Linear Algebra MAT313 Spring 2024
Professor Sormani
Lesson 15
Proofs with Matrices

If it is after Sunday, March 24,

skip this lesson and

go straight to Lesson 16

Before you start, find your team's project part 2 document and submit one last step for the project. You will start the group project part 3 after this lesson.

Students who have difficulty fitting formulas on a page may try the methods at this link.

If you work with any classmates on this lesson, be sure to write their names on the problems you completed together.

You will cut and paste the photos of your notes and completed classwork in a googledoc entitled:

MAT313S24-lesson15-lastname-firstname

and share editing of that document with me sormanic@gmail.com. You will also include your homework and any corrections to your homework in this doc.

If you have a question, type QUESTION in your googledoc next to the point in your notes that has a question and email me with the subject MAT313 QUESTION. I will answer your question by inserting a photo into your googledoc or making an extra video.

Today we have two parts:

Part I teaches basic proofs
and has five required HW problems

Part II (extra credit required for math majors)
uses sum notation for proofs and has extra credit problems

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Part I: **Watch Playlist 313F20-15-PartI**
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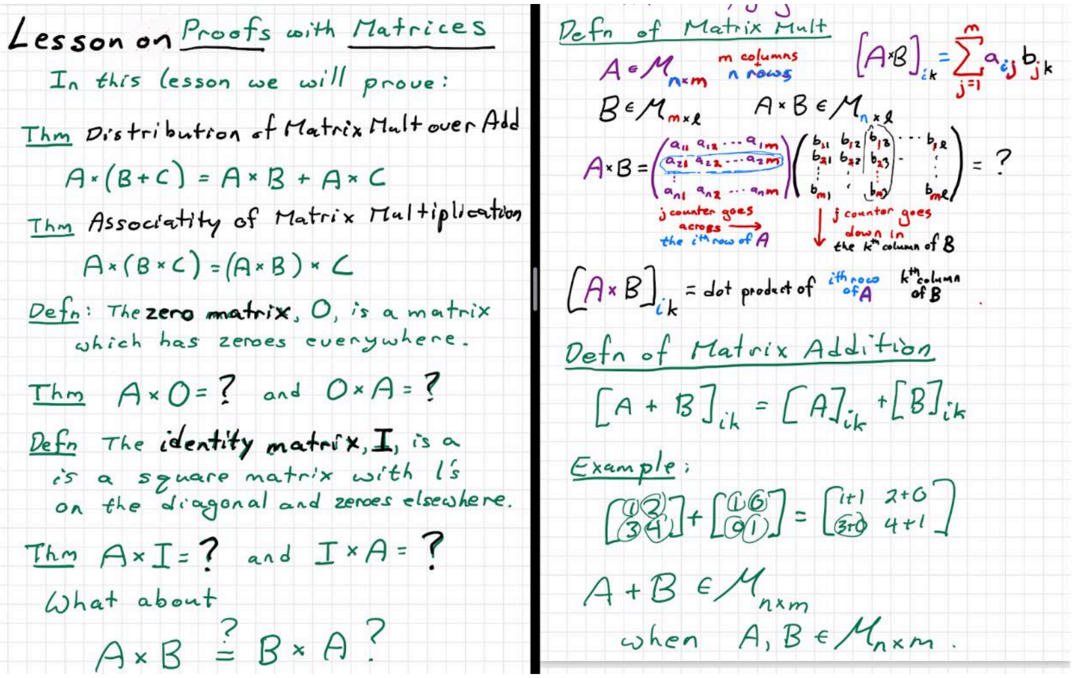


Lesson on Proofs with Matrices

In this lesson we will prove:

Thm Distribution of Matrix Mult over Add

$$A \times (B + C) = A \times B + A \times C$$

Thm Associatity of Matrix Multiplication

$$A \times (B \times C) = (A \times B) \times C$$

Defn: The **zero matrix**, O, is a matrix which has zeroes everywhere.

Thm $A \times O = ?$ and $O \times A = ?$

Defn The **identity matrix**, **I**, is a is a square matrix with 1's on the diagonal and zeroes elsewhere.

Thm $A \times I = ?$ and $I \times A = ?$

What about

$$A \times B \overset{?}{=} B \times A \, ?$$

Defn of Matrix Mult

$A \in \mathcal{M}_{n \times m}$ m columns n rows

$$[A \times B]_{ik} = \sum_{j=1}^{m} a_{ij} b_{jk}$$

$B \in \mathcal{M}_{m \times \ell}$ $\quad A \times B \in \mathcal{M}_{n \times \ell}$

$$A \times B = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1\ell} \\ b_{21} & b_{22} & b_{23} & \cdots & \vdots \\ \vdots & \vdots & \vdots & & \vdots \\ b_{m1} & & b_{m3} & & b_{m\ell} \end{pmatrix} = ?$$

j counter goes across → the i^{th} row of A

↓ j counter goes down in the k^{th} column of B

$[A \times B]_{ik}$ = dot product of i^{th} row of A k^{th} column of B

Defn of Matrix Addition

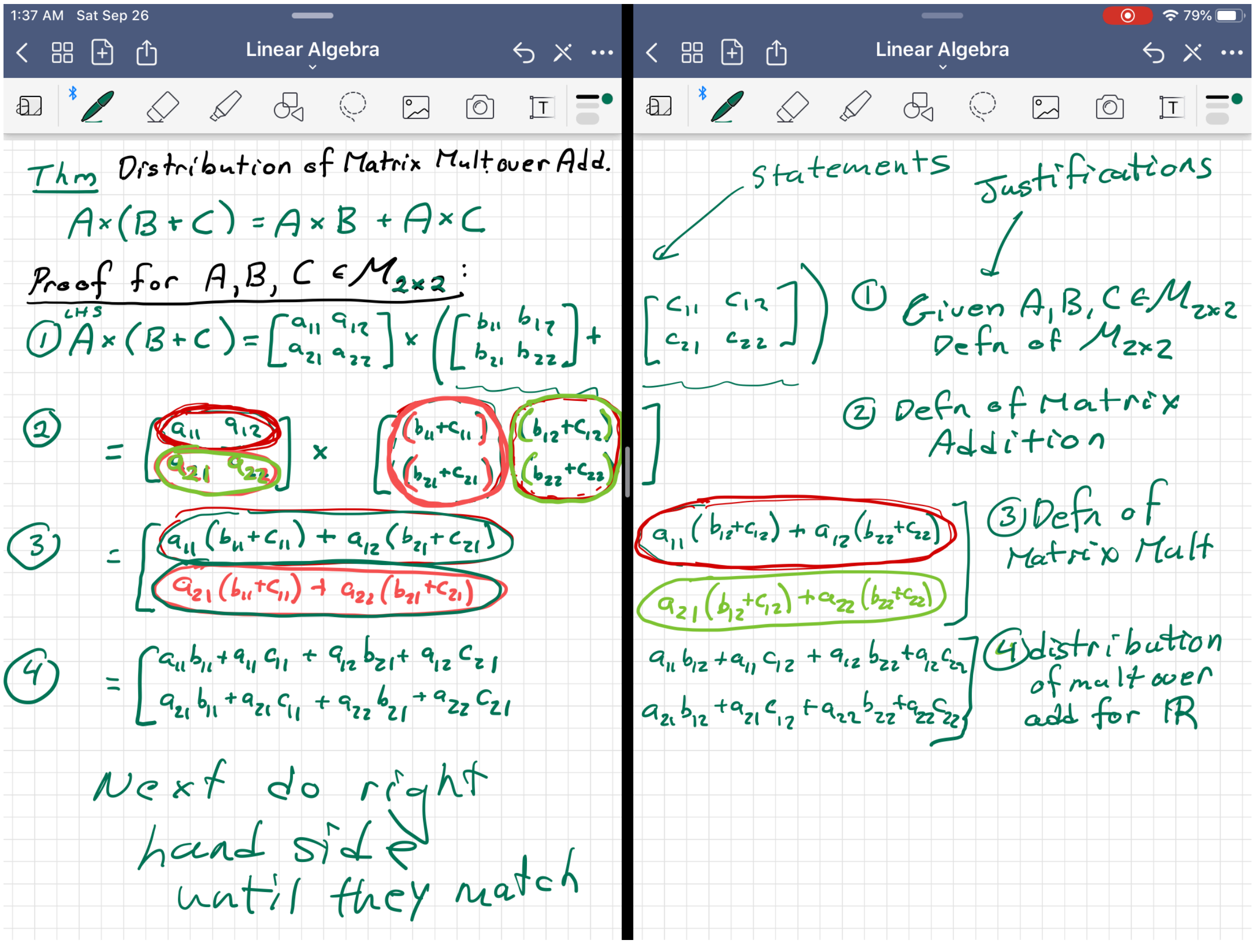
$$[A + B]_{ik} = [A]_{ik} + [B]_{ik}$$

Example:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1+1 & 2+0 \\ 3+0 & 4+1 \end{bmatrix}$$

$A + B \in \mathcal{M}_{n \times m}$

when $A, B \in \mathcal{M}_{n \times m}$.



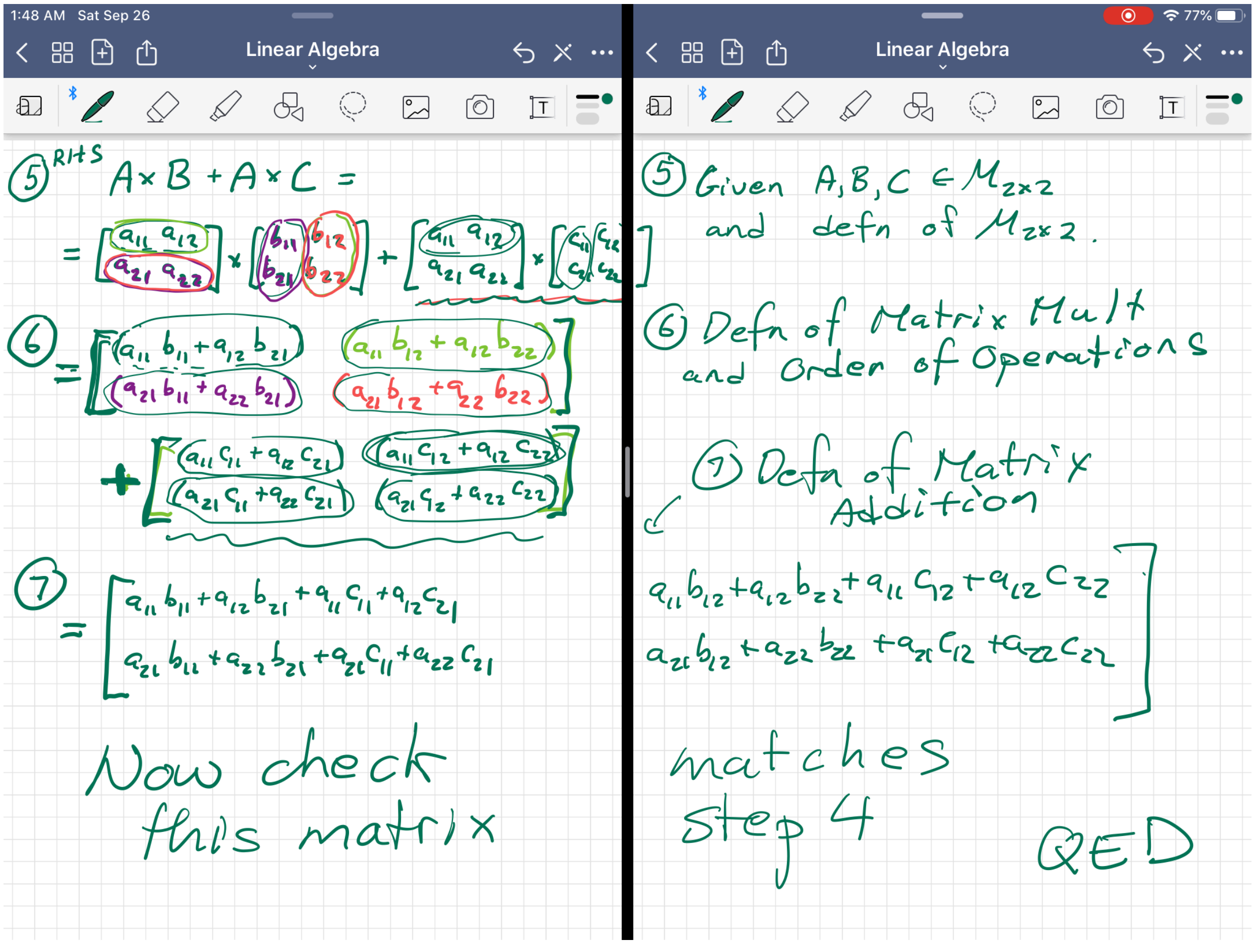
Thm Distribution of Matrix Mult over Add.

$$A\times(B+C)=A\times B+A\times C$$

Proof for $A,B,C\in M_{2\times2}$:

Statements	Justifications
① $\overset{LHS}{A}\times(B+C)=\begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{bmatrix}\times\left(\underbrace{\begin{bmatrix}b_{11}&b_{12}\\b_{21}&b_{22}\end{bmatrix}+\begin{bmatrix}c_{11}&c_{12}\\c_{21}&c_{22}\end{bmatrix}}\right)$	① Given $A,B,C\in M_{2\times2}$ Defn of $M_{2\times2}$
② $=\begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{bmatrix}\times\begin{bmatrix}(b_{11}+c_{11})&(b_{12}+c_{12})\\(b_{21}+c_{21})&(b_{22}+c_{22})\end{bmatrix}$	② Defn of Matrix Addition
③ $=\begin{bmatrix}a_{11}(b_{11}+c_{11})+a_{12}(b_{21}+c_{21})&a_{11}(b_{12}+c_{12})+a_{12}(b_{22}+c_{22})\\a_{21}(b_{11}+c_{11})+a_{22}(b_{21}+c_{21})&a_{21}(b_{12}+c_{12})+a_{22}(b_{22}+c_{22})\end{bmatrix}$	③ Defn of Matrix Mult
④ $=\begin{bmatrix}a_{11}b_{11}+a_{11}c_{11}+a_{12}b_{21}+a_{12}c_{21}&a_{11}b_{12}+a_{11}c_{12}+a_{12}b_{22}+a_{12}c_{22}\\a_{21}b_{11}+a_{21}c_{11}+a_{22}b_{21}+a_{22}c_{21}&a_{21}b_{12}+a_{21}c_{12}+a_{22}b_{22}+a_{22}c_{22}\end{bmatrix}$	④ distribution of mult over add for $\mathbb{R}$

Next do right hand side until they match



⑤ RHS $A \times B + A \times C =$

$$= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \times \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \times \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

⑥
$$= \begin{bmatrix} (a_{11} b_{11} + a_{12} b_{21}) & (a_{11} b_{12} + a_{12} b_{22}) \\ (a_{21} b_{11} + a_{22} b_{21}) & (a_{21} b_{12} + a_{22} b_{22}) \end{bmatrix}$$
$$+ \begin{bmatrix} (a_{11} c_{11} + a_{12} c_{21}) & (a_{11} c_{12} + a_{12} c_{22}) \\ (a_{21} c_{11} + a_{22} c_{21}) & (a_{21} c_{12} + a_{22} c_{22}) \end{bmatrix}$$

⑦
$$= \begin{bmatrix} a_{11} b_{11} + a_{12} b_{21} + a_{11} c_{11} + a_{12} c_{21} & a_{11} b_{12} + a_{12} b_{22} + a_{11} c_{12} + a_{12} c_{22} \\ a_{21} b_{11} + a_{22} b_{21} + a_{21} c_{11} + a_{22} c_{21} & a_{21} b_{12} + a_{22} b_{22} + a_{21} c_{12} + a_{22} c_{22} \end{bmatrix}$$

Now check this matrix

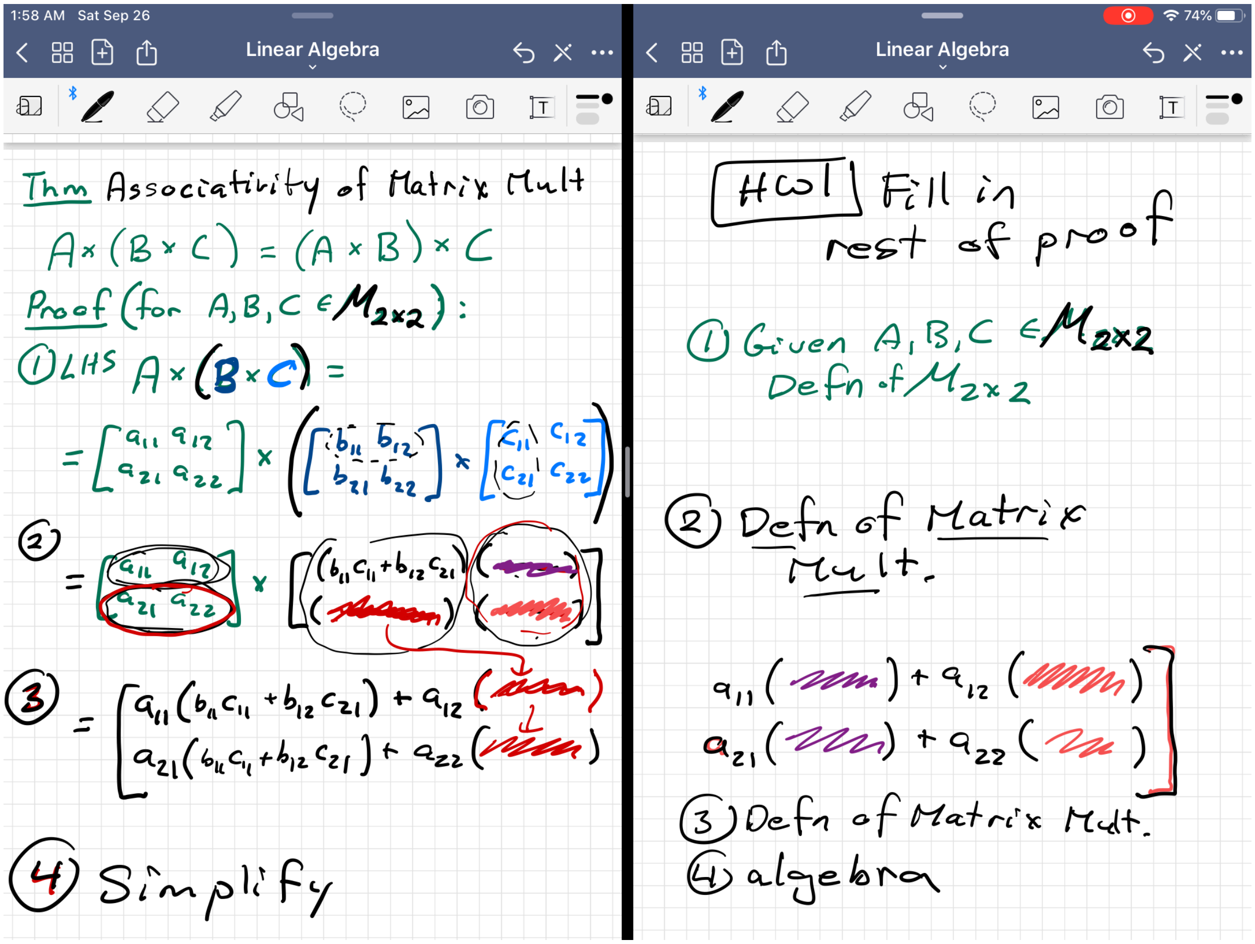
⑤ Given $A, B, C \in M_{2 \times 2}$ and defn of $M_{2 \times 2}$.

⑥ Defn of Matrix Mult and Order of Operations

✓ ⑦ Defn of Matrix Addition

matches step 4

QED



Thm Associativity of Matrix Mult

$A\times(B\times C) = (A\times B)\times C$

Proof (for $A,B,C \in M_{2\times2}$):

① LHS $A\times(B\times C) =$

$$= \begin{bmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix} \times \left(\begin{bmatrix} b_{11} & b_{12}\\ b_{21} & b_{22}\end{bmatrix}\times\begin{bmatrix}c_{11} & c_{12}\\ c_{21} & c_{22}\end{bmatrix}\right)$$

② $$= \begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\times\begin{bmatrix}(b_{11}c_{11}+b_{12}c_{21}) & (\blacksquare)\\ (\blacksquare) & (\blacksquare)\end{bmatrix}$$

③ $$= \begin{bmatrix} a_{11}(b_{11}c_{11}+b_{12}c_{21}) + a_{12}(\blacksquare) \\ a_{21}(b_{11}c_{11}+b_{12}c_{21}) + a_{22}(\blacksquare)\end{bmatrix}$$

④ Simplify

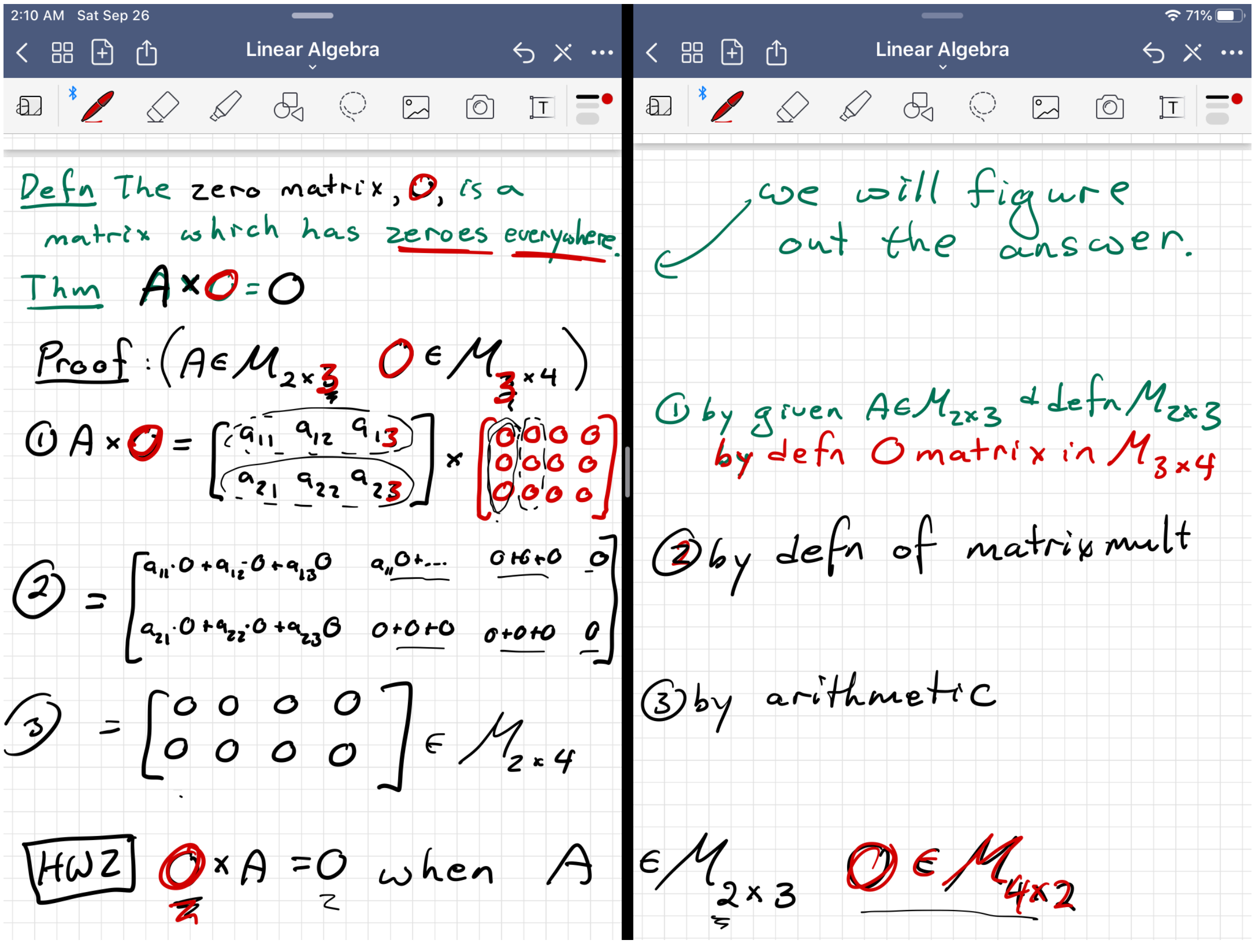
$\boxed{\text{HW1}}$ Fill in rest of proof

① Given $A,B,C \in M_{2\times2}$
Defn of $M_{2\times2}$

② Defn of Matrix Mult.

$$\begin{bmatrix} a_{11}(\blacksquare) + a_{12}(\blacksquare) \\ a_{21}(\blacksquare) + a_{22}(\blacksquare)\end{bmatrix}$$

③ Defn of Matrix Mult.
④ algebra



Defn The zero matrix, O, is a matrix which has zeroes everywhere.

Thm $A \times O = O$

Proof: $(A \in M_{2\times 3} \quad O \in M_{3\times 4})$

① $A \times O = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \times \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

② $= \begin{bmatrix} a_{11}\cdot 0 + a_{12}\cdot 0 + a_{13}0 & a_{11}0 + \dots & 0+0+0 & 0 \\ a_{21}\cdot 0 + a_{22}\cdot 0 + a_{23}0 & 0+0+0 & 0+0+0 & 0 \end{bmatrix}$

③ $= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \in M_{2\times 4}$

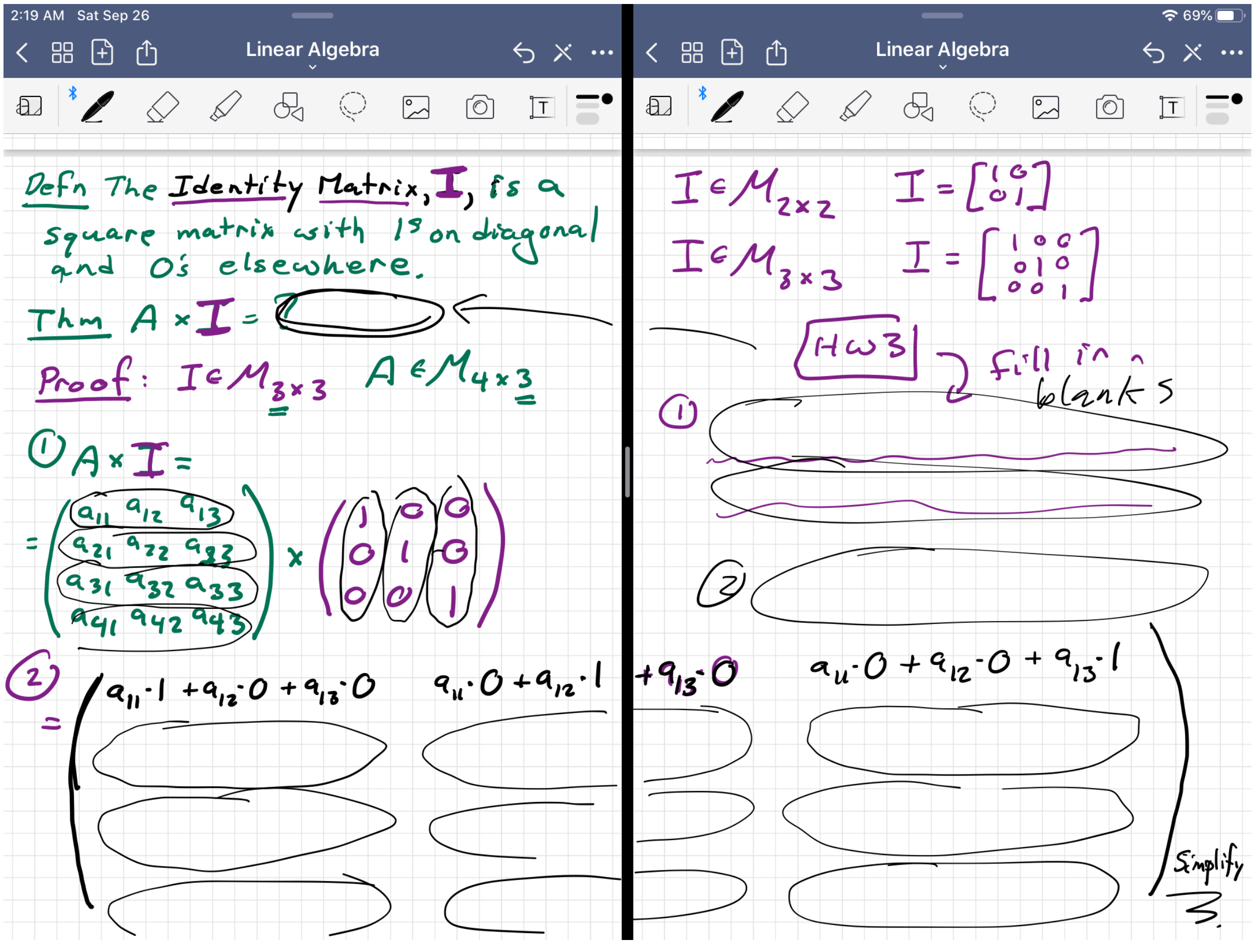
$\boxed{HW2}$ $\underset{\sim}{O} \times A = \underset{\sim}{0}$ when $A \in M_{2\times 3}$ $\quad O \in M_{4\times 2}$

we will figure out the answer.

① by given $A \in M_{2\times 3}$ & defn $M_{2\times 3}$
by defn O matrix in $M_{3\times 4}$

② by defn of matrix mult

③ by arithmetic



Defn The Identity Matrix, I, is a square matrix with 1s on diagonal and 0's elsewhere.

Thm $A \times I = ?$

Proof: $I \in M_{3\times 3}$ $A \in M_{4\times 3}$

① $A \times I =$

$$= \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{pmatrix} \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

② $= \Big(a_{11}\cdot 1 + a_{12}\cdot 0 + a_{13}\cdot 0 \quad a_{11}\cdot 0 + a_{12}\cdot 1 + a_{13}\cdot 0 \quad a_{11}\cdot 0 + a_{12}\cdot 0 + a_{13}\cdot 1 \Big)$

Simplify

$I \in M_{2\times 2}$ $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$I \in M_{3\times 3}$ $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

HW3 fill in blanks

①

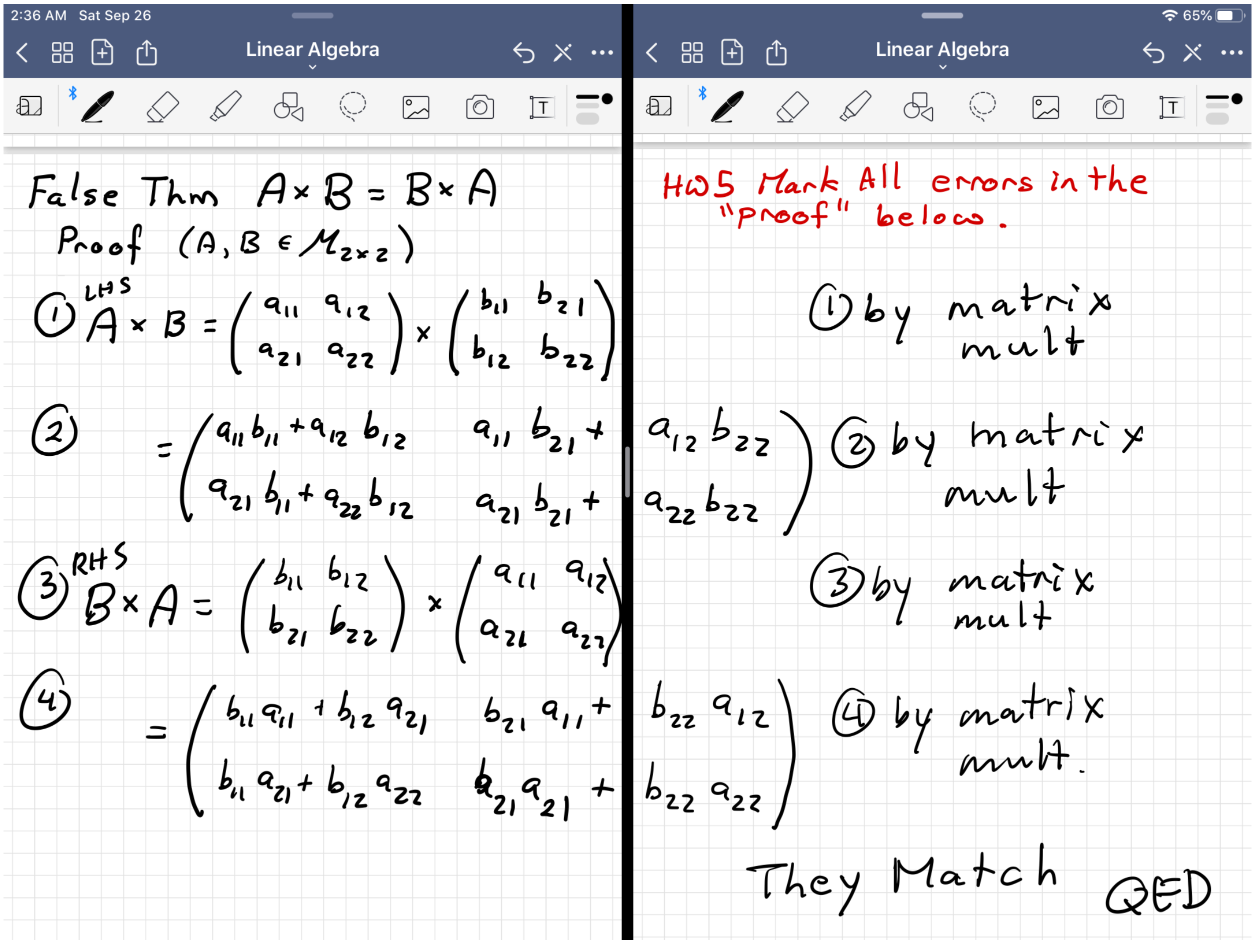
②

HW1-HW3 are described above.

HW4 Prove IxB=B when B is 2x4 and I is the correct choice of identity matrix.

HW5 is to **find the errors** in the incorrect proof below which has many errors so find all of the errors. Then either fix the proof or find a pair of specific 2x2 matrices A and B for

which this fails.



False Thm $A \times B = B \times A$

Proof $(A, B \in M_{2\times 2})$

$$① \overset{LHS}{A} \times B = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \times \begin{pmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{pmatrix}$$

$$② \quad = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{12} & a_{11}b_{21} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{12} & a_{21}b_{21} + a_{22}b_{22} \end{pmatrix}$$

$$③ \overset{RHS}{B} \times A = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \times \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$④ \quad = \begin{pmatrix} b_{11}a_{11} + b_{12}a_{21} & b_{21}a_{11} + b_{22}a_{12} \\ b_{11}a_{21} + b_{12}a_{22} & b_{21}a_{21} + b_{22}a_{22} \end{pmatrix}$$

HW5 Mark All errors in the "proof" below.

① by matrix mult

② by matrix mult

③ by matrix mult

④ by matrix mult.

They Match QED

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Part II: **Watch** **Playlist 313F20-15-PartII** **which is** **Extra Credit highly recommended for math majors and has three extra credit problems.**

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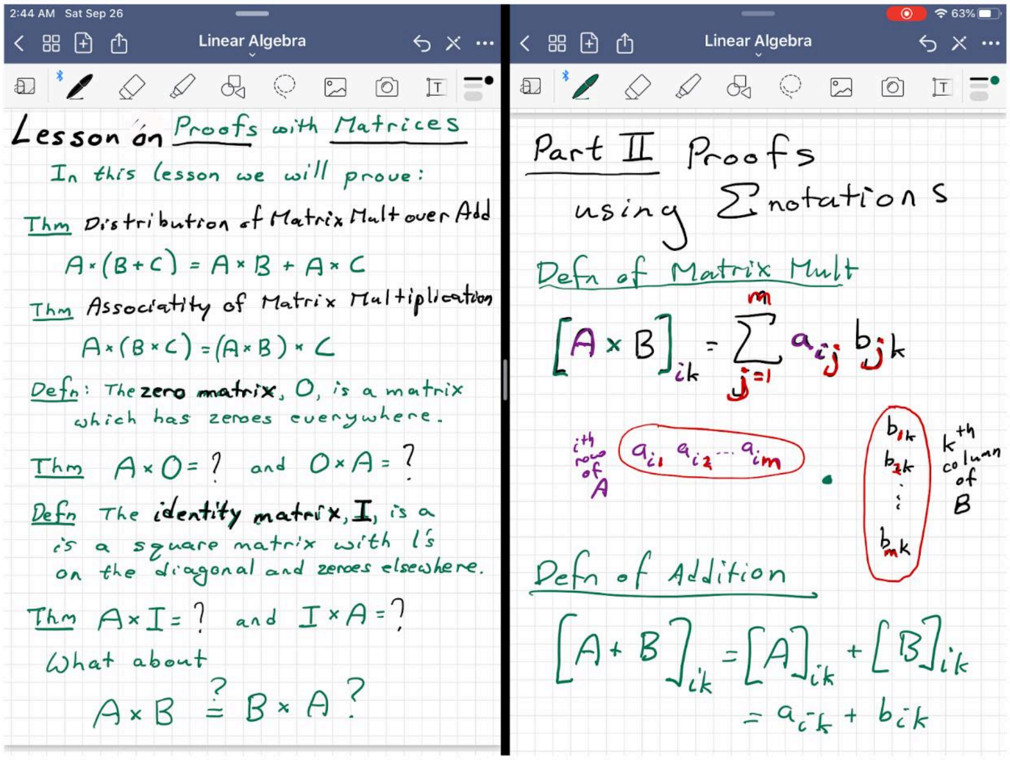

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Thm $A \times I = ?$ and $I \times A = ?$

What about

$$A \times B \overset{?}{=} B \times A \, ?$$

Part II Proofs using Σ notations

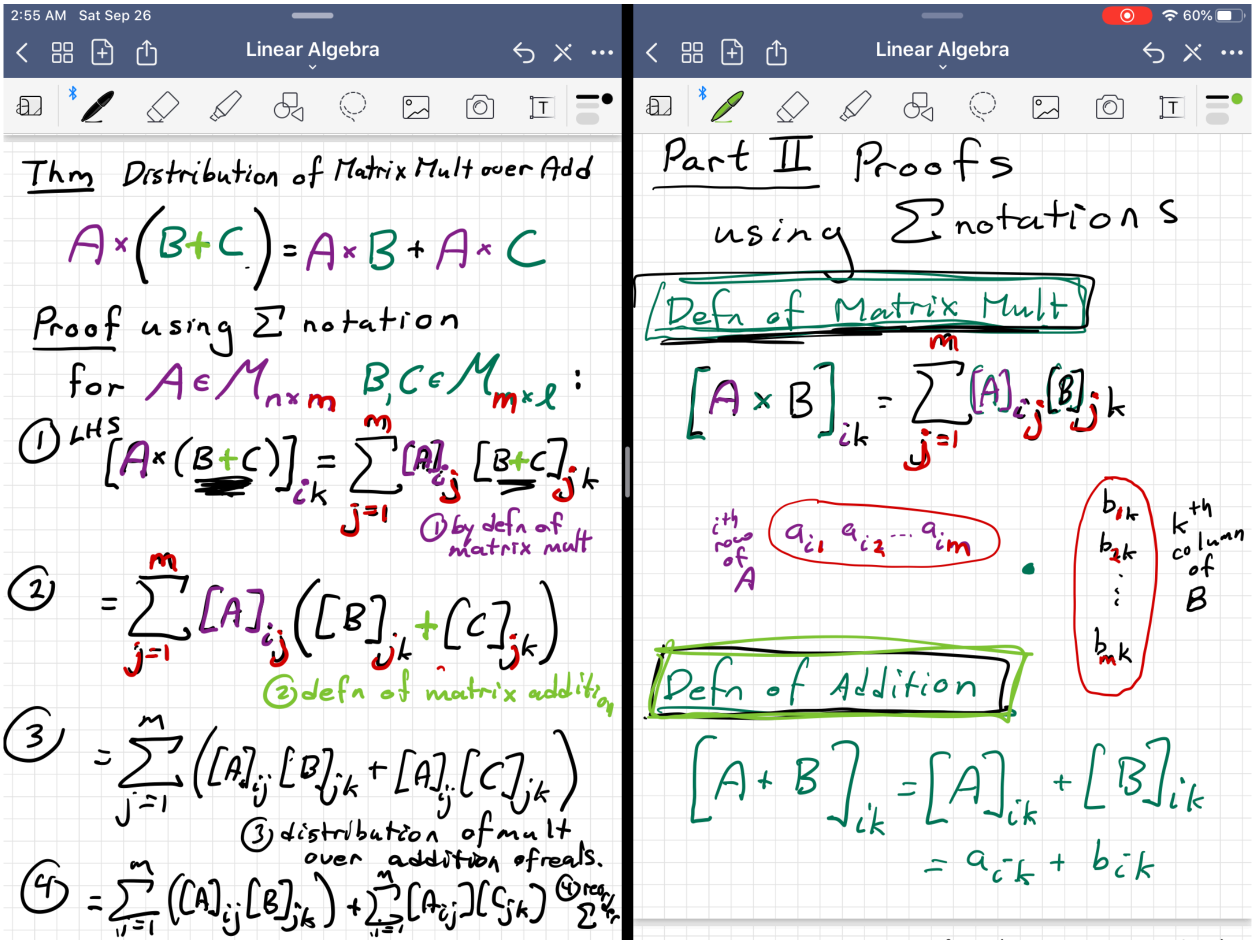
Defn of Matrix Mult

$$[A \times B]_{ik} = \sum_{j=1}^{m} a_{ij} b_{jk}$$

ith row of A $a_{i1}\ a_{i2} \cdots a_{im}$ $\cdot$ $\begin{matrix} b_{1k} \\ b_{2k} \\ \vdots \\ b_{mk} \end{matrix}$ kth column of B

Defn of Addition

$$[A + B]_{ik} = [A]_{ik} + [B]_{ik}$$
$$= a_{ik} + b_{ik}$$



Thm Distribution of Matrix Mult over Add

$$A \times (B + C) = A \times B + A \times C$$

Proof using Σ notation

for $A \in M_{n \times m}$ $B, C \in M_{m \times \ell}$:

① LHS $$[A \times (B + C)]_{ik} = \sum_{j=1}^{m} [A]_{ij} [B + C]_{jk}$$

① by defn of matrix mult

② $$= \sum_{j=1}^{m} [A]_{ij} ([B]_{jk} + [C]_{jk})$$

② defn of matrix addition

③ $$= \sum_{j=1}^{m} ([A]_{ij} [B]_{jk} + [A]_{ij} [C]_{jk})$$

③ distribution of mult over addition of reals.

④ $$= \sum_{j=1}^{m} ([A]_{ij} [B]_{jk}) + \sum_{j=1}^{m} [A_{ij}][C_{jk}]$$ ④ reorder Σ

Part II Proofs using Σ notations

Defn of Matrix Mult

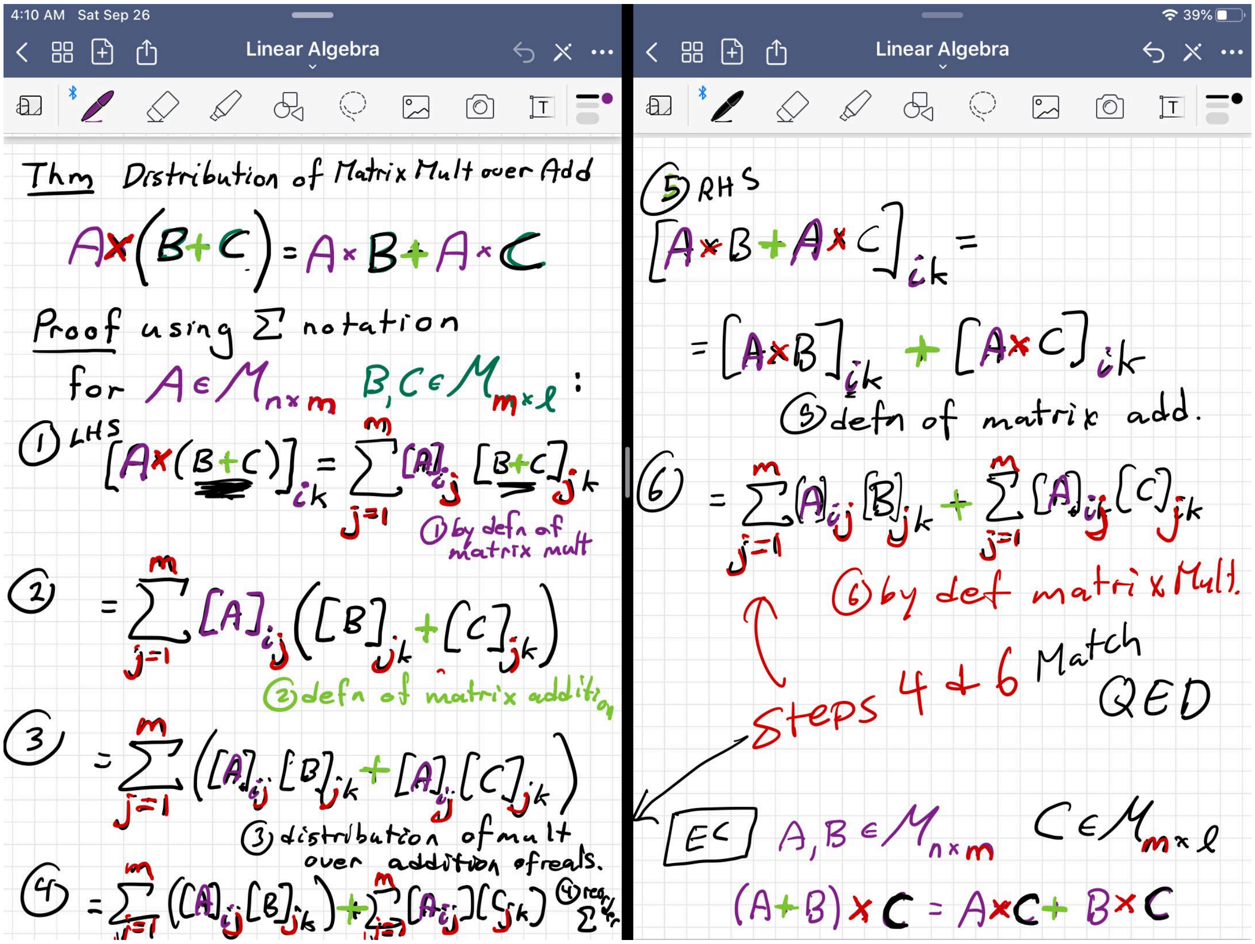
$$[A \times B]_{ik} = \sum_{j=1}^{m} [A]_{ij} [B]_{jk}$$

ith row of A $\left(a_{i1} \ a_{i2} \cdots a_{im} \right)$ $\cdot$ $\begin{pmatrix} b_{1k} \\ b_{2k} \\ \vdots \\ b_{mk} \end{pmatrix}$ kth column of B

Defn of Addition

$$[A + B]_{ik} = [A]_{ik} + [B]_{ik}$$

$$= a_{ik} + b_{ik}$$



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① LHS $[A \times (B+C)]_{ik} = \sum_{j=1}^{m} [A]_{ij} [B+C]_{jk}$

① by defn of matrix mult

② $= \sum_{j=1}^{m} [A]_{ij} \left([B]_{jk} + [C]_{jk} \right)$

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③ $= \sum_{j=1}^{m} \left([A]_{ij} [B]_{jk} + [A]_{ij} [C]_{jk} \right)$

③ distribution of mult over addition of reals.

④ $= \sum_{j=1}^{m} \left([A]_{ij} [B]_{jk} \right) + \sum_{j=1}^{m} [A_{ij}][C_{jk}]$ ④ reorder Σ

⑤ RHS

$[A \times B + A \times C]_{ik} =$

$= [A \times B]_{ik} + [A \times C]_{ik}$

⑤ defn of matrix add.

⑥ $= \sum_{j=1}^{m} [A]_{ij} [B]_{jk} + \sum_{j=1}^{m} [A]_{ij} [C]_{jk}$

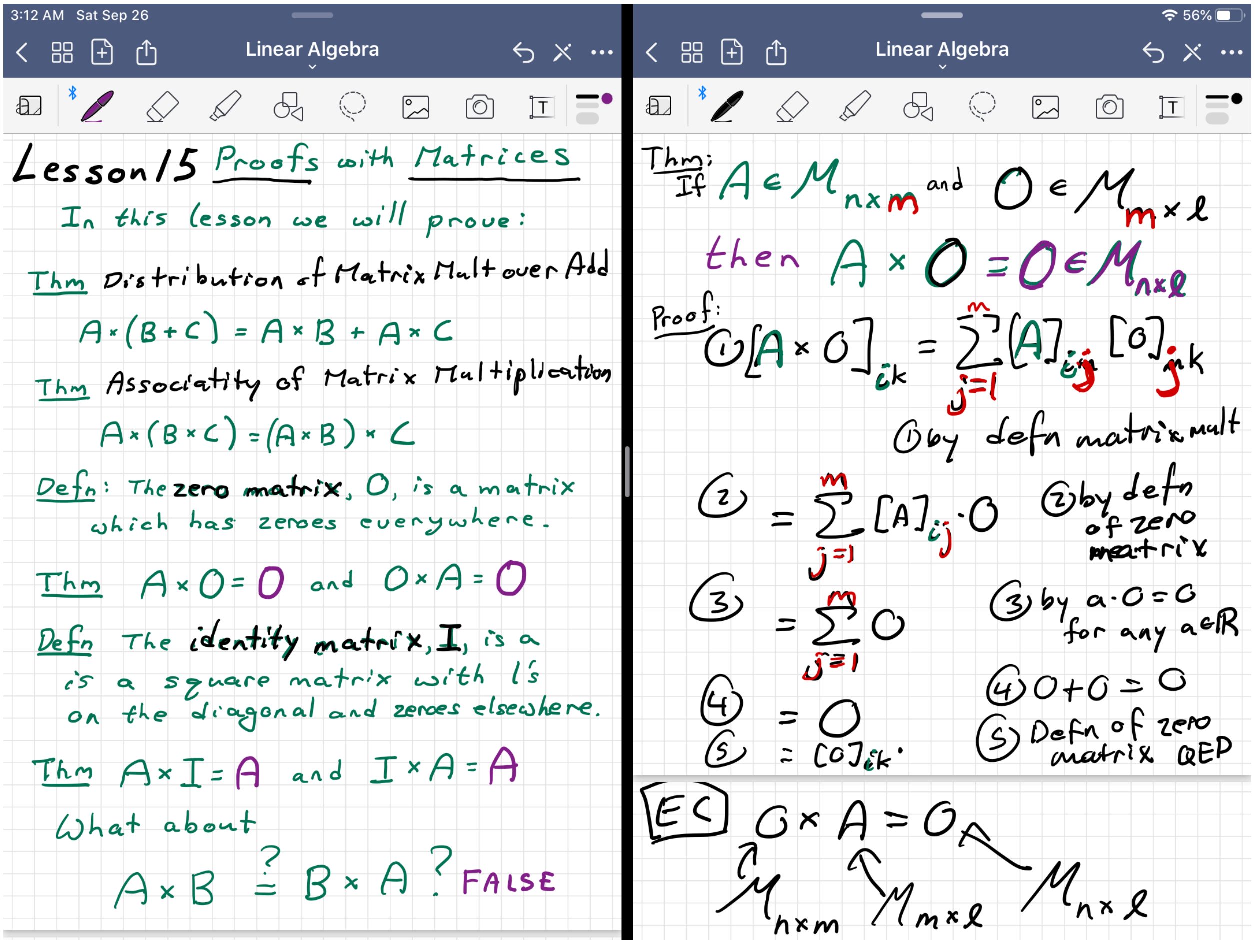
⑥ by def matrix Mult.

Steps 4 + 6 Match

QED

$\boxed{EC}$ $A, B \in M_{n \times m}$ $C \in M_{m \times \ell}$

$$(A+B) \times C = A \times C + B \times C$$



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Thm $A\times O=O$ and $O\times A=O$

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Thm $A\times I=A$ and $I\times A=A$

What about

$A\times B\overset{?}{=}B\times A$? FALSE

Thm: If $A\in M_{n\times m}$ and $O\in M_{m\times \ell}$

then $A\times O=O\in M_{n\times \ell}$

Proof:

① $[A\times O]_{ik}=\sum_{j=1}^{m}[A]_{ij}[O]_{jk}$

① by defn matrix mult

② $=\sum_{j=1}^{m}[A]_{ij}\cdot 0$ ② by defn of zero matrix

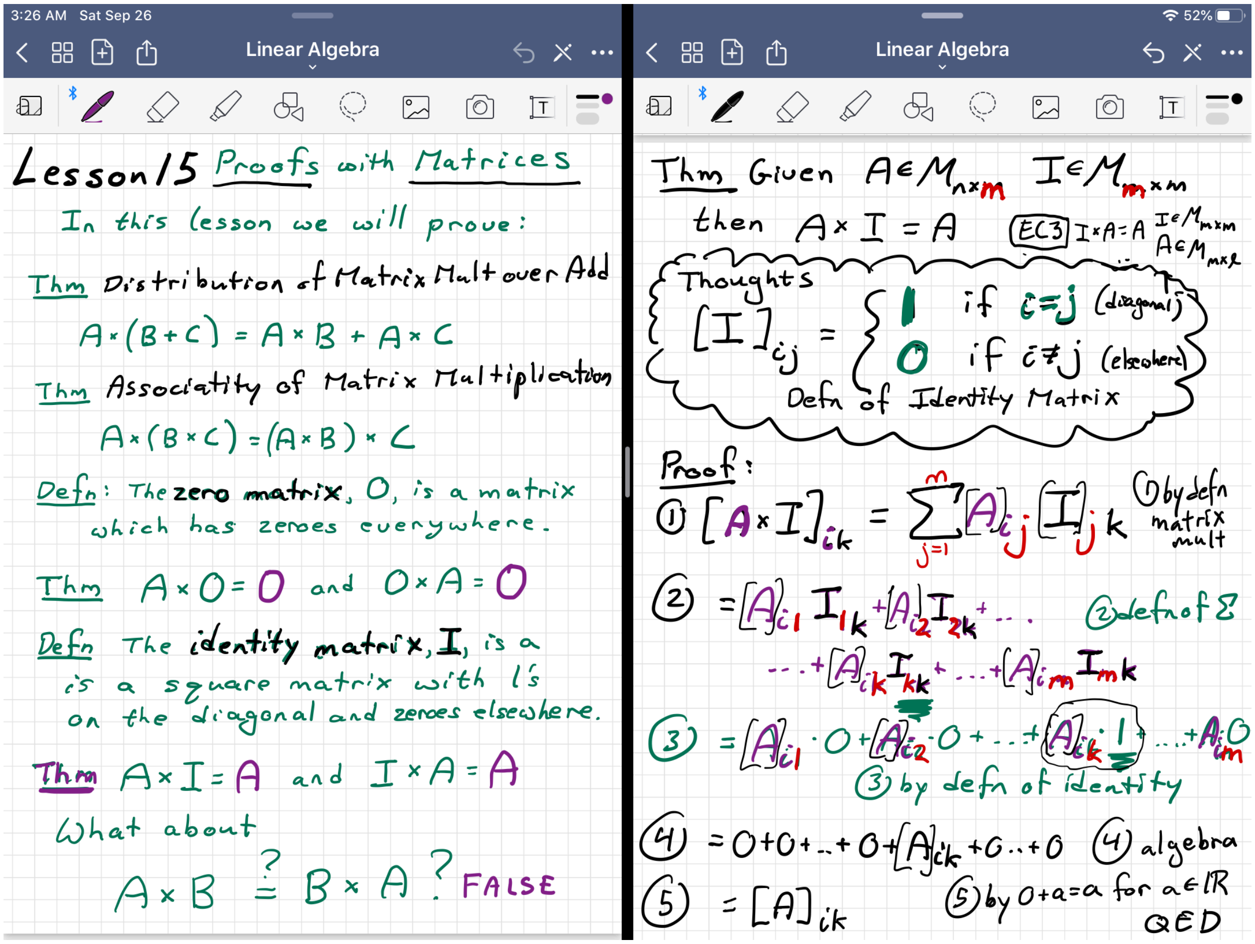
③ $=\sum_{j=1}^{m}0$ ③ by $a\cdot 0=0$ for any $a\in\mathbb{R}$

④ $=0$ ④ $0+0=0$

⑤ $=[O]_{ik}$. ⑤ Defn of zero matrix QED

$\boxed{EC}$ $O\times A=O$

$M_{n\times m}$ $M_{m\times \ell}$ $M_{n\times \ell}$



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Thm $A\times I=A$ and $I\times A=A$

What about

$A\times B \overset{?}{=} B\times A$? FALSE

Thm Given $A\in M_{n\times m}$ $I\in M_{m\times m}$ then $A\times I=A$

$\boxed{EC3}$ $I\times A=A$ $I\in M_{m\times m}$ $A\in M_{m\times \ell}$

Thoughts

$$[I]_{ij}=\begin{cases}1 & \text{if } i=j \text{ (diagonal)}\\ 0 & \text{if } i\neq j \text{ (elsewhere)}\end{cases}$$

Defn of Identity Matrix

Proof:

① $[A\times I]_{ik}=\sum_{j=1}^{m}[A]_{ij}[I]_{jk}$ ① by defn matrix mult

② $=[A]_{i1}I_{1k}+[A]_{i2}I_{2k}+\dots$ ② defn of Σ

$\dots+[A]_{ik}I_{kk}+\dots+[A]_{im}I_{mk}$

③ $=[A]_{i1}\cdot 0+[A]_{i2}\cdot 0+\dots+\boxed{[A]_{ik}\cdot 1}+\dots+A_{im}0$

③ by defn of identity

④ $=0+0+\dots+0+[A]_{ik}+0\dots+0$ ④ algebra

⑤ $=[A]_{ik}$ ⑤ by $0+a=a$ for $a\in\mathbb{R}$

QED

Extra Credit: Prove for arbitrary matrices A and B and vector v using sum notation that
A(Bv)=(AxB)v

Extra Credit: Prove for arbitrary matrices A, B, and C using sum notation that
Ax(BxC)=(AxB)xC