[Linear Algebra MAT313 Spring 2024]
[Professor Sormani]
**Lesson 15**
**Proofs with Matrices**

**If it is after Sunday, March 24,**

**skip this lesson and**

**go straight to Lesson 16**

**Before you start, find your team's project part 2 document and submit one last step for the project. You will start the group project part 3 after this lesson.**

**Students who have difficulty fitting formulas on a page may try the methods at this link.**

***If you work with any classmates on this lesson, be sure to write their names on the problems you completed together.***

***You will cut and paste the photos of your notes and completed classwork in a googledoc entitled:***

***MAT313S24-lesson15-lastname-firstname***

***and share editing of that document with me sormanic@gmail.com. You will also include your homework and any corrections to your homework in this doc.***

***If you have a question, type QUESTION in your googledoc next to the point in your notes that has a question and email me with the subject MAT313 QUESTION. I will answer your question by inserting a photo into your googledoc or making an extra video.***

**Today we have two parts:**

**Part I teaches basic proofs**
**and has five required HW problems**

**Part II (extra credit required for math majors)**
**uses sum notation for proofs and has extra credit problems**

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**Part I: Watch Playlist 313F20-15-PartI**
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# Lesson on Proofs with Matrices

In this lesson we will prove:

**Thm** Distribution of Matrix Mult over Add

$$A \times (B + C) = A \times B + A \times C$$

**Thm** Associatity of Matrix Multiplication

$$A \times (B \times C) = (A \times B) \times C$$

**Defn**: The **zero matrix**, $O$, is a matrix which has zeroes everywhere.

**Thm** $A \times O = ?$ and $O \times A = ?$

**Defn** The **identity matrix**, $I$, is a is a square matrix with 1's on the diagonal and zeroes elsewhere.

**Thm** $A \times I = ?$ and $I \times A = ?$

What about

$$A \times B \stackrel{?}{=} B \times A\ ?$$

## Defn of Matrix Mult

$A \in \mathcal{M}_{n \times m}$ m columns n rows

$$[A \times B]_{ik} = \sum_{j=1}^{m} a_{ij} b_{jk}$$

$B \in \mathcal{M}_{m \times \ell}$ $\quad A \times B \in \mathcal{M}_{n \times \ell}$

$$A \times B = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1\ell} \\ b_{21} & b_{22} & b_{23} & \cdots & \vdots \\ \vdots & \vdots & \vdots & & \vdots \\ b_{m1} & & b_{m3} & & b_{m\ell} \end{pmatrix} = ?$$

j counter goes across $\longrightarrow$ the $i^{th}$ row of $A$

$\downarrow$ j counter goes down in the $k^{th}$ column of $B$

$[A \times B]_{ik}$ = dot product of $i^{th}$ row of $A$ $k^{th}$ column of $B$

## Defn of Matrix Addition

$$[A + B]_{ik} = [A]_{ik} + [B]_{ik}$$

Example:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1+1 & 2+0 \\ 3+0 & 4+1 \end{bmatrix}$$

$$A + B \in \mathcal{M}_{n \times m}$$

when $A, B \in \mathcal{M}_{n \times m}$.

**Thm** Distribution of Matrix Mult over Add.

$$A\times(B+C)=A\times B+A\times C$$

**Proof for $A,B,C\in M_{2\times 2}$:**

Statements | Justifications

① $\underbrace{A\times(B+C)}_{\text{LHS}}=\begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{bmatrix}\times\left(\begin{bmatrix}b_{11}&b_{12}\\b_{21}&b_{22}\end{bmatrix}+\begin{bmatrix}c_{11}&c_{12}\\c_{21}&c_{22}\end{bmatrix}\right)$ — ① Given $A,B,C\in M_{2\times2}$, Defn of $M_{2\times2}$

② $=\begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{bmatrix}\times\begin{bmatrix}(b_{11}+c_{11})&(b_{12}+c_{12})\\(b_{21}+c_{21})&(b_{22}+c_{22})\end{bmatrix}$ — ② Defn of Matrix Addition

③ $=\begin{bmatrix}a_{11}(b_{11}+c_{11})+a_{12}(b_{21}+c_{21}) & a_{11}(b_{12}+c_{12})+a_{12}(b_{22}+c_{22})\\a_{21}(b_{11}+c_{11})+a_{22}(b_{21}+c_{21}) & a_{21}(b_{12}+c_{12})+a_{22}(b_{22}+c_{22})\end{bmatrix}$ — ③ Defn of Matrix Mult

④ $=\begin{bmatrix}a_{11}b_{11}+a_{11}c_{11}+a_{12}b_{21}+a_{12}c_{21} & a_{11}b_{12}+a_{11}c_{12}+a_{12}b_{22}+a_{12}c_{22}\\a_{21}b_{11}+a_{21}c_{11}+a_{22}b_{21}+a_{22}c_{21} & a_{21}b_{12}+a_{21}c_{12}+a_{22}b_{22}+a_{22}c_{22}\end{bmatrix}$ — ④ distribution of mult over add for $\mathbb{R}$

Next do right hand side until they match

⑤ RHS $A \times B + A \times C =$

$$= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \times \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \times \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

⑥

$$= \begin{bmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} \end{bmatrix} + \begin{bmatrix} a_{11}c_{11}+a_{12}c_{21} & a_{11}c_{12}+a_{12}c_{22} \\ a_{21}c_{11}+a_{22}c_{21} & a_{21}c_{12}+a_{22}c_{22} \end{bmatrix}$$

⑦

$$= \begin{bmatrix} a_{11}b_{11}+a_{12}b_{21}+a_{11}c_{11}+a_{12}c_{21} & a_{11}b_{12}+a_{12}b_{22}+a_{11}c_{12}+a_{12}c_{22} \\ a_{21}b_{11}+a_{22}b_{21}+a_{21}c_{11}+a_{22}c_{21} & a_{21}b_{12}+a_{22}b_{22}+a_{21}c_{12}+a_{22}c_{22} \end{bmatrix}$$

Now check this matrix matches step 4

QED

⑤ Given $A, B, C \in M_{2\times2}$ and defn of $M_{2\times2}$.

⑥ Defn of Matrix Mult and Order of Operations

⑦ Defn of Matrix Addition

Thm Associativity of Matrix Mult

$$A\times(B\times C) = (A\times B)\times C$$

Proof (for $A, B, C \in M_{2\times 2}$):

① LHS $A\times(B\times C) =$

$$= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \times \left( \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \times \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \right)$$

② $$= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \times \begin{bmatrix} (b_{11}c_{11}+b_{12}c_{21}) & (\quad) \\ (\quad) & (\quad) \end{bmatrix}$$

③ $$= \begin{bmatrix} a_{11}(b_{11}c_{11}+b_{12}c_{21}) + a_{12}(\quad) \\ a_{21}(b_{11}c_{11}+b_{12}c_{21}) + a_{22}(\quad) \end{bmatrix}$$

④ Simplify

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$\boxed{\text{HW1}}$ Fill in rest of proof

① Given $A, B, C \in M_{2\times 2}$
Defn of $M_{2\times 2}$

② Defn of Matrix Mult.

$$\begin{bmatrix} a_{11}(\quad) + a_{12}(\quad) \\ a_{21}(\quad) + a_{22}(\quad) \end{bmatrix}$$

③ Defn of Matrix Mult.

④ algebra

2:10 AM Sat Sep 26 — Linear Algebra | Linear Algebra 71%

**Defn** The zero matrix, $O$, is a matrix which has zeroes everywhere.

**Thm** $A \times O = O$

**Proof**: $(A \in M_{2\times 3} \quad O \in M_{3\times 4})$

① $A \times O = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \times \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

② $= \begin{bmatrix} a_{11}\cdot 0 + a_{12}\cdot 0 + a_{13} 0 & a_{11}0 + \dots & 0+0+0 & 0 \\ a_{21}\cdot 0 + a_{22}\cdot 0 + a_{23} 0 & 0+0+0 & 0+0+0 & 0 \end{bmatrix}$

③ $= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \in M_{2\times 4}$

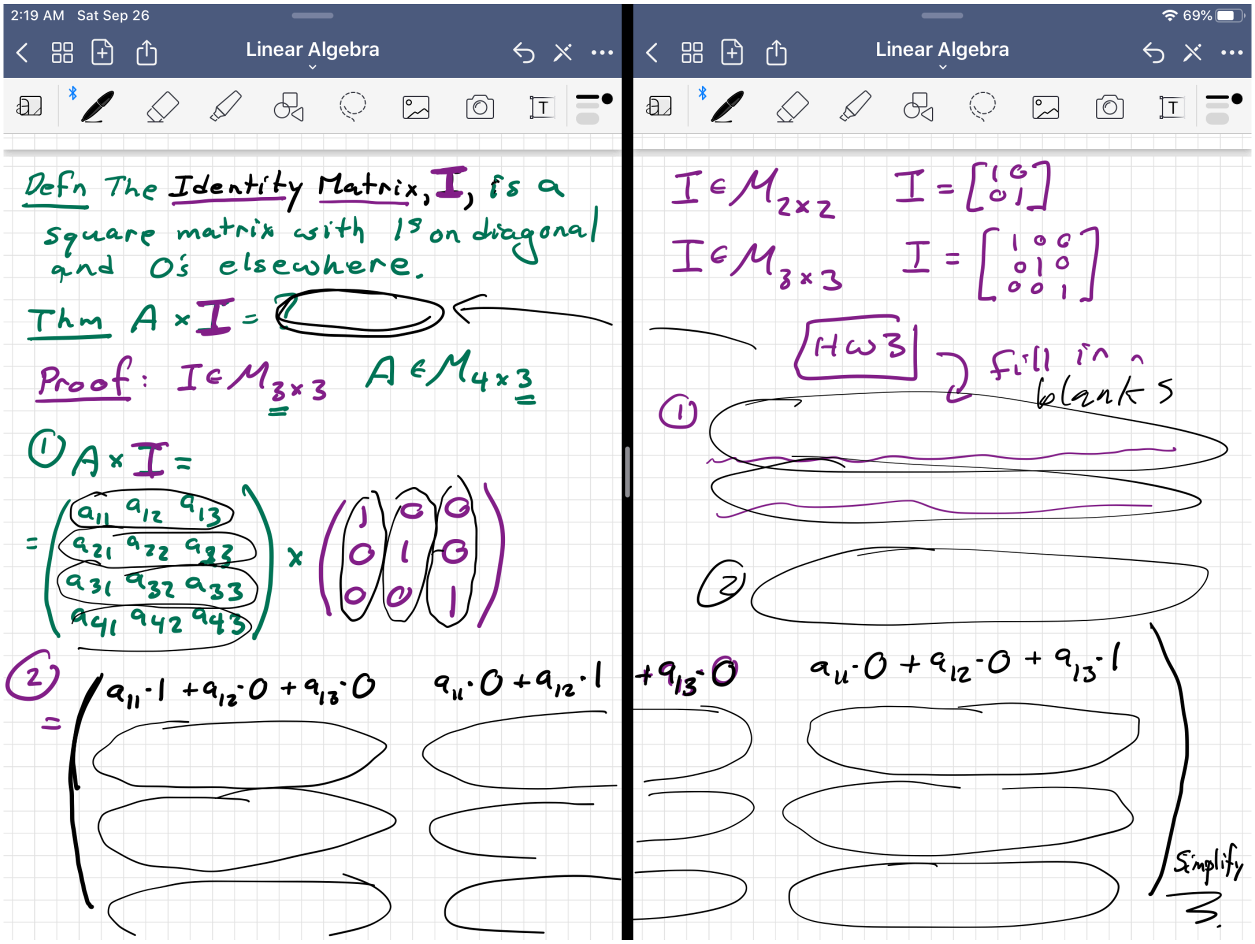
$\boxed{HW2}$ $O \times A = O$ when $A \in M_{2\times 3}$ $\quad O \in M_{4\times 2}$

we will figure out the answer.

① by given $A \in M_{2\times 3}$ & defn $M_{2\times 3}$
by defn $O$ matrix in $M_{3\times 4}$

② by defn of matrix mult

③ by arithmetic




**HW1-HW3** **are described above.**

**HW4 Prove IxB=B when B is 2x4 and I is the correct choice of identity matrix.**

**HW5 is to find the errors in the incorrect proof below which has many errors so find all of the errors. Then either fix the proof or find a pair of specific 2x2 matrices A and B for**

which this fails.

2:36 AM Sat Sep 26 — Linear Algebra

HW5 Mark All errors in the "proof" below.

False Thm $A \times B = B \times A$

Proof $(A, B \in M_{2\times 2})$

(1) LHS $A \times B = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \times \begin{pmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{pmatrix}$ — (1) by matrix mult

(2) $= \begin{pmatrix} a_{11}b_{11} + a_{12}b_{12} & a_{11}b_{21} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{12} & a_{21}b_{21} + a_{22}b_{22} \end{pmatrix}$ — (2) by matrix mult

(3) RHS $B \times A = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \times \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ — (3) by matrix mult

(4) $= \begin{pmatrix} b_{11}a_{11} + b_{12}a_{21} & b_{21}a_{11} + b_{22}a_{12} \\ b_{11}a_{21} + b_{12}a_{22} & b_{21}a_{21} + b_{22}a_{22} \end{pmatrix}$ — (4) by matrix mult.

They Match QED

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**Part II:** **Watch [Playlist 313F20-15-PartII]() which is Extra Credit highly recommended for math majors and has three extra credit problems.**

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**Thm** $A \times O = ?$ and $O \times A = ?$

**Defn** The **identity matrix**, $I$, is a is a square matrix with 1's on the diagonal and zeroes elsewhere.

**Thm** $A \times I = ?$ and $I \times A = ?$

What about

$$A \times B \overset{?}{=} B \times A \,?$$

## Part II Proofs using $\Sigma$ notations

**Defn of Matrix Mult**

$$[A \times B]_{ik} = \sum_{j=1}^{m} a_{ij} b_{jk}$$

$i^{th}$ row of $A$ $\quad (a_{i1} \; a_{i2} \cdots a_{im}) \cdot \begin{pmatrix} b_{1k} \\ b_{2k} \\ \vdots \\ b_{mk} \end{pmatrix}$ $\quad k^{th}$ column of $B$

**Defn of Addition**

$$[A + B]_{ik} = [A]_{ik} + [B]_{ik} = a_{ik} + b_{ik}$$

**Thm** Distribution of Matrix Mult over Add

$$A\times(B+C) = A\times B + A\times C$$

**Proof** using $\Sigma$ notation

for $A \in M_{n\times m}$, $B, C \in M_{m\times \ell}$:

① LHS $$[A\times(B+C)]_{ik} = \sum_{j=1}^{m} [A]_{ij}\,[B+C]_{jk}$$ ① by defn of matrix mult

② $$= \sum_{j=1}^{m} [A]_{ij}\left([B]_{jk} + [C]_{jk}\right)$$ ② defn of matrix addition

③ $$= \sum_{j=1}^{m} \left([A]_{ij}[B]_{jk} + [A]_{ij}[C]_{jk}\right)$$ ③ distribution of mult over addition of reals.

④ $$= \sum_{j=1}^{m} \left([A]_{ij}[B]_{jk}\right) + \sum_{j=1}^{m} [A_{ij}][C_{jk}]$$ ④ reg $\Sigma$

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# Part II Proofs using $\Sigma$ notations

$$\boxed{\text{Defn of Matrix Mult}}$$

$$[A\times B]_{ik} = \sum_{j=1}^{m} [A]_{ij}[B]_{jk}$$

$i^{th}$ row of $A$: $a_{i1}\ a_{i2} \cdots a_{im}$ $\cdot$ $\begin{pmatrix} b_{1k} \\ b_{2k} \\ \vdots \\ b_{mk} \end{pmatrix}$ $k^{th}$ column of $B$

$$\boxed{\text{Defn of Addition}}$$

$$[A+B]_{ik} = [A]_{ik} + [B]_{ik}$$
$$= a_{ik} + b_{ik}$$

**Thm** Distribution of Matrix Mult over Add

$$A\times(B+C)=A\times B+A\times C$$

**Proof** using $\Sigma$ notation

for $A\in M_{n\times m}$ $B,C\in M_{m\times \ell}$:

(1) LHS
$$[A\times(B+C)]_{ik}=\sum_{j=1}^{m}[A]_{ij}\,[B+C]_{jk}$$

(1) by defn of matrix mult

(2)
$$=\sum_{j=1}^{m}[A]_{ij}\left([B]_{jk}+[C]_{jk}\right)$$

(2) defn of matrix addition

(3)
$$=\sum_{j=1}^{m}\left([A]_{ij}[B]_{jk}+[A]_{ij}[C]_{jk}\right)$$

(3) distribution of mult over addition of reals.

(4)
$$=\sum_{j=1}^{m}\left([A]_{ij}[B]_{jk}\right)+\sum_{j=1}^{m}[A_{ij}][C_{jk}]$$

(4) reorder $\Sigma$

(5) RHS
$$[A\times B+A\times C]_{ik}=$$

$$=[A\times B]_{ik}+[A\times C]_{ik}$$

(5) defn of matrix add.

(6)
$$=\sum_{j=1}^{m}[A]_{ij}[B]_{jk}+\sum_{j=1}^{m}[A]_{ij}[C]_{jk}$$

(6) by def matrix Mult.

Steps 4 + 6 Match

QED

$\boxed{EC}$ $A,B\in M_{n\times m}$ $C\in M_{m\times \ell}$

$$(A+B)\times C=A\times C+B\times C$$

# Lesson 15 Proofs with Matrices

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**Defn:** The **zero matrix**, $O$, is a matrix which has zeroes everywhere.

**Thm** $A\times O = O$ and $O\times A = O$

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**Thm** $A\times I = A$ and $I\times A = A$

What about

$$A\times B \overset{?}{=} B\times A \text{ ?}$$ FALSE

**Thm:** If $A\in M_{n\times m}$ and $O\in M_{m\times \ell}$ then $A\times O = O\in M_{n\times \ell}$

**Proof:**

① $[A\times O]_{ik} = \sum_{j=1}^{m}[A]_{ij}[O]_{jk}$ — ① by defn matrix mult

② $= \sum_{j=1}^{m}[A]_{ij}\cdot 0$ — ② by defn of zero matrix

③ $= \sum_{j=1}^{m} 0$ — ③ by $a\cdot 0 = 0$ for any $a\in\mathbb{R}$

④ $= 0$ — ④ $0+0=0$

⑤ $= [O]_{ik}$. — ⑤ Defn of zero matrix QED

$\boxed{EC}$ $\underset{M_{n\times m}}{O}\times \underset{M_{m\times \ell}}{A} = \underset{M_{n\times \ell}}{O}$

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**Thm** $A\times O = O$ and $O\times A = O$

**Defn** The **identity matrix**, $I$, is a square matrix with 1's on the diagonal and zeroes elsewhere.

**Thm** $A\times I = A$ and $I\times A = A$

What about

$$A\times B \stackrel{?}{=} B\times A \;?$$ FALSE

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**Thm** Given $A\in M_{n\times m}$ $I\in M_{m\times m}$ then $A\times I = A$

$\boxed{EC3}$ $I\times A = A$ $I\in M_{m\times m}$ $A\in M_{m\times \ell}$

Thoughts

$$[I]_{ij} = \begin{cases} 1 & \text{if } i=j \text{ (diagonal)} \\ 0 & \text{if } i\neq j \text{ (elsewhere)} \end{cases}$$

Defn of Identity Matrix

**Proof:**

(1) $[A\times I]_{ik} = \sum_{j=1}^{m} [A]_{ij}[I]_{jk}$ (1) by defn matrix mult

(2) $= [A]_{i1} I_{1k} + [A]_{i2} I_{2k} + \cdots + [A]_{ik} I_{kk} + \cdots + [A]_{im} I_{mk}$ (2) defn of $\Sigma$

(3) $= [A]_{i1}\cdot 0 + [A]_{i2}\cdot 0 + \cdots + \boxed{[A]_{ik}\cdot 1} + \cdots + A_{im} 0$ (3) by defn of identity

(4) $= 0+0+\cdots+0+[A]_{ik}+0\cdots+0$ (4) algebra

(5) $= [A]_{ik}$ (5) by $0+a=a$ for $a\in\mathbb{R}$

QED

**Extra Credit:** Prove for arbitrary matrices A and B and vector v using sum notation that

**A(Bv)=(AxB)v**

**Extra Credit:** Prove for arbitrary matrices A, B, and C using sum notation that

**Ax(BxC)=(AxB)xC**