) -> 4 + 6 = 10, because for two-dog pairs, we either walk the Retriever or we do not.

Note: If students are having trouble explaining why this is true, give them more examples with different numbers.

1(e) Can you guess what (9 choose 3) + (9 choose 4) is? (Hint: refer to 1(d)). How about the value of (105 choose 77) + (105 choose 78)? What is (n choose k) + (n choose k+1) in general? Explain why.

Answer: (10 choose 4) and (106 choose 78).

The general relationship is $\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$, because a $(k + 1)$-subset is either a $(k + 1)$-subset of $\{1,\dots, n\}$ or the union of a k-subset of $\{1,\dots, n\}$ and $\{n + 1\}$.

During teamwork time identify at least teams whose presentations will, collectively build to:

- cover the numerical answers to the above questions **and** also the additional observations made above.
- If teams have made a varying amount of progress, one effective way to call on Reporters might be to start with a team who has gotten only specific cases of (d), then call on a team that has found the general version of Pascal's Formula, then (if possible) call on a team who can explain 4+6=10 in terms of combinatorial reasoning, but not the general case, then (if possible) call on a group who can use combinatorial reasoning to explain $\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$.
- Not all classes will be able to find an explanation/proof of Pascal's Formula, and it would be unusual for a student to produce an explanation/proof in polished mathematical language and abstract notation. But it's reasonable to expect a class eventually to notice the pattern that is Pascal's Formula, and that's all you need to continue to Activity 2. Of course, if some team eventually articulates an idea along the lines of "some subsets include Retriever and others don't", great!

Activity 2. Making connections (17 min, including report-out by some groups)

***Italics* in this section indicate text in the Student Handout**; plain text is for you as instructor.

2(a) Where on the number triangle do the values you got for 1(a), 1(b), and 1(c) appear?

Answer: Students are likely to circle the values 10, 4, and 6 and may refer to a row #.

2(b) What do you notice about the possible connection between the entries in the triangle and the "(a)=(b)+(c)" relationship from Activity 1? Identify a way to connect the numbers in the triangle with $\binom{n}{k}$ and label the triangle entries to explain the connection.

Answer: If we number the rows starting with 0 as the top row, and number the entries in each row starting with 0, then the 10 is in row 5 in the 2nd place (5 choose 2). More generally, (n choose k) will be entry k of row n.

Tips for Teaching–Student thinking to anticipate on 2(a) and 2(b)

- **Be patient and stick to the idea of productive struggle.** If students really get stuck, it's okay to nudge them along ("Do you see where (n choose 1) goes? How about (n choose 2)?") but in the end, the time they spend really grappling with the ideas is actually very productive.
- Many, maybe even most, teams will be able to label all of the entries on the board correctly, but the abstraction and notation involved in describing what happens in general with (n choose k) may be difficult. You may want to suggest, either as students are working or during presentations, that students try to come up with a single sentence (or a few sentences, which might be easier) that says precisely where the label for (n choose k) goes.
- One aspect of the general case that may be particularly challenging for students is the idea of numbering rows and entries starting with 0, or just thinking about the edge cases in general. 0 factorial and (0 choose 0) are particularly tricky to think about; try asking students to think about (or remember, if you've discussed this in class already) what (1 choose 0) or (2 choose 0) means in terms of sets and connect that to why (0 choose 0) is taken to equal 1.
- There will be students that already know recursion and the Binomial Triangle. Discussing the bullet point above with these students will have them think about the triangle in a different way. Ask them to explain the 1s along both sides, and at the top, in the context of the activity. You may also ask if this context makes sense for (n choose 0), (n choose n), (0 choose 0), and if they can come up with a real life context where some, or none, of these would make sense.

2(c) Now let's talk about cake. Use the triangle to determine how many different quadruple-layered cake recipes can be made from the eight common ingredients: vanilla, chocolate, strawberry, lemon, banana, coffee, carrot, and confetti.

Answer: (8 choose 4), which we can get from the extended triangle written out earlier as (7 choose 3) + (7 choose 4) = 35 + 35 = 70 (or going up more steps as necessary). Alternative answers: Students can extend the triangle to row 8 and get (8 choose 4) on the labeled triangle or use the factorial formula if they know it already.

Tips for Teaching–Student thinking to anticipate on 2(c)

Ask students who have found "the answer" by one method to keep looking for other ways to solve the problem. In presentation time, once someone presents "the answer" by one method, ask around to see if other teams have found "the answer" by a different method ("Did anyone try to solve the problem in a different way?").

Activity 3. Making more connections (14 min, including some report-outs)

***Italics* in this section indicate text in the Student Handout**; plain text is for you as instructor.

3(a) Count the number of ways to take one dog on a walk out of (Retriever, Dachshund, Shepherd, Husky, Bulldog). Then count the number of ways to take four dogs on a walk.

Answer: (5 choose 1) = 5 and (5 choose 4)=5.

3(b) Count the number of ways to take two dogs on a walk out of (Retriever, Dachshund, Shepherd, Husky, Bulldog). Then count the number of ways to take three dogs on a walk.

Answer: (5 choose 2)=10 and (5 choose 3)=10

3(c) What is the relationship between the two numbers you got in part (a)? What is the relationship between the two numbers you got in part (b)? How might you represent a generalization of this phenomenon? Explain why it works in general.

Answer: Each way of looking at binomial coefficients gives a different explanation of why (*n* choose *k*) = (*n* choose *n*-*k*).

- **Using the language of sets**: Choosing a subset of size *k* is the same as choosing its complement of size *n*-*k*.
- **In the language of the triangle**: The number triangle has a mirror symmetry across the center line. If you have already done proof by induction, try asking students who find the pattern to see if they can explain why the pattern continues (e.g., this provides a good application of induction on pattern rather than formula: If a given row is mirror-symmetric, since Pascal's Formula is also mirror-symmetric, the next row will be mirror symmetric).
- **In the language of factorials:** The denominators *k*!(*n*-*k*)! and (*n*-*k*)!(*n*-(*n*-*k*))! are equal: $\frac{5!}{1!4!} = \frac{5!}{4!1!}$.

If a team finishes 3(a), 3(b), and 3(c) with a satisfactory explanation, ask students to describe part (c) in the context of the problem. "What does (*n* choose *n*-*k*) **mean** here?"

Formative Assessment/Wrapping Up (~5 min)

As you may already know, we can also calculate binomial coefficients directly with a formula: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$*. Using this factorial formula, we could have (and you should do this for practice!) show that* $\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$ *(Pascal's Formula).*

Discussion:

1. *Imagine a lecture where Pascal's Formula is proved using algebra. What are some pros and cons of understanding Pascal's Formula algebraically versus what we did in this lesson?*
2. *Look back over the other ideas from this lesson: the relationship between the number triangle and* $\binom{n}{k}$ *from Activity 2 and the patterns found in Activity 3 (if you did Activity 3).*

   What are some pros and cons about using the number triangle? Using the factorial formula?

Tip for Teaching–Formative assessment

Have students focus on **one** of the questions. There are no correct answers; they are just meant to start a discussion. You might use some leading questions to facilitate discussion:

For #1: How does the number triangle help in understanding, remembering, or applying Pascal's Formula? How does the factorial formula help to understand, remember, or apply Pascal's Formula?

For #2: What was most helpful in explaining why the number triangle is made of (*n* choose *k*): thinking in terms of sets, looking at the number triangle, or using the factorial formula? Same question for (*n* choose *k*) = (*n* choose *n*-*k*).

Survey (~5 minutes)

To gather student feedback, give the link to the following survey. The QR code is also to the same survey https://forms.gle/t98BA78VxqFRysoaA

Ideally, the survey is done at the end of class to maximize participation. If there is no time, incentivizing participation with extra credit is another way to go. Or, you may choose to create an email or a course learning management system page (e.g., Canvas, Blackboard, Moodle) to distribute the student survey link. Send it or publish it as the lesson is approaching the end or "Schedule send" your email or schedule your course page to be published 10 minutes before the end of class.

Quiz Questions/Assessment

1(a) Using the triangle, compute (12 choose 3).

Alternatively, the numeric question can be phrased as How many ways can different trios of dogs (three different kinds of dogs) be walked from a group of 12 (Retriever, Dachshund, Shepherd, Husky, Bulldog, Chihuahua, Poodle, Maltese, Rottweiler, Pitbull, Labrador, Terrier).

Answer: (12 choose 3) = 220

1(b) Compare how hard it is to find (12 choose 3) using the triangle versus the formula $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

Answer: Will vary. Some students will prefer the method of extending the triangle.

1(c) How would your answer be different if you had to compute (28 choose 3) instead?

Answer: Will vary but much more likely students will say using the formula is "easier."

2. Using algebra, compute (12 choose 3). How does this compare with using the triangle.

Answer: (12 choose 3) = (12 × 11 × 10) ÷ (3 × 2 × 1) = 220.

[NOTE: The following is too prolonged a question for a quiz or exam, but might be good for homework or even for a short project.]

Try adding up all of the numbers in each row of the triangle. Can you find a pattern? Can you explain this pattern in terms of sets? Using the triangle? Using the factorial formula for (n choose k), if you have seen that already?

Answer: Row n sums to 2^n. Explanations:

a. In terms of sets, the sum of row n is the number of subsets of $\{1,..., n\}$, and this is 2^n. (Proof: For a given subset S, each element of $\{1,..., n\}$ is either in S or it isn't, and these n independent yes/no choices produce the 2^n different subsets.)
b. In terms of the triangle, by induction: In the base case, the sum of the top row (n=0) is 1. In the induction step, each entry in row n contributes to two entries in row $n + 1$, so row $n + 1$ is a sum of the entries of row n, but with each entry n appearing twice, and therefore has double the sum.
c. Algebraic: One can turn the above triangle argument into an algebraic induction proof, using summations and Pascal's Formula.