Linear Algebra MAT313 Spring 2024
Professor Sormani
Lesson 1 Linear Systems and Vectors

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Carefully take notes on pencil and graph paper while watching the lesson videos. Pause the lesson to try classwork before watching the solutions video going over that classwork. If you work with any classmates, be sure to write their names on the problems you completed together.

Next take photos of your notes and completed classwork and a selfie taken holding up the first page of your work. Copy and paste the photos into a googledoc (not a pdf) with the name:
MAT313S24-lesson1-lastname-firstname
and share editing of that doc with me: sormanic@gmail.com. You can use your Lehman id and hand instead of your face in your selfie.

If you have a question, type **QUESTION** ***in your googledoc next to the point in your notes that has a question and then email me sormanic@gmail.com with the subject*** **MAT313 QUESTION**. ***I will answer your question by inserting a photo into your googledoc or making an extra video.***

Before the deadline, you will also complete your homework and add photos of your homework at the end of the same doc then share editing of that doc with me again: sormanic@gmail.com.

After the deadline, I will send feedback on your notes, classwork, and homework and then you will have the opportunity to resubmit (if necessary) adding missing work and correcting errors. After this lesson's notes, classwork and homework are completed correctly, then you may continue to the next lesson. After Lessons 1-3 are completed correctly, then you may take Quiz 1.

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Linear Algebra Welcome Video Playlist
(if you have not watched it yet, please watch it now)

The videos and classwork today will take more than two hours but the homework is shorter than usual. There are 10 HW problems. Usually a lesson including homework will take 6 hours.

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This lesson has two parts
and each part has its own playlist:

Part I: Linear Systems

Part II Vectors

Part I: Linear Systems

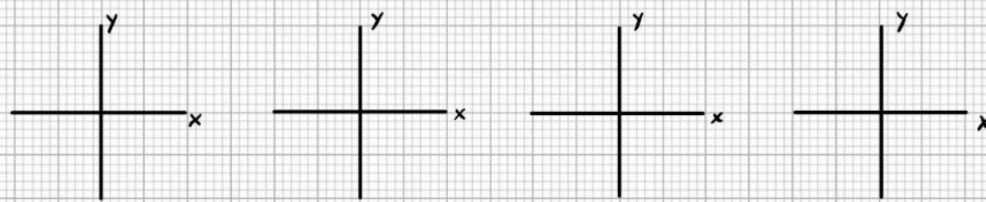
Lesson 1 Playlist 313S23-L1-P1 with 12 videos and 20 classwork problems. If you wish you may first watch the videos about linear systems and take a break before watching the videos about vectors.

Class notes including solutions to classwork followed by homework:

We begin with Linear Equations

Examples of Linear Equations

(1) $x+y=4$ (2) $x-y=4$ (3) $x+y=8$ (4) $2x+2y=8$



one equation
two variables : x and y
graph the solution to each of these

(1) $x+y=4$

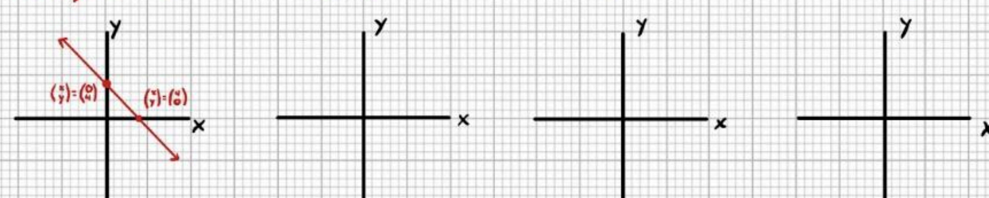
$y=4-x$ (basic algebra) slope $=-1$
$y=-x+4$ y intercept $=4$

$y=mx+b$ ← will not use this in this course because it does not work with more variables

graph by finding two points on the line and drawing the line between them

Examples of Linear Equations

(1) $x+y=4$ (2) $x-y=4$ (3) $x+y=8$ (4) $2x+2y=8$



one equation
two variables : x and y

$\boxed{ax+by=z}$ The solution is a line in the plane.

graph the solution to each of these

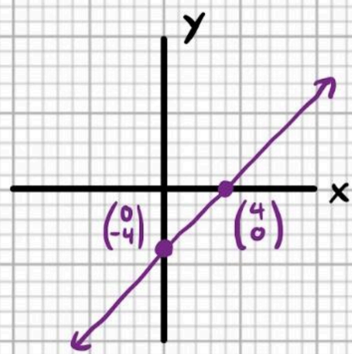
(1) $x+y=4$ Solve graphically (know it is a line)

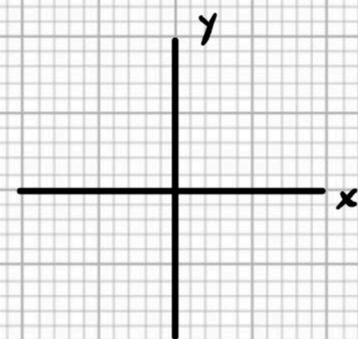
x	y
0	4
4	0

$0+y=4$
$x+0=4$

Classwork
Use this method
to graph (2) (3) + (4)

② $x - y = 4$





② $x - y = 4$

$0 - y = 4$
$y = -4$

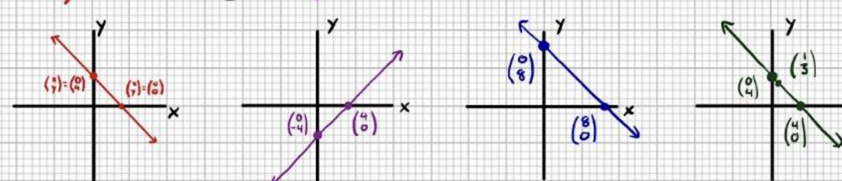
x	y
0	-4
4	0

$\binom{0}{-4}$

$x - 0 = 4$
$x = 4$

Examples of Linear Equations

① $x + y = 4$ ② $x - y = 4$ ③ $x + y = 8$ ④ $2x + 2y = 8$



All points here solve the equation

③ $x + y = 8$

$0 + y = 8$
$x + 0 = 8$

x	y
0	8
8	0

④ $2x + 2y = 8$

$2 \cdot 1 + 2y = 8$
$2y = 6$
$y = 3$

$2x + 2 \cdot 0 = 8$
$2x = 8$
$x = 4$

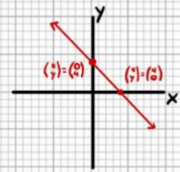
x	y
1	3
4	0

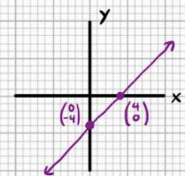
Just need any two points

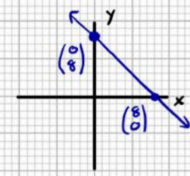
Next Solve systems of linear equations.

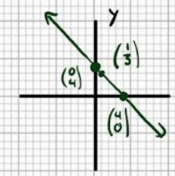
Examples of Linear Equations

① $x+y=4$ ② $x-y=4$ ③ $x+y=8$ ④ $2x+2y=8$









All points here solve the equation

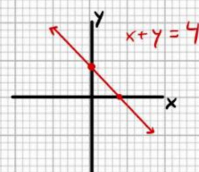
Examples of Systems of Linear Equations

⑤ $x+y=4$ AND $x-y=4$

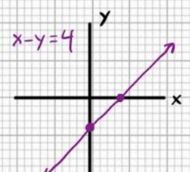
solutions to both equations

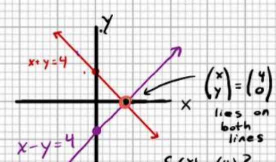
$\boxed{\begin{array}{l} x+y=4 \\ x-y=4 \end{array}}$

Find points that lie on both lines.



and





Solution set $\left\{\binom{x}{y}=\binom{4}{0}\right\}$

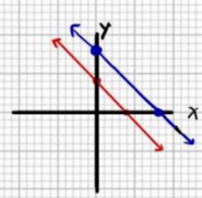
Check $x=4$ $y=0$ solves the system

$4+0=4$ ✓ $4-0=4$ ✓

⑥ $x+y=4$ AND $x+y=8$

$\boxed{\begin{array}{l} x+y=4 \\ x+y=8 \end{array}}$

Pause + Solve graphically



$x+y=4$
$x+y=8$
are parallel
never meet

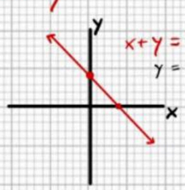
} no solution

$\emptyset$

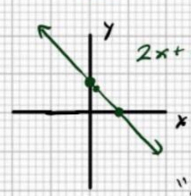
empty set

Pause + solve!

⑦ $x+y=4$ AND $2x+2y=8$



$x+y=4$
$y=4-x$



$2x+2y=8$
$2y=8-2x$
$y=4-x$

x can have any value in $\mathbb{R}$
"x is free"

The same line!

Every point on the line is a solution!

$\left\{\binom{x}{y}=\binom{x}{4-x} \,\middle|\, x\in\mathbb{R}\right\}$

real line

Notice $2x+2y=8$

divide both sides by 2

$x+y=4$

multiply both sides by 2

$2x+2y=8$

$x+y=4 \iff 2x+2y=8$

iff symbol "if and only if"

(8) $x+2y=8$ AND $x-2y=0$

(9) $x+2y=8$ AND $2x+4y=16$

(10) $x+2y=8$ AND $2x+4y=0$

pause + try

solutions are in the classnotes.

Solutions

(8) $x+2y=8$ AND $x-2y=0$

$$\boxed{\begin{aligned} x+2y&=8 \\ x-2y&=0 \end{aligned}}$$

x	y
0	4
8	0

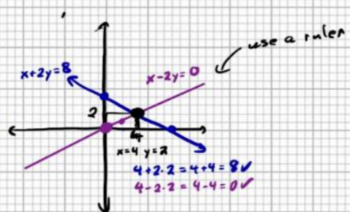
$0+2y=8$, $2y=8$, $y=4$

$x+2\cdot0=8$, $x=8$

x	y
0	0
2	1

$0-2y=0$, $y=0$

$2-2y=0$, $y=1$



Solution Set $= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} \right\}$

only one point

(9) $x+2y=8$ AND $2x+4y=16$

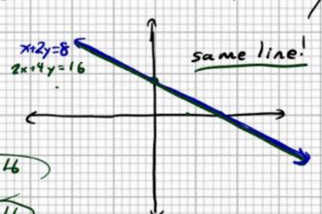
$$\boxed{\begin{aligned} x+2y&=8 \\ 2x+4y&=16 \end{aligned}}$$

$2x+4y=16$

x	y
0	4
8	0

$2\cdot0+4y=16$, $y=4$

$2x+4\cdot0=16$, $x=8$



a line

Solution Set $= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 4-\frac{1}{2}x \end{pmatrix} \middle| \; x \in \mathbb{R} \right\}$

$x+2y=8$

$2y=8-x$

$y=4-\frac{1}{2}x$

(10) $x+2y=8$ AND $2x+4y=0$

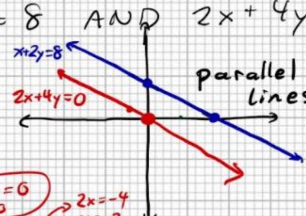
$$\boxed{\begin{aligned} x+2y&=8 \\ 2x+4y&=0 \end{aligned}}$$

$2x+4y=0$

x	y
0	0
-2	1

$2\cdot0+4y=0$, $y=0$

$2x+4\cdot1=0$, $2x=-4$, $x=-2$



$2y=8-x$

$y=-\frac{1}{2}x+4$

$4y=0-2x$

$y=-\frac{2}{4}x=-\frac{1}{2}x$

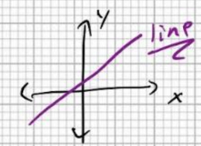
same slope $=\frac{-1}{2}$

Solution Set $=\emptyset$

no solution

Definition of a Linear Equation

a linear equation with two variables:



coefficients

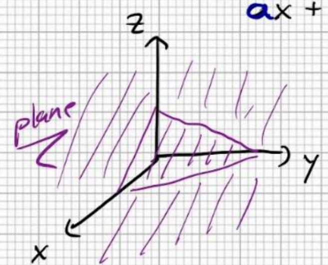
$ax + by = c$ where a, b, c are given real numbers and x, y are unknown variables

the solution is always a line.

with three variables:

coefficients

$ax + by + cz = d$ where a, b, c, d are given and x, y, z are unknown



Solution set is a plane (will discuss more later)

more generally we will have m variables with m very large.

Defn a linear equation with m variables

$x_1, x_2, x_3 \dots x_m$ are unknown variables

each has a coefficient a_i in $\mathbb{R}$

$$a_1 x_1 + a_2 x_2 + a_3 x_3 + \dots + a_m x_m = d$$

we are given $a_1, a_2 \dots a_m \in \mathbb{R}$ and $d \in \mathbb{R}$

find $x_1 \dots x_m$ in $\mathbb{R}$

Sum notation:

$$\sum_{j=1}^{m} a_j x_j = d$$

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} \in \mathbb{R}^m \quad m\text{-vector}$$

(11) $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 8 \\ 9 \end{pmatrix}$ $a_1 = 2 \quad a_2 = 4 \quad a_3 = 6$

$d = 10$

Rewrite $\sum_{j=1}^{m} a_j x_j = d$ using these values: check if true or false

Solution:

Try before looking at the solution

(11) $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5 \\ 8 \\ 9 \end{pmatrix}$ $a_1 = 2 \quad a_2 = 4 \quad a_3 = 6$

$d = 10$

Rewrite $\sum_{j=1}^{3} a_j x_j = d$ using these values: and check if true or false

Solution: $a_1 x_1 + a_2 x_2 + a_3 x_3 = d$ by sum notation

stop at $\underline{3}$ top of the sum

$2 \cdot 5 + 4 \cdot 8 + 6 \cdot 9 = 10$ false

(12) $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 6 \\ 8 \end{pmatrix}$ $a_1 = 1 \quad a_2 = 3 \quad a_3 = 5 \quad a_4 = 7$

Find the value of d

$d = \sum_{j=1}^{4} a_j x_j$ solutions in notes

(12) $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 6 \\ 8 \end{pmatrix}$ $a_1 = 1 \quad a_2 = 3 \quad a_3 = 5 \quad a_4 = 7$

Find the value of d

$d = \sum_{j=1}^{4} a_j x_j$ solutions in notes

Try before looking at the solution!

SOLUTION

(12) $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 6 \\ 8 \end{pmatrix}$ $a_1 = 1 \quad a_2 = 3 \quad a_3 = 5 \quad a_4 = 7$

Find the value of d

$$d = \sum_{j=1}^{4} a_j x_j = \sum_{j=1}^{4} a_j x_j = a_1 x_1 + a_2 x_2 + a_3 x_3 + a_4 x_4$$

stop at 4

$$= 1 \cdot 2 + 3 \cdot 4 + 5 \cdot 6 + 7 \cdot 8$$

$$= 2 + 12 + 30 + 56 = 44 + 56 = 100 \quad \boxed{d = 100}$$

Defn: A system of $\underline{n}$ linear equations with m unknown variables

$$\boxed{\begin{aligned} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,m}x_m &= d_1 &&\leftarrow \text{equation 1} \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,m}x_m &= d_2 &&\leftarrow \text{equation 2} \\ &\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{n,m}x_m &= d_n &&\leftarrow \text{equation } n \end{aligned}}$$

Can also use sum notation

$$\boxed{\sum_{j=1}^{m} a_{i,j}x_j = d_i \quad \text{for } i = 1 \text{ to } n}$$

(13) $a_{1,1} = 1$ $a_{1,2} = 2$ $a_{1,3} = 3$ $d_1 = 10$

$a_{2,1} = 4$ $a_{2,2} = 5$ $a_{2,3} = 6$ $d_2 = 11$

Rewrite the system $\sum_{j=1}^{3} a_{ij} x_j = d_i$ for $i=1,2$

with $x_1 = x$
$x_2 = y$
$x_3 = z$

pause + try

(13) $a_{1,1} = 1$ $a_{1,2} = 2$ $a_{1,3} = 3$ $d_1 = 10$

$a_{2,1} = 4$ $a_{2,2} = 5$ $a_{2,3} = 6$ $d_2 = 11$

Rewrite the system $\boxed{\sum_{j=1}^{3} a_{ij} x_j = d_i \text{ for } i=1,2}$

with $x_1 = x$
$x_2 = y$
$x_3 = z$

pause + try

2 equations:

$i=1$ $\sum_{j=1}^{3} a_{1j} x_j = d_1$ $a_{1,1} x_1 + a_{1,2} x_2 + a_{1,3} x_3 = d_1$

$1x + 2y + 3z = 10$

$i=2$ $\sum_{j=1}^{3} a_{2j} x_j = d_2$ $a_{2,1} x_1 + a_{2,2} x_2 + a_{2,3} x_3 = d_2$

$4x + 5y + 6z = 11$

$$\boxed{\begin{aligned} 1x + 2y + 3z &= 10 \\ 4x + 5y + 6z &= 11 \end{aligned}}$$

(14) Rewrite the system $\boxed{\begin{array}{l} 5x+2y=10 \\ 2x-4y=0 \\ x+y=8 \end{array}}$

using sum notation

and tell us the value of each a_{ij}, d_i, x_j

use sum notation

coefficients

$a_{i,j}$

i^{th} equation

j^{th} variable x_j

$$\boxed{\sum_{j=1}^{m} a_{i,j} x_j = d_i \quad \text{for } i=1 \text{ to } n}$$

pause + solve

(14) Rewrite the system $\boxed{\begin{array}{l} 5x+2y=10 \\ 2x-4y=0 \\ 1x+1y=8 \end{array}}$ ✓ ✓ ✓

using sum notation

and tell us the value of each a_{ij}, d_i, x_j

use sum notation

coefficients

$a_{i,j}$

i^{th} equation

j^{th} variable x_j

$$\boxed{\sum_{j=1}^{m} a_{i,j} x_j = d_i \quad \text{for } i=1 \text{ to } n}$$

$\sum_{j=1}^{2}$ ← because we have $m=2$ variables $x_1=x$ $x_2=y$

$$\sum_{j=1}^{2} a_{ij} x_j = d_i \quad \text{for } i=1 \text{ to } n=3$$

because we have 3 equations

equation $i=1$ $\quad a_{1,1}=5 \quad a_{1,2}=2 \quad d_1=10$

equation $i=2$ $\quad a_{2,1}=2 \quad a_{2,2}=-4 \quad d_2=0$

equation $i=3$ $\quad a_{3,1}=1 \quad a_{3,2}=1 \quad d_3=8$

because $x=1\cdot x$

If you wish you can do the first five homework problems now and do Part II another day.

Lesson 1 Homework

HW 1 Graph and Solve (write the solution set)

$$\boxed{\begin{aligned} 2x+3y&=12 \\ 4x+9y&=36 \end{aligned}}$$

HW 2 Graph and Solve

$$\boxed{\begin{aligned} 2x+3y&=12 \\ 4x+6y&=36 \end{aligned}}$$

HW 3 Graph and Solve

$$\boxed{\begin{aligned} 2x+3y&=12 \\ 4x+6y&=24 \end{aligned}}$$

HW 4

Rewrite the system

$$\boxed{\begin{aligned} 2x+3y+4z&=5 \\ 6x+7y+8z&=9 \\ x-y&=0 \end{aligned}}$$

as a sum

$$\sum_{j=1}^{m} a_{ij} x_j = d_i \text{ for } i=1 \text{ to } n$$

with $x_1=x$ $x_2=y$ $x_3=z$

What is m? What is n?

What are the $a_{i,j}$ and d_i?

HW 5 Write the system $\sum_{j=1}^{4} a_{i,j} x_j = d_i$ for $i=1$ to 3

with $x_1=x$ $x_2=y$ $x_3=z$ $x_4=w$

$d_i = i^2$ (that is $d_1=1$ $d_2=2^2=4$ $d_3=3^2=9\ldots$)

and $a_{i,j}=(i+j)$ ($a_{1,1}=1+1=2$ $a_{1,2}=1+2=3\ldots$)

(

Part II Vectors

Watch [Playlist 313S23-L1-P2.]

Vectors in $\mathbb{R}^m$

$m=2$ vector $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$

Example $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ 3

$m=3$ vector $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3$ seen in vector calculus

vector $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} \in \mathbb{R}^m$

$\mathbb{R}$ reals $(-\infty, \infty)$

Linear Algebra 0

Scratchwork

Linear Algebra 0

Vectors in $\mathbb{R}^m$

$m=2$ vector $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$

Example $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ 3

$m=3$ vector $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3$ seen in vector calculus

vector $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} \in \mathbb{R}^m$

$\mathbb{R}$ reals $(-\infty, \infty)$

Example

$\begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix} \in \mathbb{R}^3$ $\qquad$ $\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \in \mathbb{R}^8$

we will have $m = 10{,}000$ or higher

$\vec{x} \in \mathbb{R}^4 \qquad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \qquad \vec{z} \in \mathbb{R}^5 \qquad \vec{z} = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{pmatrix}$

$\vec{y} \in \mathbb{R}^4 \qquad \vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ x_3 \\ y_4 \end{pmatrix}$

y_i is the i^{th} "entry" of vector $\vec{y}$
"component"

$\boxed{\text{Addition of Vectors:}}$

Definition: Given $\vec{x}$ and $\vec{y} \in \mathbb{R}^m$ then

$$\vec{x} + \vec{y} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_m + y_m \end{pmatrix} \in \mathbb{R}^m$$

$\in$ in symbol

Rewrite

Addition of Vectors:

Defn: Given $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}$ and $\vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}$ in $\mathbb{R}^m$ then

$$\vec{x} + \vec{y} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_m + y_m \end{pmatrix} \in \mathbb{R}^m$$

(15)
$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 1+4 \\ 2+5 \\ 3+6 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \\ 9 \end{pmatrix} \in \mathbb{R}^3$$

(16)
$$\begin{pmatrix} 1 \\ 3 \\ 5 \\ 7 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1+0 \\ 3+0 \\ 5+0 \\ 7+0 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 5 \\ 7 \end{pmatrix} \in \mathbb{R}^4$$

Definition: Scalar Multiplication of Vectors

Given a vector $\vec{v} \in \mathbb{R}^m$ and

a scalar $t \in \mathbb{R}$

we define the scalar product

$$t\vec{v} = t\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{pmatrix} = \begin{pmatrix} tv_1 \\ tv_2 \\ \vdots \\ tv_m \end{pmatrix}$$

Rescales the vector by t

(17) $2\begin{pmatrix} 1 \\ 4 \end{pmatrix} = ?$ graph

(18) $-4\begin{pmatrix} 3 \\ 2 \end{pmatrix} = ?$ graph

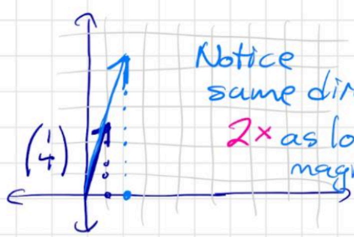
pause + try

we define the scalar product

$$t\vec{v} = t\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{pmatrix} = \begin{pmatrix} tv_1 \\ tv_2 \\ \vdots \\ tv_m \end{pmatrix}$$

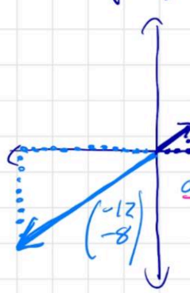
Rescales the vector by t

(17) $2\begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 2\cdot 1 \\ 2\cdot 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \end{pmatrix}$



Notice same direction
2× as long a magnitude

(18) $-4\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} -4\cdot 3 \\ -4\cdot 2 \end{pmatrix} = \begin{pmatrix} -12 \\ -8 \end{pmatrix}$



negative scalar
opposite direction
4× as long

Definition: Subtraction of vectors: Given $v, w \in \mathbb{R}^m$

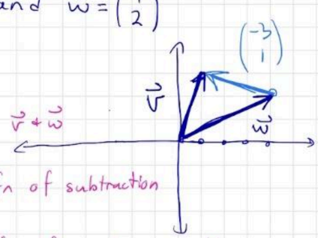
$$\vec{v} - \vec{w} = \vec{v} + \underbrace{(-1)\vec{w}}_{\substack{\text{scalar product} \\ \text{(reverses direction)}}} \in \mathbb{R}^m$$

(19) Graph $\vec{v} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\vec{w} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$ and find $\vec{v} - \vec{w}$

(pause + try)

Solution:

(19) Graph $\vec{v} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\vec{w} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$ and find $\vec{v} - \vec{w}$



$\vec{v} - \vec{w} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} - \begin{pmatrix} 4 \\ 2 \end{pmatrix}$ by our $\vec{v} + \vec{w}$

$= \begin{pmatrix} 1 \\ 3 \end{pmatrix} + (-1)\begin{pmatrix} 4 \\ 2 \end{pmatrix}$ by defn of subtraction

$= \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \begin{pmatrix} (-1)4 \\ (-1)2 \end{pmatrix}$ by defn of scalar mult.

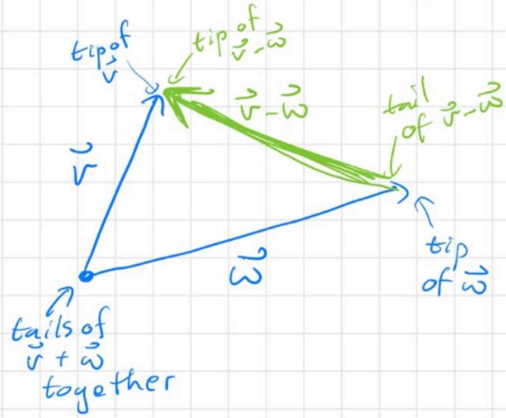
$= \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \begin{pmatrix} -4 \\ -2 \end{pmatrix}$ by arithmetic $\begin{matrix} (-1)4 = -4 \\ (-1)2 = -2 \end{matrix}$

$= \begin{pmatrix} 1 + (-4) \\ 3 + (-2) \end{pmatrix}$ by defn of vector addition

$= \begin{pmatrix} -3 \\ 1 \end{pmatrix}$ by arithmetic $\begin{matrix} 1 - 4 = -3 \\ 3 - 2 = 1 \end{matrix}$

Justified my steps

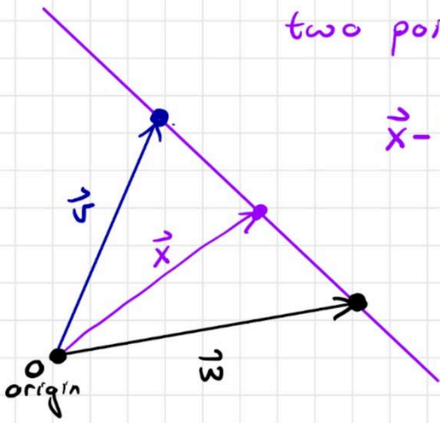
In fact $\vec{v}-\vec{w}$ is always a vector whose tip is at the tip of $\vec{v}$ and whose tail is at the tip of $\vec{w}$



For more about vector addition and scalar multiplication, see Additional Resources at the end of this.lesson

Lines written in Vector Notation

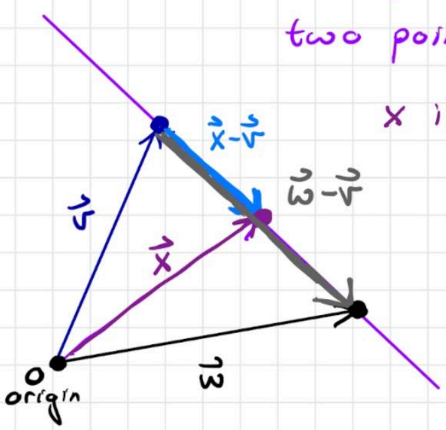
Find a formula for a line through two points $\vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix}$ and $\vec{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_m \end{pmatrix}$



$\vec{x} - \vec{v}$

Lines written in Vector Notation

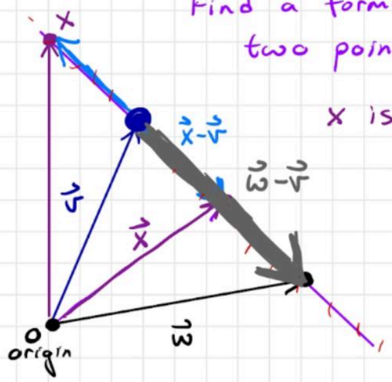
Find a formula for a line through two points $\vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix}$ and $\vec{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_m \end{pmatrix}$



x is a typical point on the line

$$\vec{x} - \vec{v} = t\,(\vec{w} - \vec{v})$$

Lines written in Vector Notation



Find a formula for a line through two points $\vec{v}=\begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix}$ and $\vec{w}=\begin{pmatrix} w_1 \\ \vdots \\ w_m \end{pmatrix}$

x is a typical point on the line

$$\vec{x}-\vec{v} = t\,(\vec{w}-\vec{v})$$

$$\Big\{\vec{x} = \vec{v} + t\,\underbrace{(\vec{w}-\vec{v})}\ \Big|\ t\in\mathbb{R}\Big\}$$

position vector $\vec{v}$ on the line

direction vector $(\vec{w}-\vec{v})$ where $\vec{w}$ is also on the line

as we change the value of t we get different points $\vec{x}$ lying on the line

(20) (a) Write this line in vector notation

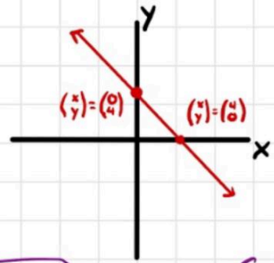
$x+y=4$

Solution below

First choose two points to be $\vec{v}$ and $\vec{w}$.

(20) (a) Write this line in vector notation

$x + y = 4$ using $\vec{v} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$ and $\vec{\omega} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$



$$\left\{ \vec{x} = \underset{\text{position}}{\vec{v}} + t\,\underbrace{(\vec{\omega} - \vec{v})}_{\text{direction}} \;\middle|\; t \in \mathbb{R} \right\}$$

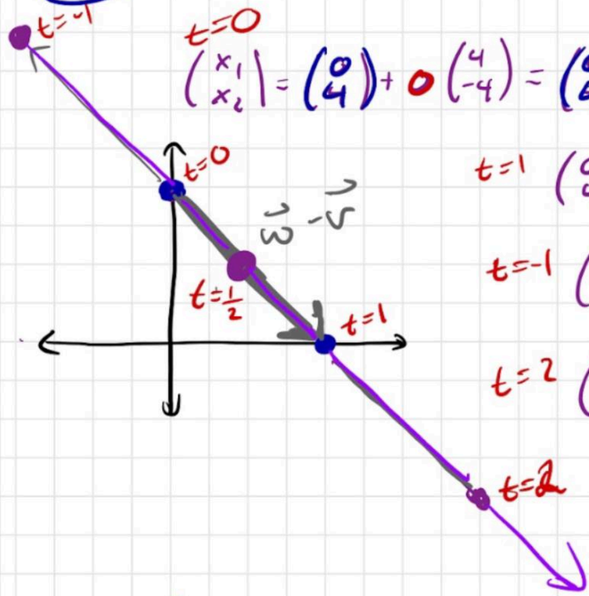
position $\begin{pmatrix} 0 \\ 4 \end{pmatrix}$

direction $\begin{pmatrix} 4 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 4-0 \\ 0-4 \end{pmatrix} = \begin{pmatrix} 4 \\ -4 \end{pmatrix}$

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} + t \begin{pmatrix} 4 \\ -4 \end{pmatrix} \;\middle|\; t \in \mathbb{R} \right\}$$

(20b) Plot the vectors for $t=0, t=1, t=-1, t=2, t=4$ and $t = \frac{1}{2}$



$t=0$: $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} + 0 \begin{pmatrix} 4 \\ -4 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$ position $\vec{v}$

$t=1$: $\begin{pmatrix} 0 \\ 4 \end{pmatrix} + 1 \begin{pmatrix} 4 \\ -4 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \leftarrow \vec{\omega}$

$t=-1$: $\begin{pmatrix} 0 \\ 4 \end{pmatrix} + (-1) \begin{pmatrix} 4 \\ -4 \end{pmatrix} = \begin{pmatrix} -4 \\ 8 \end{pmatrix}$

$t=2$: $\begin{pmatrix} 0 \\ 4 \end{pmatrix} + 2 \begin{pmatrix} 4 \\ -4 \end{pmatrix} = \begin{pmatrix} 8 \\ -4 \end{pmatrix}$

$t=\frac{1}{2}$: $\begin{pmatrix} 0 \\ 4 \end{pmatrix} + \left(\frac{1}{2}\right) \begin{pmatrix} 4 \\ -4 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

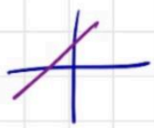
$$\left\{ \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \end{pmatrix} + t \begin{pmatrix} 4 \\ -4 \end{pmatrix} \;\middle|\; t \in \mathbb{R} \right\}$$

Lines in $\mathbb{R}^3$, $\mathbb{R}^4$ and even $\mathbb{R}^m$

$$\{\vec{x} = \underset{\text{position}}{\vec{v}} + t\,\underbrace{(\vec{w}-\vec{v})}_{\text{direction}} \mid t \in \mathbb{R}\}$$

works in all dimensions.

$m=2$ in $\mathbb{R}^2$



$m=3$ in $\mathbb{R}^3$

$m=4$ and higher cannot be drawn but we can still describe the set.

Example in $\mathbb{R}^4$

Find the line through $\vec{v} = \begin{pmatrix}1\\2\\3\\4\end{pmatrix}$ and $\vec{w} = \begin{pmatrix}0\\1\\5\\8\end{pmatrix}$

Solution: try first if you with

Direction $\vec{w}-\vec{v} = \begin{pmatrix}0\\1\\5\\8\end{pmatrix} - \begin{pmatrix}1\\2\\3\\4\end{pmatrix} = \begin{pmatrix}0-1\\1-2\\5-3\\8-4\end{pmatrix} = \begin{pmatrix}-1\\-1\\2\\4\end{pmatrix}$

$$\left\{\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} = \begin{pmatrix}1\\2\\3\\4\end{pmatrix} + t\begin{pmatrix}-1\\-1\\2\\4\end{pmatrix} \;\middle|\; t \in \mathbb{R}\right\}$$

Check $\vec{v}$ and $\vec{w}$ are on the line

at $t=0$ $\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} = \begin{pmatrix}1\\2\\3\\4\end{pmatrix} + 0\begin{pmatrix}-1\\-1\\2\\4\end{pmatrix} = \begin{pmatrix}1\\2\\3\\4\end{pmatrix} + \begin{pmatrix}0\\0\\0\\0\end{pmatrix} = \begin{pmatrix}1\\2\\3\\4\end{pmatrix}$ ✓ $\vec{v} \in$ line

at $t=1$ $\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} = \begin{pmatrix}1\\2\\3\\4\end{pmatrix} + 1\begin{pmatrix}-1\\-1\\2\\4\end{pmatrix} = \begin{pmatrix}1-1\\2-1\\3+2\\4+4\end{pmatrix} = \begin{pmatrix}0\\1\\5\\8\end{pmatrix}$ ✓ $\vec{w} \in$ line

at $t=2$ $\begin{pmatrix}x_1\\x_2\\x_3\\\ldots\end{pmatrix} = \begin{pmatrix}1\\2\\3\\\ldots\end{pmatrix} + 2\begin{pmatrix}-1\\-1\\2\\\ldots\end{pmatrix} = \begin{pmatrix}1-2\\2-2\\3+4\\\ldots\end{pmatrix} = \begin{pmatrix}-1\\0\\7\\\ldots\end{pmatrix}$ on the line.

Check to verify that you have watched all the videos in both playlists before doing your homework.

Additional Resources:

Rootmath Linear Algebra Section 1.1: about vector addition and scalar multiplication:
https://www.youtube.com/playlist?list=PLA738885C1D6E75A4

Lines for Linear Algebra videos and notes

Homework:

At the top of your lesson 1 googledoc, write your full name as on the Lehman registration and also any preferred names or pronouns if you wish. Let me know a little about your career and education goals. It is fine to mention a few possible directions of interest. It is always great to have options and to double major or have a minor.

Include photos of your notes and then do the ten homework problems below showing all work:

HW1 Graph and Solve

$$\boxed{\begin{array}{l} 2x+3y=12 \\ 4x+6y=24 \end{array}}$$

HW2 Graph and Solve

$$\boxed{\begin{array}{l} 2x+3y=12 \\ 4x+6y=36 \end{array}}$$

HW3 Graph and Solve (write the solution set)

$$\boxed{\begin{aligned}2x+3y&=12\\4x+9y&=36\end{aligned}}$$

HW4

Rewrite the system
$$\boxed{\begin{aligned}5x+1y+2z&=3\\4x+2y+1z&=4\\x-y&=0\end{aligned}}$$
as a sum

$$\sum_{j=1}^{m} a_{ij}x_j = d_i \text{ for } i = 1 \text{ to } n$$

with $x_1 = x \quad x_2 = y \quad x_3 = z$

What is m? What is n?

What are the $a_{i,j}$ and d_i?

HW5

Write the system $\sum_{j=1}^{4} a_{i,j}x_j = d_i$ for $i=1$ to 3

with $x_1 = x \quad x_2 = y \quad x_3 = z \quad x_4 = w$

$d_i = i^2$ (that is $d_1 = 1 \quad d_2 = 2^2 = 4 \quad d_3 = 3^2 = 9 \ldots$)

and $a_{i,j} = (i+j)$ ($a_{1,1} = 1+1 = 2 \quad a_{1,2} = 1+2 = 3 \ldots$)

$\boxed{\text{HW6}}$ Let $\vec{v} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$ $\vec{w} = \begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix}$

find $\vec{v} + \vec{w} =$

$\vec{w} - \vec{v} =$

$-3\vec{v} =$

$5\vec{v} + 2\vec{w} =$

$x\vec{v} + y\vec{w} =$

$\boxed{\text{HW7}}$ Plot the line with position $\begin{pmatrix} 5 \\ 0 \end{pmatrix}$ and direction $\begin{pmatrix} -2 \\ -3 \end{pmatrix}$

Write it using set notation.

$\boxed{\text{HW8}}$ Plot the line through $\vec{v} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ and $\vec{w} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$

Write it using set notation.

HW 9 Find the solution set for the line with position $\begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}$ and direction $\begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}$ (No plot needed) and find the points on the line at $t=0$, $t=1$, and $t=2$.

HW 10 Find the line through

$$\vec{v} = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \quad \text{and} \quad \vec{w} = \begin{pmatrix} 5 \\ 0 \\ 0 \\ 6 \\ 0 \end{pmatrix}$$

and write the solution set (do not plot)

To submit homework follow the directions at the top of this document.

Do not continue to the next lesson until I have checked this work and given feedback.

****

Extra work if this lesson was not too difficult for you:
Read [Beezer Preliminaries on Complex numbers] and then practice at [IXL]
and then do the following 5 problems:

Find
$(2-3i)+(4+5i)=$
$(2+5i)-(5-2i)=$
$(2-3i)(4+5i)=$
$(2+3i)(2-3i)=$
$\dfrac{2+3i}{4+5i}=$

Submit this in your Lesson 1 googledoc at well. If this lesson was difficult for you then you may do this later in the course. We will only need complex numbers later.