Overview

Given the generalizability and predictive accuracy of Jack Daniels' VDOT system, I thought it would be interesting to explore applying it using the framework of the critical speed model. Specifically, if an optimal test protocol is used, what is the critical speed based on a person's VDOT value, how long can that critical speed be maintained, and at what percent of VDOT and 5km speed does it occur?

Constructing the Critical Speed Model Using VDOT

Given the optimal critical speed test protocol of 5 efforts spanning 3-20 minutes, I chose to use 5 times over equal duration intervals. So, the set of times used for the calculation, in minutes, is:

$$Time = \{3.00,\ 7.25,\ 11.50,\ 15.75,\ 20.00\}$$

Using VDOT values from 30-85, the speed and distance were calculated for the above time durations. This allowed for the fitting of a linear model to both estimate the parameters of the hyperbolic model, but also use in later calculations. Since I'm using VDOT values from 30-85, there are 56 models, one of which follows as an example.

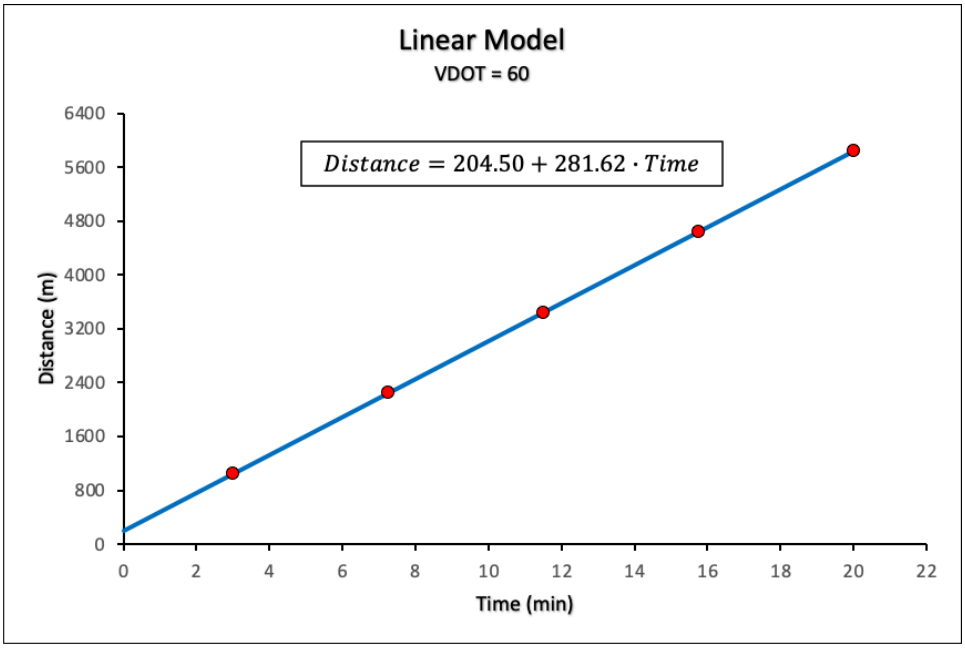

Figure 1

Once the linear model was established, both D Prime and critical speed associated with the 56 VDOT values could be used to form the hyperbolic models for each VDOT value. Again, an example of one of these models follows.

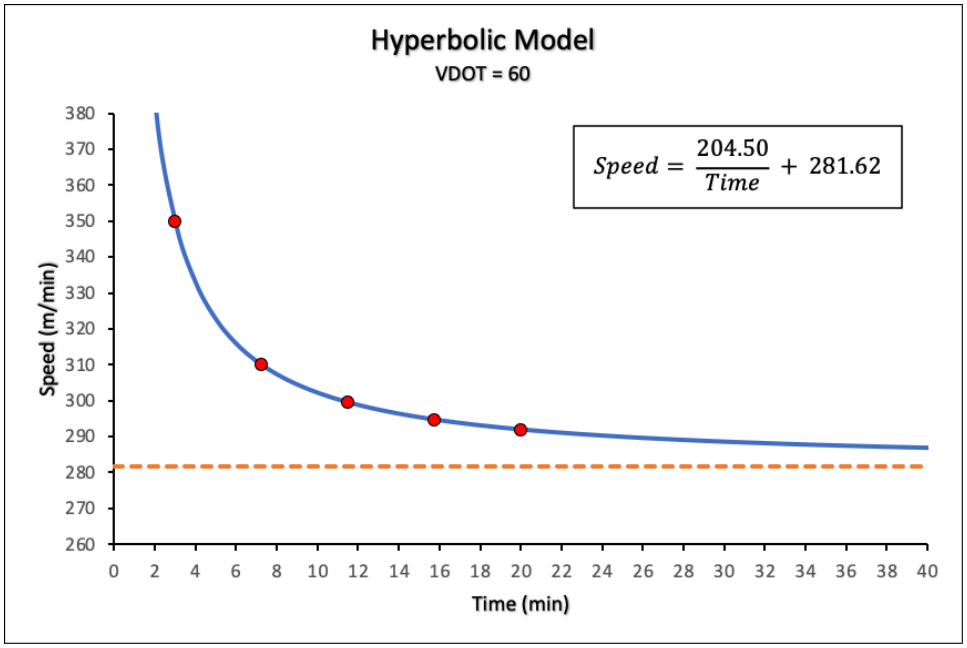

Figure 2

Applications of Critical Speed Using VDOT

Two features which are not calculable using the classic critical speed model are a specific duration critical speed can be maintained and, given the large range in abilities, a specific percentage of 5km pace critical speed can be conveyed as. However, this can be accomplished using the VDOT system.

Below are two graphs. The first describes the percent of VDOT at which critical speed occurs for different VDOT levels. The second describes the duration, based on the percent of VDOT, critical speed can be sustained for different VDOT levels. Notice that while the relationships are both non-linear, the range in which they occur is rather narrow.

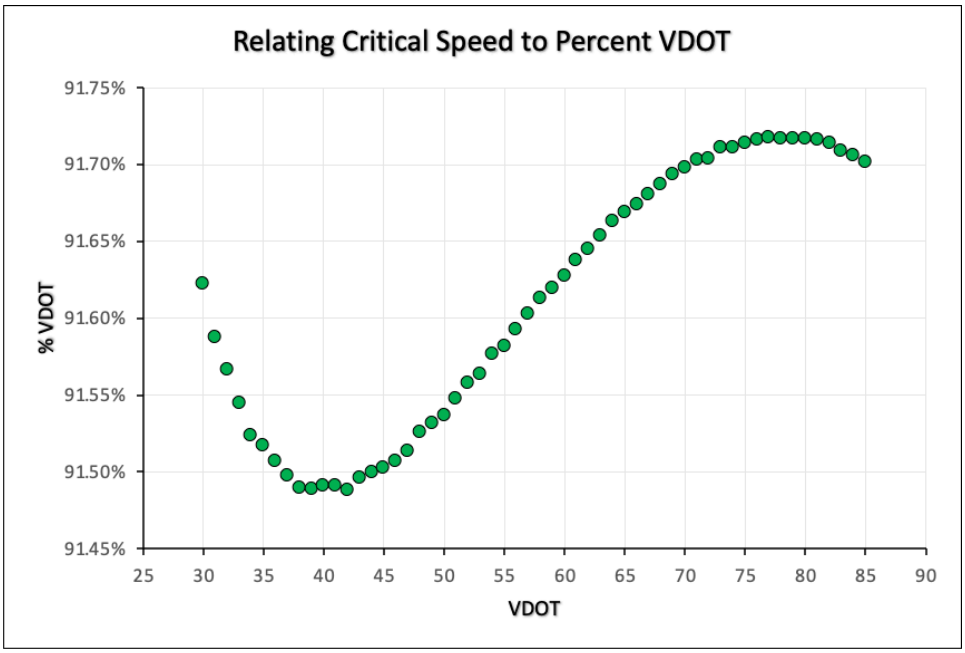

Figure 3

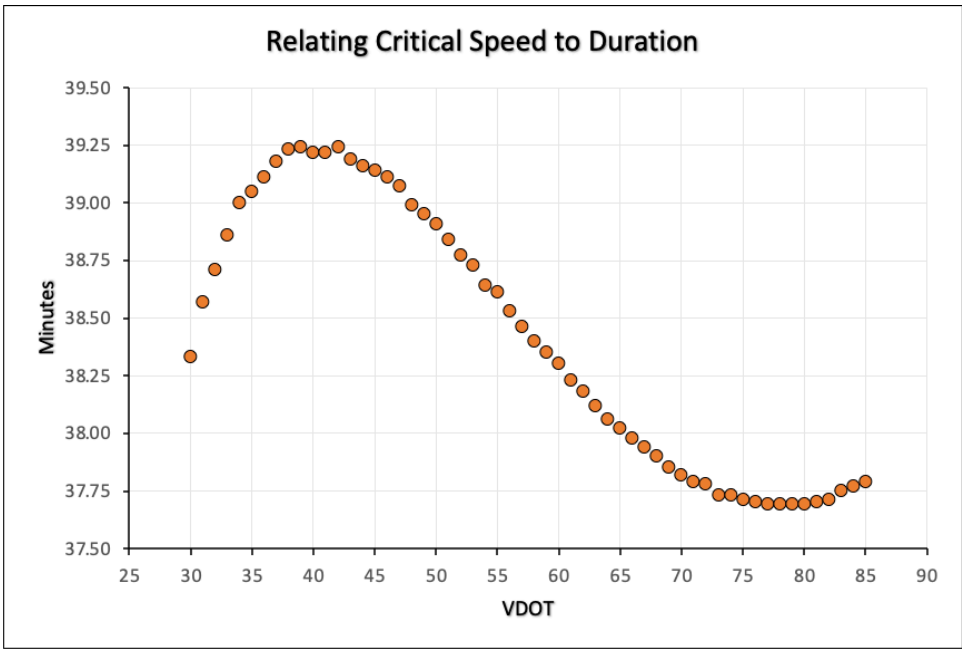

Figure 4

Lastly, the following graph displays the relationship between 5km speed and critical speed, in percentage terms. Expectedly, as the VDOT value increases, the percent of 5km race speed that critical speed occurs at declines. This relationship is particularly helpful in using 5km pace alone to establish an accurate critical speed, based on the optimal test protocol, without having to actually carry out the testing.

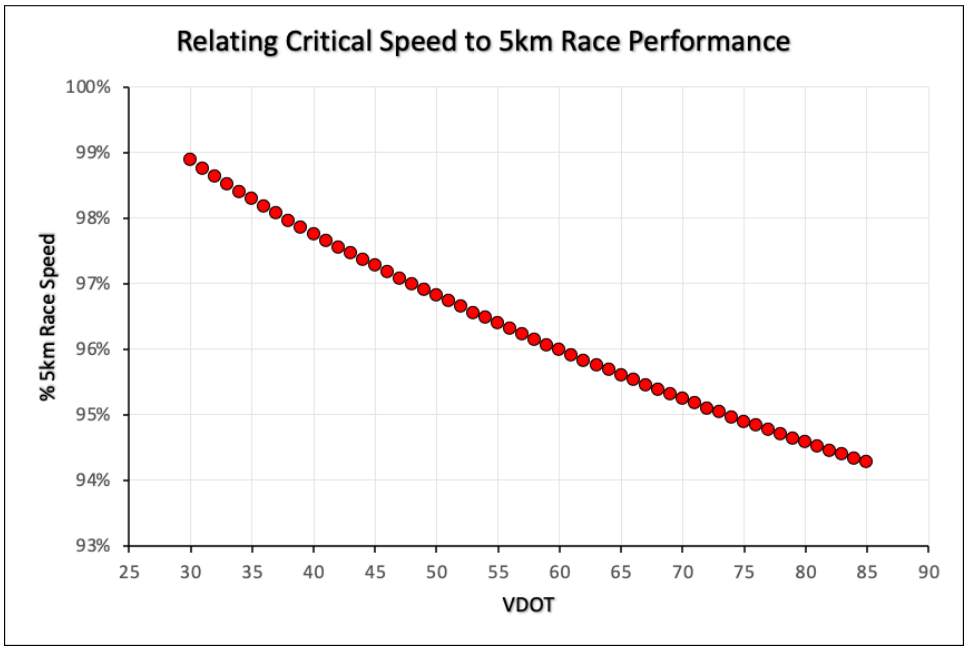

Figure 5

That concludes the analytical part, but wait! That's not all! Below are two tables and an appendix.

- **Table 1** - Displays 5km time, 5km pace, CS pace, D', and CS values associated with each VDOT value.
- **Table 2** - Provides paces for CS, 60 min, 90 min, and 150 min race pace for those interested in using a broader range of threshold paces, based on VDOT.
- **Appendix 1** - Provides the three VDOT equations attributable to Jack Daniels and Jimmy Gilbert
- **Appendix 2** - Provides a more detailed look at the mathematics behind the analytical process. This will likely be of interest to few, but I included for any who enjoy that level of detailed explanation. Also, this could be an interesting undergraduate research project for those in applied mathematics and/or the sciences, as there are additional concepts to explore both in the modeling but also in application, I think.

VDOT	5K Time	5K Pace	CS Pace	D'	Critical Speed
30	30:40	6:08.1	6:12.2	113.86	161.22
31	29:51	5:58.3	6:02.7	117.32	165.41
32	29:05	5:49.0	5:53.8	120.65	169.60
33	28:21	5:40.2	5:45.3	124.00	173.76
34	27:39	5:31.9	5:37.2	127.37	177.91
35	26:59	5:23.9	5:29.5	130.59	182.07
36	26:22	5:16.4	5:22.2	133.83	186.20
37	25:46	5:09.2	5:15.3	137.14	190.32
38	25:11	5:02.3	5:08.6	140.38	194.42
39	24:39	4:55.8	5:02.2	143.52	198.52
40	24:08	4:49.5	4:56.1	146.64	202.61
41	23:38	4:43.5	4:50.3	149.79	206.68
42	23:09	4:37.8	4:44.7	152.94	210.73
43	22:41	4:32.3	4:39.4	155.95	214.78
44	22:15	4:27.0	4:34.2	159.03	218.81
45	21:50	4:21.9	4:29.3	162.10	222.83
46	21:25	4:17.0	4:24.5	165.15	226.83
47	21:02	4:12.4	4:19.9	168.10	230.83
48	20:39	4:07.8	4:15.5	170.98	234.82
49	20:17	4:03.5	4:11.3	173.94	238.79
50	19:56	3:59.3	4:07.2	176.92	242.74
51	19:37	3:55.3	4:03.2	179.74	246.69
52	19:17	3:51.4	3:59.4	182.58	250.62
53	18:58	3:47.6	3:55.7	185.45	254.54
54	18:40	3:44.0	3:52.2	188.17	258.45
55	18:22	3:40.5	3:48.7	191.01	262.34
56	18:05	3:37.1	3:45.4	193.75	266.22
57	17:49	3:33.8	3:42.1	196.47	270.09
58	17:33	3:30.6	3:39.0	199.14	273.95
59	17:17	3:27.5	3:36.0	201.83	277.80
60	17:02	3:24.5	3:33.1	204.50	281.62
61	16:48	3:21.6	3:30.2	207.09	285.45
62	16:34	3:18.8	3:27.4	209.71	289.25
63	16:20	3:16.1	3:24.7	212.22	293.05
64	16:07	3:13.4	3:22.1	214.76	296.84
65	15:54	3:10.8	3:19.6	217.31	300.60
66	15:41	3:08.3	3:17.1	219.81	304.36
67	15:29	3:05.9	3:14.7	222.25	308.10
68	15:17	3:03.5	3:12.4	224.67	311.84
69	15:06	3:01.2	3:10.1	227.10	315.56
70	14:55	2:59.0	3:07.9	229.51	319.26
71	14:44	2:56.8	3:05.8	231.87	322.96
72	14:34	2:54.7	3:03.7	234.24	326.63
73	14:23	2:52.6	3:01.6	236.46	330.31
74	14:13	2:50.6	2:59.7	238.80	333.96
75	14:03	2:48.7	2:57.7	241.01	337.61
76	13:54	2:46.7	2:55.8	243.26	341.24
77	13:44	2:44.9	2:54.0	245.44	344.86
78	13:35	2:43.1	2:52.2	247.66	348.47
79	13:26	2:41.3	2:50.4	249.77	352.06
80	13:18	2:39.6	2:48.7	251.91	355.65
81	13:10	2:37.9	2:47.0	254.00	359.22
82	13:01	2:36.2	2:45.4	256.07	362.77
83	12:53	2:34.6	2:43.8	258.18	366.31
84	12:45	2:33.0	2:42.2	260.20	369.84
85	12:37	2:31.5	2:40.7	262.17	373.37

Table 1

VDOT	5K Time	5K Pace	CS Pace	60 min Pace	90 min Pace	150 min Pace
30	30:40	6:08.1	6:12.2	6:21.2	6:30.8	6:42.5
31	29:51	5:58.3	6:02.7	6:11.6	6:21.0	6:32.4
32	29:05	5:49.0	5:53.8	6:02.4	6:11.6	6:22.8
33	28:21	5:40.2	5:45.3	5:53.8	6:02.8	6:13.8
34	27:39	5:31.9	5:37.2	5:45.5	5:54.4	6:05.1
35	26:59	5:23.9	5:29.5	5:37.7	5:46.4	5:56.9
36	26:22	5:16.4	5:22.2	5:30.2	5:38.7	5:49.1
37	25:46	5:09.2	5:15.3	5:23.1	5:31.5	5:41.6
38	25:11	5:02.3	5:08.6	5:16.3	5:24.5	5:34.5
39	24:39	4:55.8	5:02.2	5:09.8	5:17.8	5:27.6
40	24:08	4:49.5	4:56.1	5:03.6	5:11.5	5:21.1
41	23:38	4:43.5	4:50.3	4:57.6	5:05.4	5:14.8
42	23:09	4:37.8	4:44.7	4:51.9	4:59.5	5:08.8
43	22:41	4:32.3	4:39.4	4:46.4	4:53.9	5:03.0
44	22:15	4:27.0	4:34.2	4:41.2	4:48.5	4:57.5
45	21:50	4:21.9	4:29.3	4:36.1	4:43.3	4:52.2
46	21:25	4:17.0	4:24.5	4:31.2	4:38.3	4:47.0
47	21:02	4:12.4	4:19.9	4:26.5	4:33.6	4:42.1
48	20:39	4:07.8	4:15.5	4:22.0	4:28.9	4:37.3
49	20:17	4:03.5	4:11.3	4:17.7	4:24.5	4:32.7
50	19:56	3:59.3	4:07.2	4:13.5	4:20.2	4:28.3
51	19:37	3:55.3	4:03.2	4:09.4	4:16.0	4:24.1
52	19:17	3:51.4	3:59.4	4:05.5	4:12.0	4:19.9
53	18:58	3:47.6	3:55.7	4:01.8	4:08.1	4:15.9
54	18:40	3:44.0	3:52.2	3:58.1	4:04.4	4:12.1
55	18:22	3:40.5	3:48.7	3:54.6	4:00.8	4:08.3
56	18:05	3:37.1	3:45.4	3:51.2	3:57.3	4:04.7
57	17:49	3:33.8	3:42.1	3:47.9	3:53.9	4:01.2
58	17:33	3:30.6	3:39.0	3:44.7	3:50.6	3:57.8
59	17:17	3:27.5	3:36.0	3:41.5	3:47.4	3:54.6
60	17:02	3:24.5	3:33.1	3:38.5	3:44.3	3:51.4
61	16:48	3:21.6	3:30.2	3:35.6	3:41.3	3:48.3
62	16:34	3:18.8	3:27.4	3:32.8	3:38.4	3:45.3
63	16:20	3:16.1	3:24.7	3:30.0	3:35.6	3:42.4
64	16:07	3:13.4	3:22.1	3:27.4	3:32.8	3:39.5
65	15:54	3:10.8	3:19.6	3:24.8	3:30.2	3:36.8
66	15:41	3:08.3	3:17.1	3:22.2	3:27.6	3:34.1
67	15:29	3:05.9	3:14.7	3:19.8	3:25.0	3:31.5
68	15:17	3:03.5	3:12.4	3:17.4	3:22.6	3:29.0
69	15:06	3:01.2	3:10.1	3:15.1	3:20.2	3:26.5
70	14:55	2:59.0	3:07.9	3:12.8	3:17.9	3:24.1
71	14:44	2:56.8	3:05.8	3:10.6	3:15.6	3:21.7
72	14:34	2:54.7	3:03.7	3:08.4	3:13.4	3:19.5
73	14:23	2:52.6	3:01.6	3:06.3	3:11.2	3:17.2
74	14:13	2:50.6	2:59.7	3:04.3	3:09.1	3:15.1
75	14:03	2:48.7	2:57.7	3:02.3	3:07.1	3:13.0
76	13:54	2:46.7	2:55.8	3:00.4	3:05.1	3:10.9
77	13:44	2:44.9	2:54.0	2:58.5	3:03.2	3:08.9
78	13:35	2:43.1	2:52.2	2:56.6	3:01.3	3:06.9
79	13:26	2:41.3	2:50.4	2:54.8	2:59.4	3:05.0
80	13:18	2:39.6	2:48.7	2:53.1	2:57.6	3:03.1
81	13:10	2:37.9	2:47.0	2:51.3	2:55.8	3:01.3
82	13:01	2:36.2	2:45.4	2:49.7	2:54.1	2:59.5
83	12:53	2:34.6	2:43.8	2:48.0	2:52.4	2:57.8
84	12:45	2:33.0	2:42.2	2:46.4	2:50.7	2:56.1
85	12:37	2:31.5	2:40.7	2:44.8	2:49.1	2:54.4

Table 2

Appendix 1 - VDOT Equations

EQ1: $VDOT = 0.000104 \cdot Velocity^2 + 0.182258 \cdot Velocity - 4.60$

EQ2: $Velocity = -0.007546 \cdot VDOT^2 + 5.000663 \cdot VDOT + 29.54$

EQ3: $Percent\ of\ VDOT = 0.1894393e^{-0.012778 \cdot Time} + 0.2989558e^{-0.1932605 \cdot Time} + 0.80$

Appendix 2 - Derivation of Critical Speed Model Using VDOT

1. Test Durations

$$Time = \{3.00,\ 7.25,\ 11.50,\ 15.75,\ 20.00\}$$

2. Calculating Speed and Distance

$$Distance = Speed \cdot Time$$

$$Distance = (-0.007546 \cdot VDOT^2 + 5.000663 \cdot VDOT + 29.54) \cdot Time$$

Using EQ3, we can obtain the percent of VDOT associated with the five test durations and represent that as follows:

$$pVDOT = (Percent\ of\ VDOT) \cdot (VDOT)$$

While the VDOT values range from 30-85, the percent of VDOT given the time duration will be the same, so we just simply multiply the VDOT value by the associated percent for the time duration we are calculating and arrive at the final form.

$$Distance = (-0.007546 \cdot (pVDOT)^2 + 5.000663 \cdot (pVDOT) + 29.54) \cdot Time$$

Therefore, given the VDOT value, we can calculate the speed and distance for that VDOT value using the 5 test times established.

3. Fit the Linear Model Using Time and Distance

Each VDOT value will have its associated D prime and critical speed which can be computed as follows using the least squares method.

$$SSE = \sum_{i=1}^{5} (d_i - \widehat{d})^2 \text{ , where } \widehat{d} = D' + CS \cdot T_i$$

$$SSE = \sum_{i=1}^{5} [d_i - (D' + CS \cdot T_i)]^2$$

Minimizing SSE with respect to D prime and critical speed results in the following two equations for the parameter estimates.

$$CS = [\sum_{i=1}^{5} T_i \cdot d_i - 5 \cdot (\overline{T} \cdot \overline{d})] \div [\sum_{i=1}^{5} T_i^2 - 5 \cdot (\overline{T})^2]$$

$$D' = \overline{d} - CS \cdot \overline{T}$$

This calculation for the two parameters would then be applied to each VDOT value, preferably through a coded algorithm, in a language such as Python, or in Excel.

4. Calculating the Percent of VDOT at which Critical Speed Occurs

$$Percent_{VDOT} = (0.000104 \cdot (CS_{VDOT})^2 + 0.182258 \cdot (CS_{VDOT}) - 4.60) \div VDOT$$

5. Calculating Critical Speed Duration

Using an iterative technique, in this case Newton-Raphson Method, we can determine for what duration of time critical speed can be maintained. First, redefine EQ3 as Follows:

$$f(t) = Percent\ of\ VDOT$$

Using the Newton-Raphson Method, we can determine the duration at critical speed as follows:

$$t_{n+1} = t_n - \frac{f(t_n) - Percent_{VDOT}}{f'(t_n)} \text{ , where } f'(t) = \frac{d}{dt} f(t)$$

The iteration process will run until the following condition is met:

$$\left| f(t_j) - Percent_{VDOT} \right| \leq Error$$

6. Calculating Percent of 5km Race Speed at which Critical Speed Occurs

Define a new function to calculate distance run, based on the VDOT value and time.

$$d(t) = [-0.007546 \cdot p(t)^2 + 5.000663 \cdot p(t) + 29.54] \cdot t$$

$$where\ p(t) = VDOT \cdot f(t)$$

Again, we will use the Newton-Raphson Method to solve for the time, remembering the distance is 5000m.

$$t_{n+1} = t_n - \frac{d(t_n) - 5000}{d'(t_n)}$$

Given the compound function, the derivative this time is more nuanced, but manageable. So, for clarity and ease of those interested and following along, the derivative is:

$$d'(t) = [2 \cdot (-0.007546) \cdot p(t) \cdot p'(t) + 5.000663 \cdot p'(t)] \cdot t + \frac{1}{t} \cdot d(t)$$

$$where\ p'(t) = VDOT \cdot f'(t)$$

This time, the iteration process will run until the following condition is met:

$$\left| d(t_j) - 5000 \right| \leq Error$$

In addition to obtaining the time for the distance, we will want to know the speed. That is achieved by simply dividing the distance by the time.

$$Speed_{VDOT} = \frac{5000}{Time_{VDOT}}$$

Lastly, calculating the percent of 5km race speed that critical speed occurs at:

$$Percent\ of\ Speed_{VDOT} = \frac{Critical\ Speed_{VDOT}}{Speed_{VDOT}}$$