PRESIDENT'S OFFICE

REGINAL ADMINISTRATION AND LOCAL GOVERNMENT

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FORM SIX JOINT PRE NECTA EXAMINATIONS

MARKING SCHEME

Advanced Mathematics 2

Question no.1

(a). Let X be the event that the stock market will go up, and Y be the event that the economy will do well in the coming year

$$\text{Then } P(X) = P\left(\frac{X}{Y}\right)P(Y) + P\left(\frac{X}{\bar{Y}}\right)P(\bar{Y})$$

$$P(X) = (0.65)(0.75) + (0.30)(0.30)$$

$$P(X) = 0.4875 + 0.09$$

$$P(X) = 0.5775 \qquad (02\ marks)$$

∴The probability that stock market will go up next year is 0.5775

(b).(i). For P.d.f $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\int_0^9 b(81 - x^2)dx = 1$$

$$b\left[81x - \frac{x^3}{3}\right]_0^9 = 1$$

$$b[(729 - 243) - 0] = 1 \qquad (02\ marks)$$

$$486b = 1$$

$$\therefore b = \frac{1}{486} \quad \text{shown}$$

(ii).When $x < 0$ then $F(x) = 0$,

When $0 \leq X \leq 9$ then $\int_0^x \frac{1}{486}(81 - x^2)dx$

$$F(x) = \frac{1}{486}\left[81x - \frac{x^3}{3}\right]_0^x$$

$$F(x) = \frac{1}{486}(81x - \frac{x^3}{3})$$

$$F(x) = \frac{x}{6} - \frac{x^3}{1458} \qquad (02\ marks)$$

When $x > 1$, then $F(x) = 1$

∴The cumulative density function is

$$F(x) = \{0, \qquad x < 0\ \frac{x}{6} - \frac{x^3}{1458}, \quad 0 \leq x \leq 9\ 1 \ , \qquad x > 9$$

(iii).$P(X > 3) = 1 - P(X < 3)$

$$= 1 - F(3)$$

$$= 1 - (\frac{3}{6} - \frac{3^3}{1458})$$

$$= \frac{14}{27}$$

$$.P(X > 3) = 0.518519 \qquad (01\ mark)$$

∴The probability that a customer will queue for longer than 3 minutes is 0.5185

(c).(i).For discrete random variable , $\sum_{x=0}^{n} f(x) = 1$

$$0.05 + 0.15 + 2k + 0.25 + 0.25 + k = 1$$

$$3k + 0.70 = 1$$

$$3k = 1 - 0.70 \qquad (01\ mark)$$

$$3k = 0.3$$

$$\therefore k = 0.1$$

The probability distribution table will be as follows

x	0	1	2	3	4	5
$P(x)$	0.05	0.15	0.20	0.25	0.25	0.10

(ii). $E(x) = \sum_{0}^{5} xf(x)$ $(01\ mark)$

$$E(x) = (0\times0.05) + (1\times0.15) + (2\times0.20) + (3\times0.25) + (4\times0.25) + (5\times0.10)$$

$$E(x) = 0 + 0.15 + 0.40 + 0.75 + 1.00 + 0.50$$

$$E(x) = 2.8$$

Then $E(5x + 7) = 5E(x) + 7$ but $E(x) = 1.8$

$$E(5x + 7) = 5(2.8) + 7 = 14 + 7 = 21$$

$\therefore E(5x + 7) = 16$ $(01\ mark)$

(iii). $var(3x - 5) = 3^2 var(x)$ but $var(x) = E\left(x^2\right) - (E(x))^2$

and

$$E\left(x^2\right) = \sum_{0}^{5} x^2 f(x) = (0^2\times0.05) + \left(1^2\times0.15\right) + \left(2^2\times0.20\right) + \quad ($$

$E(x^2) = 0 + 0.15 + 0.8 + 2.25 + 4 + 2.5 = 9.7$ $(01\ mark)$

$$var(x) = (9.7) - (1.8)^2 = 1.86$$

$$var(3x - 5) = 3^2 \times 1.86 = 16.74$$

$$\therefore S.D = \sqrt{var(3x - 5)} = \sqrt{16.74} = 4.09 \qquad (01\ mark)$$

(d).For Poisson distribution $P(X = x) = \frac{\lambda^x e^{-\lambda}}{x!}$, $mean$ λ=
$np = 2000 \times 0.001 = 2$

(i).$P(x > 2) = 1 - P(x \leq 2) = 1 - [P(x = 0) + P(x = 1) + P(x = 2)$

$$P(x > 2) = 1 - \left(\frac{2^0 e^{-2}}{0!} + \frac{2^1 e^{-2}}{1!} + \frac{2^2 e^{-2}}{2!}\right)$$

$$P(x > 2) = 1 - \left(\frac{e^{-2}}{1} + \frac{2e^{-2}}{1} + \frac{2e^{-2}}{1}\right)$$

$$P(x > 2) = 1 - \left(\frac{5e^{-3}}{1}\right)$$

$(02\ marks)$

$$P(x > 2) = 1 - 0.6767 = 0.3233$$

∴The probability that out of 2000 people more than 2 suffer from malaria is 0.3233

(ii).$P(x = 3) = \frac{2^3 e^{-3}}{3!} = 0.1804$

∴The probability that out of 2000 people exactly 3 suffer from malaria is 0.1804 $(01\ mark)$

4.(a).(i).From $x(3 + 4i) - y(1 + 2i) + 5 = 0$

$$3x + 4xi - y - 2yi = -5 + 0i$$

Comparing Re-part: $3x - y = -5$ ……(i)

Im.Part: $4x - 2y = 0$ ……(ii) $(02\ marks)$

Solving (i) and (ii) simultaneously, $x = -5,\ y = -10$

(ii).From $3\left(\frac{1+i}{1-i}\right)^2 - 2\left(\frac{1-i}{1+i}\right)^3$ …..(i)

Multiply the numerator and denominator by the conjugate of the denominator

$\frac{1+i}{1-i} = \frac{(1+i)(1+i)}{(1-i)(1+i)} = \frac{1+i+i+i^2}{1+i-i-i^2} = \frac{2i}{2} = i$ ….(ii) $(01\ mark)$

$\frac{1-i}{1+i} = \frac{(1-i)(1-i)}{(1+i)(1-i)} = \frac{1-i-i+i^2}{1+i-i-i^2} = \frac{2i}{2} = i$ ….(iii)

Equation (i) becomes $3(i)^2 - 2(i)^3 =- 3 + 2i$ $(02\ marks)$

$\therefore\ 3\left(\frac{1+i}{1-i}\right)^2 - 2\left(\frac{1-i}{1+i}\right)^3 =- 3 + 2i$

4. (b).Requiredb to show $tan\theta = \frac{3tan\theta - tan^3\theta}{1-3tan^2\theta}$,

Use de-Moivre's theorem, $Cos3\theta + iSin3\theta = (Cos\theta + iSin\theta)^3$

Expanding RHS by using Binomial theorem,

$(Cos\theta + iSin\theta)^3 = Cos^2\theta - 3sin^2\theta cos\theta) + i\left(3sin\theta Cos^2\theta - Sin^3\theta\right)$ $(01\ mark)$

Equate Re.parts: $Cos3\theta = Cos^2\theta - 3sin^2\theta cos\theta)$……..(i)
$(01\ mark)$

Equate Im.parts: $Sin3\theta = 3sin\theta Cos^2\theta - Sin^3\theta$ but $Cos^2\theta = 1 - Sin^2\theta$

$Sin3\theta = 3Sin\theta - 4Sin^3\theta$ ………..(ii)

Then $tan\theta = \frac{Sin3\theta}{cos3\theta} = \frac{3Sin\theta Cos^2\theta - Sin^3\theta}{Cos^2\theta - 3Sin^2\theta Cos\theta}$ $(01\ mark)$

Dividing each term in numerator and denominator by $Cos^2\theta$

$\therefore\ tan\theta = \frac{3tan\theta - tan^3\theta}{1-3tan^2\theta}$

Rearranging the given equation $t^3 - 3t^2 - 3t + 1 = 0$ then $\frac{3t-t^2}{1-3t^2} = 1$ (01 *mark*)

$tan3\theta = 1$

$$3\theta = n\pi + tan^{-1}(1)$$

$$\theta = \frac{n\pi}{3} + \frac{\pi}{12}$$

When $n = 0, 1, 2, 3$ then the values of $\theta = 0.268,\ 3.73, 1.00$ respectively (01 *mark*)

4.(c). (i).If w is one of the complex cube roots of unit, then $1 + w + w^2 = 0$ then
$w + w^2 =- 1$

$\left(a + wb + w^2c\right)\left(a + w^2b + wc\right) = a^2 + abw^2 + awc + awb^2 + w^3b^3 + w^2cb^2 + ac$
$w^2 + w^4bc + c^2w^2$

$= a^2 + w^3b^2 + w^3c^2 + ab\left(w + w^3\right) + cb\left(w^3 + w^4\right) + ac$
$\left(w^2 + w\right)$

$$= a^2 + w^3(b^2 + c^2) + ab\left(w + w^3\right) + cb\left(w^3 + w^4\right) + ac\left(w^2 + w\right)$$

$$= a^2 + b^2 + c^2 + ab(-1) + cb(-1) + ac(-1)$$

(03 *marks*)

$$= a^2 + b^2 + c^2 - ab - cb - ac$$

$$\therefore\left(a + wb + w^2c\right)\left(a + w^2b + wc\right) = a^2 + b^2 + c^2 - ab - bc - ca$$

(ii).From complex number $|z - 1 + 2i| = 3$ let $z = x + iy$

$$|x + iy - 1 + 2i| = 3$$

$$|(x - 1) + i(y + 2)| = 3$$ (02 *marks*)

$$\sqrt{(x-1)^2+(y+2)^2}=\sqrt{3}^2$$

∴The locus is circle centred at (1,2) with radius of $\sqrt{3}$ units

5.(a).(i).Given $sin\theta$ for $\theta = 1^\circ, 2^\circ, 3^\circ, 4^\circ, 5^\circ$

From the mathematical table the values are as follows

$Angle, \theta$	1°	2°	3°	4°	5°
$Sin\theta$	0.0175	0.0349	0.0523	0.0698	0.0872

$(02\ marks)$

(ii).In radians, the values are computed by using formula: $\theta_{Radian} = \theta_{degree} \times \frac{\pi}{180}$

$Angle, \theta$	$1^\circ = 0.01745$	$2^\circ = 0.0349$	$3^\circ = 0.0523$	$4^\circ = 0.0698$	$5^\circ = 0.0872$
$Sin\theta$	0.0175	0.0349	0.0523	0.0698	0.0872

$(02\ marks)$

Conclusion: For very small angle $Sin\theta$ is almost equal to θ in radians, with minimal difference

∴$Sin\theta \approx \theta$, when θ is in radians $(01\ mark)$

(b).Given $tan\left(tan^{-1}(3x) - tan^{-1}(2)\right) + tan\left(tan^{-1}(3) - tan^{-1}(2x)\right) = \frac{3}{8}$

Let $A = tan^{-1}(3x), then\ tanA = 3x,\ B = tan^{-1}(2), then\ tanB = 2,$

$C = tan^{-1}(3), then\ tanC = 3\ and\ D = tan^{-1}(2x) then\ tanD = 2x$

$$tan(A-B) + tan(C-D) = \frac{3}{8}$$

$$\frac{tanA - tanB}{1 + tanAtanB} + \frac{tanC - tanD}{1 + tanCtanD} = \frac{3}{8}$$

$$\frac{3x-2}{1+3x(2)} + \frac{3-2x}{1+3(2x)} = \frac{3}{8}$$

$$\frac{3x-2}{1+6x} + \frac{3-2x}{1+6x} = \frac{3}{8}$$

$$\frac{3x-2+3-2x}{1+6x} = \frac{3}{8}$$

$$8(3x - 2 + 3x - 2x) = 3(1 + 6x)$$

$\therefore x = \frac{1}{2}$ $(05\ marks)$

(c). Given $x = asin\theta + bcos\theta$ and $y = asin\theta - bcos\theta$

Adding the two equations: $x + y = 2aSin\theta$ then $Sin\theta = \frac{x+y}{2a}$

Subtracting the two equations: $x - siny = 2aCos\theta$ then $Cos\theta = \frac{x-y}{2b}$

Using the identity $Sin^2\theta + Cos^2\theta = 1$ $(05\ marks)$

$\left(\frac{x+y}{2a}\right)^2 + \left(\frac{x-y}{2b}\right)^2 = 1$, $\therefore\theta$ is eliminated

(d).Required to show if $\frac{1-coscos\ 2\theta}{1+cos2\theta} = tan^2\theta$

From LHS: $\frac{1-coscos\ 2\theta}{1+cos2\theta} = \frac{1-(Cos^2\theta - Sin^2\theta)}{1+(Cos^2\theta-Sin^2\theta)} = \frac{1-Cos^2\theta+Sin^2\theta}{1+Cos^2\theta-Sin^2\theta} = \frac{Sin^2\theta+Sin^2\theta}{Cos^2\theta-Cos^2\theta} = \frac{2Sin^2\theta}{2Cos^2\theta} = tan^2\theta$

RHS: $tan^2\theta$ $(05\ marks)$

$\therefore \frac{1-coscos\ 2\theta}{1+cos2\theta} = tan^2\theta$, SHOWN

6.(a).(i).Let $7^{2n+1} + 1 = 8A$, where A is an integer

S_1: Let $n = 1$, $7^{2(1)+1} + 1 = 8A$ $(01\ mark)$

$7^{2+1} + 1 = 8A$

343+1=8A

$43(8) = 8A$, it is true (01 $mark$)

S_2: Assume $n = k, 7^{2k+1} + 1 = 8A$, then $7^{2k+1} = 8A - 1$ ……(i)

S_3: Let $n = k + 1, 7^{2(k+1)+1} + 1 = 8A$

$7^{2k+2+1} + 1 = 8A$ (01 $mark$)

$7^{2k+1}.7^2 + 1 = 8A$ but $7^{2k} = 8A - 1$ from(i) above

$(8A - 1).7^2 + 1 = 8A$

$8A.49 - 49 + 1 = 8A$ (01 $mark$)

$8A.49 - 48 = 8A$

$8(A.49 - 6) = 8A$, It is also true (01 $mark$)

Since the statement is proven true for S_1, S_2 and S_3, it is true that $7^{2n+1} + 1$ is a multiple of 8

(ii).Let $\frac{3x+1}{(x-1)(x^2+1)} \equiv \frac{A}{x-1} + \frac{Bx+c}{x^2+1}$ where A, B and $C$are the constants to be determined

$\frac{3x+1}{(x-1)(x^2+1)} \equiv \frac{A(x^2+1)+(x-1)(Bx+C)}{(x-1)(x^2+1)}$ (01 $mark$)

Compare the LCM: $3x + 1 = A(x^2 + 1) + (x - 1)(Bx + C)$

When $x = 1$, $then\ 4 = 2A \therefore A = 2$ (01 $mark$)

Expanding the equation, $3x + 1 = A(x^2 + 1) + (Bx^2 + Cx - Bx - C)$

Coef.x^2: $0 = A + B$ but $A = 2$ then $B =- 2$ (01 $mark$)

Coef.x: $3 = C - B$ but $B =- 2$ then $C = 1$ (01 $mark$)

$\therefore \frac{3x+1}{(x-1)(x^2+1)} = \frac{2}{x-1} + \frac{-2x+1}{x^2+1}$ (01 $mars$)

6 (b) .From $|a\ h\ g\ h\ b\ f\ g\ f\ c\,| = a|b\ f\ f\ c\,| - h|h\ f\ g\ c\,| + g|h\ b\ g\ f\,|$
$(03\ marks)$

$$= a\left(bc - f^2\right) - h(hc - gf) + g(hf - gb)$$

$$= abc - af^2 - h^2c + hgf + ghf - g^2b$$

$$= abc + 2hfg - af^2 - g^2b - ch^2 \qquad (02\ marks)$$

$$\therefore \quad |a\ h\ g\ h\ b\ f\ g\ f\ c\,| = abc + 2hfg - af^2 - bg^2 - ch^2$$

(c). If log$log_4 m = a$ and $log_{12} m = b$, prove that $log_3 48 = \frac{a+b}{a-b}$

Using logarithm definitions: $m = 4^a$ and $m = 12^b$

Taking log_3 on both sides: $log_3\left(4^a\right) = log_3\left(12^b\right)$

$$alog_3 4 = blog_3 12 \qquad (05\ marks)$$

$$2alog_3 2 = blog_3\left(2^2 + 3\right)$$

$$2alog_3 2 = 2blog_3 2 + blog_3 3$$

$$2alog_3 2 - 2blog_3 2 = blog_3 3$$

6.(d).The Binomial expansion for $(a + x)^n = \sum_{k=0}^{n} (n\ k\,)a^{n-k}x^k$ $(01\ mark)$

Then for$(2 + x)^5$, we set $a = 2$, $n = 5$

$$(2 + x)^5 = \sum_{k=0}^{5} (5\ k\,)2^{5-k}x^k \qquad (01\ mark)$$

Expanding the first three terms: when $k = 0$, $\sum_{k=0}^{5} (5\ 0\,)2^{5-k0}x^0 = 1\times2^5\times1 = 32$

When $k = 1$, $\sum_{k=1}^{5} (5\ 1)2^{5-1}x^{1} = 5 \times 2^{4} \times x = 80x$

When $k = 2$, $\sum_{k=2}^{5} (5\ 2)2^{5-2}x^{1} = 10 \times 2^{3} \times x^{2} = 80x^{2}$

Thus, the first three terms are $(2 + x)^{5} \approx 32 + 80x + 80x^{2}$ $(02\ mark)$

Substitute $x = 0.001$, to find $(2 + x)^{5}$:

$(2 + 0.001)^{5} \approx 32 + 80(0.001) + 80(0.001)^{2} = 32.08008$

$\therefore (2.001)^{5} \approx 32.08008$ $(01\ mark)$

8.(a).(i).O$rder$ is the highest derivative of a given differential equation while $degree$ is an integer whereby the highest derivative of a differential equation is raised$(01\ mark)$

For example: $\frac{dy}{dx} + 5y = 0$,it is first order and of degree 1

$\left(\frac{d^{2}y}{dx^{2}}\right)^{5} - 7\frac{dy}{dx} + 3y = 5x - 2$, it is second order and of degree 5 $(01\ mark)$

(ii). Re-arrange the equation $x^{2}\frac{dy}{dx} + 2xy = 0$

$x^{2}\frac{dy}{dx} =- 2xy$

Simplifying, $\frac{dy}{dx} =- \frac{2y}{x}$

Separating the variables, $\frac{dy}{y} =- \frac{2dx}{x}$ $(01\ mark)$

Integrating in both sides $\int\frac{dy}{y} =- 2\int\frac{dx}{x}$

$ln|y| =- 2ln|x| + C$ $(01\ mark)$

Solve for y, $ln|y| = ln\left|x^{-2}\right| + C$

$y = x^{-2}e^{C}$

Choose $e^{C} = A$ (or any constant), then $y = Ax^{-2}$ or $y = \frac{A}{x^2}$ $(01\ mark)$

8.(b). $y = ACosh3x + BSinh3x + x^2 + 2$……..(i) $(01\ mark)$

$y^{'} = 3ASinh3x + 3BCosh3x + 2x$ ………..(ii)

$y^{''} = 9ACosh3x + 9BSinh3x + 2$ ……….(iii) $(01\ mark)$

From eqn(i), $ACosh3x = y - BSinh3x - x^2 - 2$ and multiply by 9 to get

$9ACosh3x = 9y - 9BSinh3x - 9x^2 - 18$ …..(iv) $(01\ mark)$

Substitute (iv) into (iii), $y^{''} = 9y - 9BSinh3x - 9x^2 - 18 + 9BSinh3x + 2$

Simplifying, $y^{''} = 9y - 9x^2 - 16$ $(01\ mark)$

$y^{''} - 9y =- 9x^2 - 16$

∴The required differential equation is $\frac{d^2y}{dx^2} - 9\frac{dy}{dx} =- 9x^2 - 16$ $(01\ mark)$

(c).From $\frac{d^2x}{dx^2} + 2\frac{dy}{dx} + 5y = 26 + 15x$,

The A.Q.E is $m^2 + 2m + 5 = 0$, solving for m then $m = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$$m = \frac{-2\pm\sqrt{2^2-4(1)(5)}}{2(1)}$$

$m =- 1\pm2i$ implying that $p =- 1$, $q = 2$

Complimentary solution is $y_c = e^{-x}(ACos2x + BSin2x)$ $(01\ mark)$

For particular solution, let $y = ax + c$

$y^{'} = a$

$y^{''} = 0$

Putting these into $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 26 + 15x$, $(01\ mark)$

$$0 + 2(a) + 5(ax + b) = 26 + 15x$$

Coef. x: $5a = 15, a = 3$

Constant: $2a + 5b = 26$ but $a = 3$ $then$ $6 + 5b = 26$, $b = 4$ $(01\ mark)$

Then $y_p = 3x + 4$,solution by $y = y_c + y_p$

$$y = e^{-x}(ACos2x + BSin2x) + 3x + 4$$

When $= 0, y = 7, 7 = e^{-0}(ACos0 + BSin0) + 0 + 4$, $A + 4 = 7$, $A = 3$
$(01\ mark)$

$$y' = - e^{-x}(- 2ASin2x + 2BCos2x) + 3$$

When $\frac{dy}{dx} = 3$, $y = 7$ and $x = 0$, $3 = - e^{-0}(0 + 2B) + 3$, $3 = 2B + 3$, $B = 0$

$\therefore The\ solution\ is\ y = 3e^{-x}Cos2x + 3x + 4$ $(01\ mark)$

8.(d).Let $P_0 =$ Initial population18700000

$P_t =$ Population at time t $= 1.5 \times P_0 = 1.5 \times 18700000 = 28050000$

$k =$ Rate constant (growth constant) $(01\ mark)$

$t =$ time$(1988 - 1978 = 10 years)$

$$\frac{dP}{dt} \propto P_t \text{ then } \quad \frac{dP}{dt} = k\,P_t$$

Separating the variable and Integrating, $\int_{P_0}^{P} \frac{1}{P} dP = k\int_0^t dt$

$$(lnp)_{p_0}^{p} = k(t)_0^t \quad (01\ mark)$$

$$lnP - lnP_0 = kt$$

$\frac{P}{P_0} = e^{kt}$ then $k = \frac{1}{t} ln\left(\frac{P}{P_0}\right) = \frac{1}{10} ln\left(\frac{28050000}{18700000}\right) = 0.0405$

∴The growth constant is 0.0405 per year (01 $mark$)

$$P = p_0 e^{kt}$$

Then $= 1998 - 1978 = 20\ years$,

$P = 1870000e^{(0.0405\times20)} = 42035879$ (02 $marks$)

∴The size of the population in 1998 will be equal to 42035879