OPTIMAL TARIFFS AND INTERNATIONAL TRADE

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Abstract

Optimal tariff strategies, with and without optimal responses, are derived, based on a linear partial equilibrium trade model, A, and a nonlinear general equilibrium trade model with two nations, B. Under free trade, models A and B are applied to prove that it is always profitable for one nation, N1, to introduce a strictly positive tariff on imports, T1, as long as other nations do not introduce tariffs. With model B, it is proved that it is rational also for nation N2, to introduce a tariff, T2, if N1, introduces a tariff, T1. The Nash equilibrium tariff combination is unique and stable. The economic results, in both nations, are however higher with free trade, than in the Nash equilibrium tariff solution. If both nations know that the other nations will respond to increasing tariffs, and select the optimal tariff, then it is not rational for any nation to leave free trade and introduce tariffs. Free trade agreements are rational for the participants. It is also shown that it may be rational for a dictator, to introduce a tariff, even if this is not rational for the consumers in the same nation. The dictator may keep the tariff revenues and use these for other purposes. In such a case, the tariff reduces the consumer surpluses in both nations.

Keywords: Optimal tariffs, international trade, Nash equilibrium, Free trade, Dynamic non-zero-sum game theory.

1. Introduction

Krugman and Obstfeld [1] is a famous book that covers the field international economics, theory and policy. One of the interesting parts in this book contains a section where they show that the net welfare effect of a tariff is a strictly concave and quadratic function of the size of the tariff. For sufficiently small tariffs, the function is strictly positive. This means that the optimal tariff is strictly positive. In other words, free trade is not the optimal solution according to that analysis. Krugman obtained the Nobel Memorial Prize in Economic Sciences in 2008.

In this paper, it will also be demonstrated that the optimal tariff is strictly positive, in case the other nations do not introduce tariffs. The present analysis will also show that there is a stable Nash equilibrium with positive tariffs, but that the nations have an economically better situation if they cooperate and select free trade. In other words, free trade is a better solution for the participants than a world with tariffs.

The optimal tariff problem is not a new topic. One classical approach is to derive a static formula that gives the optimal tariff as a simple function of the export supply elasticity. Dávila et al [2] give a good description of the earlier literature on the subject, and give an interesting description of assumptions behind the optimal tariff formulas from the past. They generalize a static optimal tariff formula to a dynamic version, where changing export supply elasticities can be handled.

In this analysis, the topic will be extended even further. As Dávila et al [2] mention, for most functions, export supply elasticities are usually not constant when the export volume changes. Hence, a new optimal tariff formula is developed, which is not a function of the elasticities, but a function of the parameters of the supply and demand functions. This function also makes it possible to derive general results concerning the optimal tariff and several other things.

Chacholiades [3] gives a very good presentation of the many general theories of international trade and policy. He also touches several important issues connected to tariffs, such as the fact that several countries may introduce tariffs. He calls that retaliation. Chacholiades [3] does not analyze the dynamic retaliation game further but suggests that if one nation introduces a tariff, other nations will be tempted to make tariff changes in, as he says "an infinite regression". He also asks the question if a tariff equilibrium exists, where the different nations would not be interested to change the tariffs any more.

In this paper, the questions by Chacholiades [3] are answered. The optimal dynamic tariff adjustment game is solved and the Nash equilibrium is presented. Furthermore, it is found that the Nash equilibrium, which is the result of individual and independent adjustments of the tariffs, is a stable equilibrium, but that the economic situations of the different nations in this equilibrium are worse than in the free trade solution, completely without tariffs. It is shown that the different nations, when they realize this fact, have a strong reason to negotiate tariff reductions until they reach the free trade solution, where the involved nations are better off.

Relevant theories should be tested with relevant data. Empirical data of high quality and relevance to empirical estimations of effects of tariffs are often difficult to find. Furthermore, dynamics with asymmetric effects over time may sometimes be observed. Irwin [4] describes this field and Bowden [5] gives updated comments on the same topic. Broda el al [6] however find that market power explains tariff variation very well. Prior to World Trade Organization membership, countries set import tariffs 9 percentage points higher on inelastically supplied imports relative to those supplied elastically.

When a nation introduces a tariff, this may be interpreted as a form of economic warfare, namely an attack. The objective is to gain some economic advantage that should benefit the attacker and most likely harm the attacked nation. If also the attacked nation introduces a tariff, that may be considered as defense, with the objective to improve the economic situation of the attacked nation, and most likely also hurt the attacker. In the literature, the defense is often called retaliation. These changes of tariffs may continue in several steps. Lohmander [7] presents general approaches to dynamic interactions and games of the different kinds of relevance in conflict situations.

Game theory is central to the understanding of rational tariff decisions. If we initially have free trade and one nation introduces a tariff, it is reasonable to expect that the other nations will also consider the options to introduce tariffs. This may be considered and analyzed as a dynamic tariff game.

Nash [8] and Nash [9] develop the general theory of noncooperative games and equilibria. Rasmusen [10] writes a very instructive book on applications of the game theory concepts in many different and typical applications.

Isaacs [11] defines the concept differential games and applies that to solve many typical dynamic game problems, mostly from the military and mostly in continuous time. In the continuous time solutions, it is assumed that the games are zero sum games, which means that one player has the same objective function as the other player, but that one player wants to maximize and the other player wants to minimize the same objective function.

Washburn [12] continues in the same tradition, and develops zero sum game theory with applications to military battles, also including random strategies. The Washburn examples mostly concern problems from the navy and air force. Lohmander [13] and Lohmander [14] continue in the field of Washburn [12] but primarily focus on and solve game problems in the army. Washburn and Kress [15] cover a wide area of military operations analysis including zero sum games. There are however strong reasons to admit that many games of conflict cannot be considered as zero sum games. This is obvious in wars as Lohmander [16] describes, where large areas, cities and infrastructure are destroyed and people are killed. In such cases, zero sum game theory is simply not relevant. For this reason, the final parts of the analyses in this paper are performed with dynamic non-zero-sum game theory.

Lubik [17] gives the general recommendation that tariffs should not be introduced. He advocates international cooperation. He states that tariffs distort markets, make resource allocation inefficient and often lead to destructive trade conflicts and trade wars. In 2025, however, President Trump in USA did not follow the advice from Lubik. Chu et al [18] describes the tariffs introduced by USA during 2025. These tariffs strongly influence world trade and the general economic situation in many parts of the world. The present analysis may be viewed as a theoretical structure that can be helpful when these empirical observations are interpreted.

The following three sections focus on these topics:

Section 2: The optimal tariff in partial equilibrium.

Section 3: General equilibrium analysis.

Section 4: The optimal tariffs without and with cooperation in general equilibrium.

2. The optimal tariff in partial equilibrium

First, the free trade solution in an importing nation, N1 is derived, based on a linear demand function and a linear supply function from the world market. The solution is based on partial equilibrium analysis. Then, consumer surplus maximization, is used to derive the optimal import volume. It is shown that it is always optimal to import less than according to the free trade solution. When the optimal import volume is selected, the marginal consumer surplus is equal to the marginal cost, which means that the marginal consumer surplus, which equals the price level in N1, exceeds the import price. The optimal import volume can be obtained in different ways. One option is to let the government introduces a tariff that corresponds to the difference between the optimal price in N1 and the import price. Then, in some way, the government transfers the tariff revenue to the consumers. In general, however, it is not easy to demonstrate that and how governments really distribute the tariff revenues to the consumers. Furthermore, it is not easy to show if such possible transfers really are distributed to the consumers in a fair way. The tariff that maximizes the sum of the

consumer surplus and tariff gain (equal to the consumer surplus in case the tariff revenue is really transferred to the consumers) is derived. The optimal tariff is presented as an explicit function of all the parameters in the demand and supply functions. It is also proved that the optimal tariff always is unique and strictly positive. An explicit numerical case is used to explicitly illustrate and confirm the general results.

Linear approximations of demand and supply:

The demand function in the importing nation, N1, is found in (1). Q_D is the imported volume and P is the price. a and b are strictly positive parameters.

$$Q_D = a - bP \quad (1)$$

The export supply, Q_S, from the exporting country N2 (or the World), is given in (2). g > 0 and h > 0.

$$Q_S = -g + hP \quad (2)$$

In equilibrium, Q_S equals Q_D, which leads to (3). This leads to (4).

$$-g + hP = a - bP \quad (3)$$

$$(b + h)P = a + g \quad (4)$$

The equilibrium price is found in (5).

$$P = \frac{a + g}{b + h} \quad (5)$$

Figure 1 illustrates the equations (1) to (5).

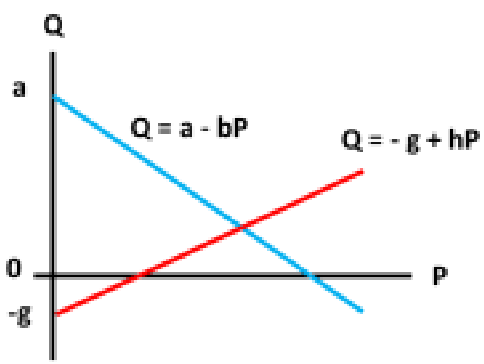

Figure 1.

Illustration of the demand and supply functions. Only solutions in the first quadrant are relevant.

Observe that g > 0, which implies that the supply can only be strictly positive in case the price is strictly positive. Of course, real functions can often be expected to be nonlinear. The linear functions used here can be considered as approximations. In case the reader has access to empirically estimated nonlinear supply and demand functions, the reader is encouraged to generalize the following analysis to such cases.

Since alternative price solutions will be derived in the following analysis, the free trade equilibrium price, corresponding to (5) is defined as (6).

$$P_F = \frac{a+g}{b+h} \tag{6}$$

The demand level in N1 in free trade equilibrium, is found in (7). This can be derived via (1) and (6). This is further developed in (8) and (9).

$$Q_{DF} = a - b\left(\frac{a+g}{b+h}\right) \tag{7}$$

$$Q_{DF} = \frac{a(b+h) - b(a+g)}{b+h} \tag{8}$$

$$Q_{DF} = \frac{ah - bg}{b+h} \tag{9}$$

The export supply from N2 (or from World) in free trade equilibrium, is found in (10). This is derived via (2) and (6). It is further developed in (10), (11) and (12).

$$Q_{SF} = -g + h\left(\frac{a+g}{b+h}\right) \tag{10}$$

$$Q_{SF} = \frac{-g(b+h) + h(a+g)}{b+h} \tag{11}$$

$+\, b\, h \quad +$

The export supply from N2 (or from World) in free trade equilibrium is found in (12). This equals the import demand in N1.

$$Q_S^F = Q_D^F = \frac{ah - bg}{b + h} \qquad (12)$$

The demand function in N1, (1), can be rewritten as (13). Here, the subscript is removed.

$$bP = a - Q \qquad (13)$$

Equation (14) is the inverse demand function in N1. This is developed from (13).

$$P = \frac{a}{b} - \frac{1}{b}Q \qquad (14)$$

The export supply function (2) is transformed to (15), The subscript is removed.

$$hP = g + Q \qquad (15)$$

The inverse supply function from N2 is found in (16). This follows from (15).

$$P = \frac{g}{h} + \frac{1}{h}Q \qquad (16)$$

The inverse demand and supply functions are illustrated in Figure 2.

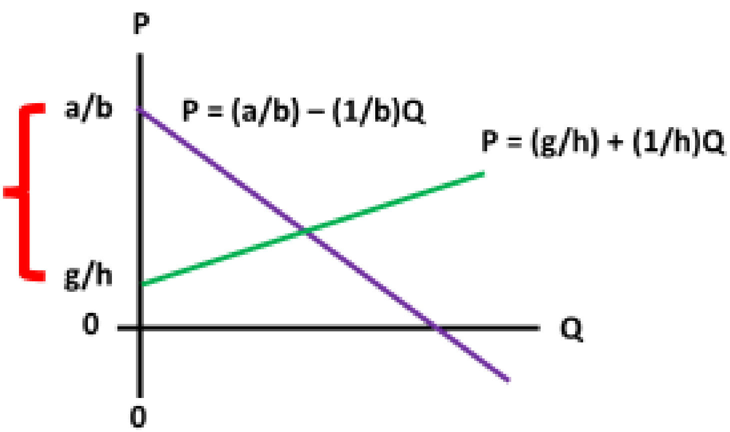

Figure 2. Inverse Demand function in N1 and Inverse Supply function from N2.

Strictly positive trade implies that the intersection occurs in the first quadrant. This implies that equation (17) holds. Equation (17) is an essential result that is needed in later derivations.

$>$(17)

$a\ g$

$b\ h$

The consumer surplus in N1, S, is found in (18).

$1\ 1\ ^{Q}$

$a\ g\ S\ q\ dq\ Q\ Q$

$(\ )\ (\ )$

$= - - ^{+} |\ ||\ ^{|} (\ )\ (\ ) \int$(18)

$b\ b\ h\ h\ _{0}$

The consumer surplus in N1 in free trade equilibrium is found in (19).

^{Q}

$a\ g\ ^{ah\ bg}\ S\ q\ dq\ Q\ Q\ Q\ Q$

$1\ 1$

$(\ )\ (\ ) -$

^{F}

$= - - + ^{= =} |\ ||\ | (\ )\ (\ ) +$

$\int$(19)

$,\ ^{D}$

$b\ b\ h\ h\ b\ h\ _{0}$

This is graphically illustrated as the read area in Figure 3.

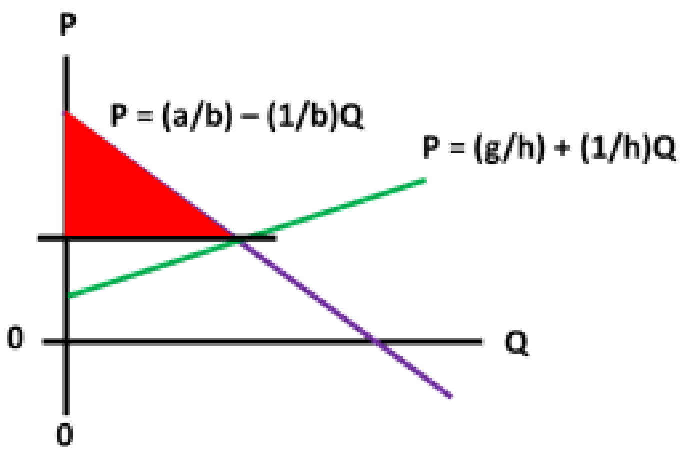

Figure 3.

The consumer surplus in N1 in free trade equilibrium. This is the area marked by red color.

Equation (18) gives (20). This is further developed to (21).

^{Q}

$a\ g\ S\ q\ q\ Q\ Q$

$1\ 1$

$\lceil\ \rceil$

$2\ 2$

$= - - - | \ | \lfloor\ \rfloor$ (20) $b\ b\ h\ h$

$2 \quad 0$

Since all parameters in (21) are strictly positive, S is strictly concave quadratic function of Q. Furthermore, equation (17) implies that the coefficient of Q, in equation (22) is strictly positive. These conditions imply that there is a unique and strictly positive value of Q that maximizes S.

$a\ g\ S\ Q\ Q$

$1\ 1\ _2$

$(\ )(\ )$

$= - - + | \ || \ |(\ )(\ )$ (21) $b\ h\ b\ h$

2

An explicit analysis of the consumer surplus in free trade equilibrium:

Let S^F be the consumer surplus in N1 in free trade equilibrium. (19) and (21) give (22).

2

$_F a\ g\ ah\ bg\ ^{ah\ bg}\ S$

$(\ )(\ )(\ )(\ ) - - 1\ 1$

$= - - ^+ | \ || \ || \ ^{||}\ |(\ )(\ )(\ )(\ ) + +$ (22) $b\ h\ b\ h\ b\ h\ b\ h$

2

If free trade should always be desired, it should be possible to show that the consumer surplus with free trade always is the highest obtainable consumer surplus.

The following analysis investigates if some other import volume, Q, gives a larger consumer surplus than the free trade equilibrium. Figures 4, 5 and 6 illustrate how the consumer surplus (red area) changes is the imported volumes change. When the imported volumes change, the prices change, in N1 and in the world market.

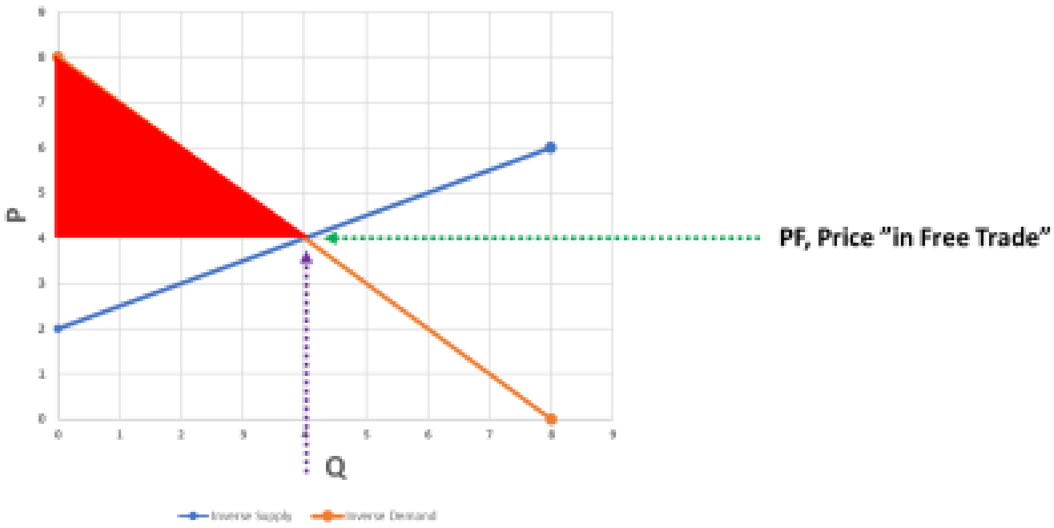

Figure 4.

Illustration of the consumer surplus (red area) in free trade and the free trade price.

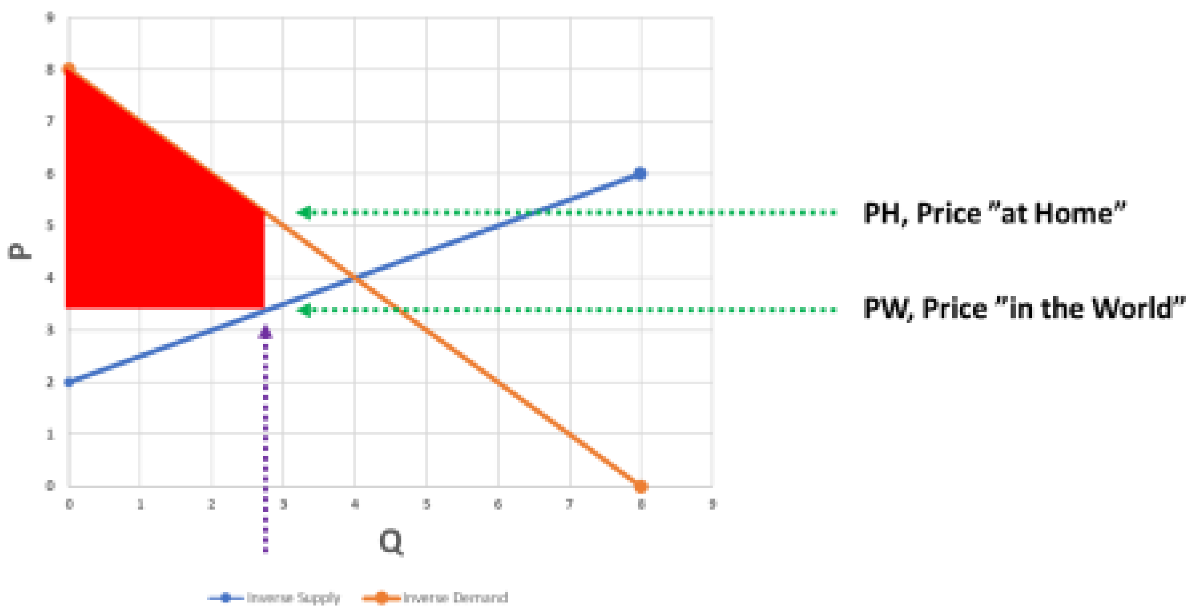

Figure 5.

Illustration of the consumer surplus (red area), PH and PW, in case Q = 2.7.

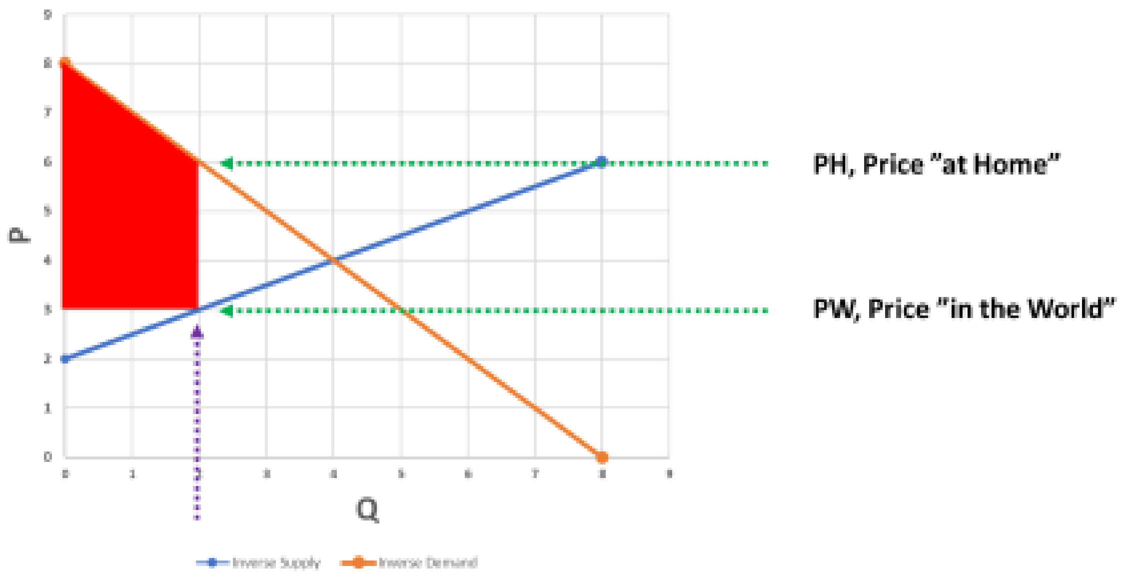

Figure 6.
Illustration of the consumer surplus (red area), PH and PW, in case Q = 2.0.

Now, it is time for maximization of consumer surplus in N1, via control of the total import, Q. The objective function, S, is found in equation (21).

The first order optimum condition is equation (23).

$$\frac{dS}{dQ} = \left(\frac{a}{b} - \frac{g}{h}\right) - \left(\frac{1}{b} + \frac{2}{h}\right)Q = 0 \qquad (23)$$

Equation (24) is the second order maximum condition. This is always satisfied, since b and h are strictly positive

$$\frac{d^2S}{dQ^2} = -\left(\frac{1}{b} + \frac{2}{h}\right) < 0 \qquad (24)$$

Let us determine the optimal import volume from the first order optimum condition. (23) gives

The optimal import level (27) is found via (25) and (26).

$$\frac{dS}{dQ} = \left(\frac{ah-bg}{bh}\right) - \left(\frac{h+b}{bh}\right)2Q = 0 \tag{25}$$

$$Q = Q^* = \frac{\left(\frac{ah-bg}{bh}\right)}{2\left(\frac{h+b}{bh}\right)} \tag{26}$$

$$Q^* = \frac{ah-bg}{2(b+h)} \tag{27}$$

Let us investigate the ratio, z, in equation (28). This is the optimal import volume divided by the free trade import volume. The free trade import level is found in (12).

$$0 < z = \frac{Q^*}{Q_F} = \frac{\left(\frac{ah-bg}{2(b+h)}\right)}{\left(\frac{ah-bg}{b+h}\right)} = \frac{1}{2} < 1 \tag{28}$$

Observation:

It is always optimal to import strictly less, than according to the free trade solution. (It is important to remember that this result comes from a partial equilibrium analysis.)

Comparative statics analysis of z:

It is useful to know how the value of z, (29), is affected by the parameter values.

$$z = \frac{Q^*}{Q^F} = \left(\frac{b+h}{2b+h} \right) \quad (29)$$

In Figure 7, the parameter b increases.

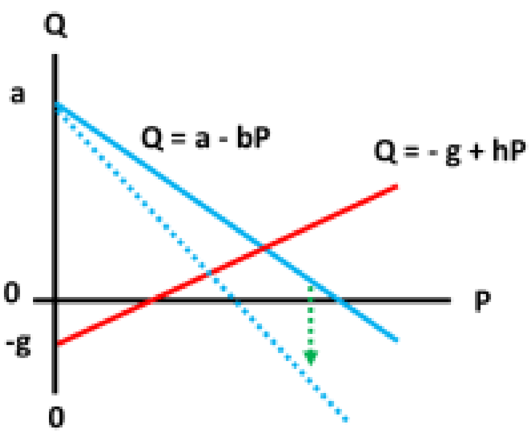

Figure 7.

Illustration of how the graph changes if parameter b increases.

Equations (30) to (32) show that the optimal import volume divided by the free trade import volume, decreases if parameter b increases. Compare Figure 7.

$$\frac{dz}{db} = \frac{(1)(2b+h) - (b+h)(2)}{(2b+h)^2} \quad (30)$$

$$\frac{dz}{db} = \frac{2b+h-2b-2h}{(2b+h)^2} \quad (31)$$

db $b\ h$

2

$dz\ h$

$-$

$= <$

$(\ )^2 0$

$+ (32)\ db\ \ b\ h$

2

Observation:

If b increases, the negative slope of the demand function increases. Then, z decreases.

In Figure 8, the parameter h increases.

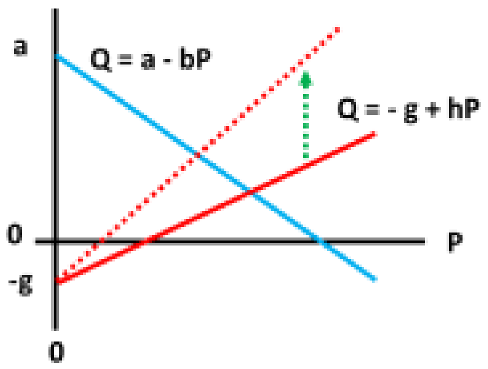

Figure 8.

Illustration of how the graph changes if parameter h increases.

Equations (33) to (35) show that the optimal import volume divided by the free trade import volume, increases if parameter h increases. Compare Figure 8.

$dz\ b\ h\ b\ h$

$(1)(2\ )\ (\ )(1)$

$+ - +$

$= + (33)\ (\ )^2$

dh $b\ h$

2

$dz\ b\ h\ b\ h$

$2 \qquad + - -$

$$\frac{dz}{dh} = \frac{b}{2(b+h)^2} \quad (34)$$

$$\frac{dz}{dh} = \frac{b}{2(b+h)^2} > 0 \quad (35)$$

Conclusion:

If h increases, the slope of the supply function increases. Then, z increases.

Important variables determined as functions of the optimized import volume:

Several variables may be considered as determined by the optimal import volume, in equation (27). One of these, equation (36), is the optimal price for the customers inside N1, here denoted "price at home (H)". This is determined from the inverse demand function (14).

$$P_H^{**} = \frac{a}{b} - \frac{1}{b}Q^{**} \quad (36)$$

The optimal price for the exporters from the outside World (W), is found in equation (37). This is determined from the inverse supply function, (16).

$$P_W^{**} = \frac{g}{h} + \frac{1}{h}Q^{**} \quad (37)$$

The optimal tariff in equation (38) is determined as the difference between (36) and (37).

$$T^{**} = P_H^{**} - P_W^{**} \quad (38)$$

A deeper investigation of the optimal tariff, gives (39). The optimal tariff is determined as an explicit function of the parameters via the algebra in equations (39) to (47).

$$T^{**} = \left(\frac{a}{b} - \frac{g}{h}\right) - \left(\frac{1}{b} + \frac{1}{h}\right)Q^{**} \quad (39)$$

** a g 1 ^{1} T Q Q

= − − − (40)

b b h h

* * *a g* 1 1 *T Q*

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= − − + | | | | ( ) ( ) (41)

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ah bg b h b h *T*

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= | | | | ( ) ( ) + (45) *bh b h*

2

ah bg b *T*

( ) ( ) −

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= | | | | ( ) ( ) + (46) *bh b h*

The explicit function of the optimal tariff is found in (47).

$$T^* = \frac{ah - bg}{2bh + h^2} \tag{47}$$

As reported in (48), it is possible to determine the sign of the optimal tariff. This result is surprisingly strong. Remember equation (17).

$$\left(b > 0 \wedge h > 0 \wedge ah - bg > 0 \right) \Rightarrow T^* > 0 \tag{48}$$

(47) and (48) imply that the optimal tariff *always* is strictly positive. This result holds, at least if other nations cannot respond with adjusted tariffs, if one nation is increasing the tariffs. In the later part of this paper, also other nations can introduce and adjust tariffs. Then, with such a problem structure, it is found that the optimal tariffs are not always strictly positive.

Comparative statics analysis of the optimal tariff:

The optimal tariff (47) always increases if parameter a increases. Compare (49).

$$\frac{dT^*}{da} = \frac{1}{2b + h} > 0 \tag{49}$$

Hence, if the level of the demand function increases, the optimal tariff increases. The optimal tariff (47) always decreases if parameter b increases. Compare (50) to (52).

$$\frac{dT^*}{db} = \frac{-g\left(2bh + h^2\right) - \left(ah - bg\right)\left(2h\right)}{\left(2bh + h^2\right)^2} \tag{50}$$

$dT\ bgh\ gh\ ah\ bgh - - - +$

$2\ 2\ 2$

$=$

$+$ (51)

$2\ \ 2$

$db\ \ bh\ h$

$(\ )$

2

$*\ 2$

$(\ )$

$dT\ a\ g\ h$

$- +$

2

$= < \quad 0$

$+$ (52)

2

$db\ \ bh\ h\ 2$

$(\ )$

2

Thus, if the slope of the demand function becomes steeper, b increases, then the optimal tariff decreases.

The optimal tariff (47) always decreases if parameter g increases. Compare (53) and (54).

$2\ 2\ *\ dT\ b\ bh\ h\ bh\ h - + - + \times 2\ 2\ 0$

$(\ )(\ )\ (\ ) =$

(53)

$2\ \ 2$

$dg\ \ bh\ h$

$(\ )$

$2 \qquad -$

$* \qquad +$

$dT\ b$

$= <$

$_2 0$

+(54) $dg\ bh\ h$

2

Equation (54) means that if the level of the supply function increases, g decreases, then the optimal tariff increases. In general, it is not clear how the optimal tariff changes, in case the slope of the supply function increases. The sign of the derivative is a nonlinear function of all the parameters. Compare equations (55) to (57).

2 *

dT $a\ bh\ h\ ah\ bg\ b\ h$

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+ − − +

( )( ) ( )( ) =

(55)

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dh $bh\ h$

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$dT\ abh\ ah\ abh\ ah\ b\ g\ bgh$ 2 2 2 2 2

+ − − + + =

(56)

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dh $bh\ h$

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* 2 2

$dT\ ah\ b\ g\ bgh$ − + +

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+(57)

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dh $bh\ h$ 2

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2

Hence, compare (57), without exact knowledge of all parameter values, we cannot know if the optimal tariff increases, is unchanged or decreases, if the slope of the supply function increases.

A concrete example:

Below, a numerically specified example is introduced to confirm and illustrate the general results reported in equations (1) to (57). The parameters are found in (58).

$(, , ,) (8,1, 4, 2)\ a\ b\ g\ h =$(58) The inverse demand function based on (14) and (58), is presented in (59).

$P\ Q = -8$(59) The inverse supply function, based on (16) and (58), is given in (60).

1

$P\ Q = +$(60)

$2 \qquad 2$

The free trade demand volume (61) and supply volume (62) are derived from (12) and (58).

$ah\ bg\ Q$

$- -$

$16\ 4\ 4$

F

$= = =$

$+$(61) D

$b\ h \qquad h$

$ah\ bg\ Q_b\ 3$

$F \qquad - \qquad 4$

$= =$

$+$(62) S

In Figure 9, the concrete free trade solution is illustrated, based on the parameters in (58), and in Figure 10, the corresponding concrete free trade consumer surplus is found.

Figure 9.

The concrete example of a free trade solution: Inverse supply and inverse

demand. The consumer surplus, as a quadratic function of Q in (21) gives in (63).

$a\ g\ S\ Q\ Q$

$1\ 1\ _2$

$(\ )(\ )$

$= - - + |\ ||\ |(\ )(\ )$ (63) $b\ h\ b\ h$

2

Using the parameter values from (58), we get equation (64).

$S\ Q\ Q = -6$ (64)

2

With the free trade equilibrium quantity, we get the value of the consumer surplus in (65). $S = \times$

$- = 6\ 4\ 4\ 8$ (65)

$(\ )^{2}$

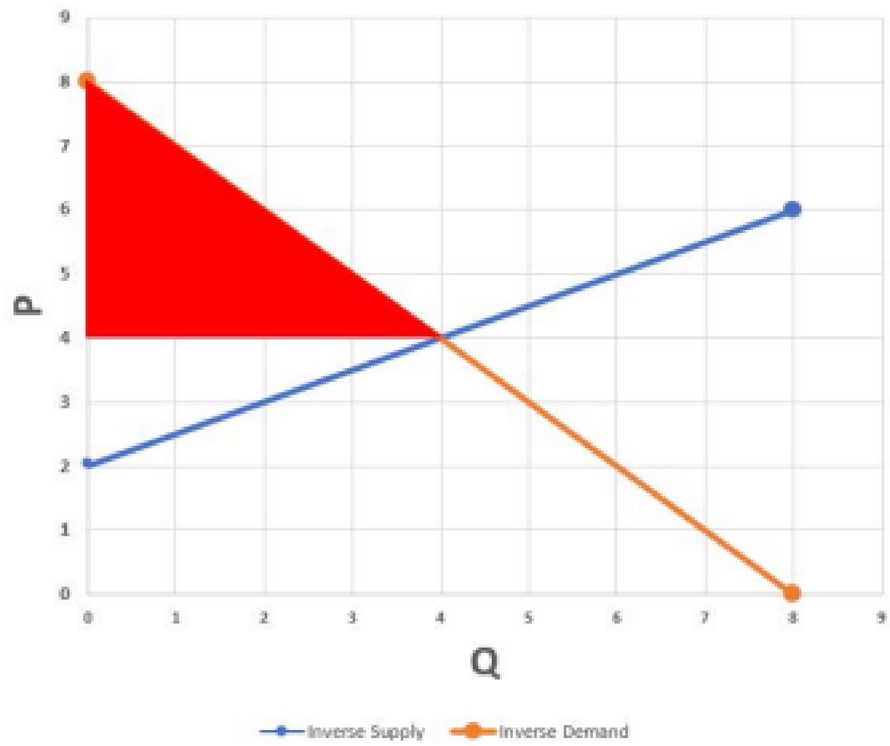


Figure 10.

The concrete free trade solution: The consumer surplus (red area).

Optimization of the import level:

Here, we optimize the import volume. A tariff is used to reduce the import volume, which also creates a tariff revenue. The objective function is the sum of the consumer surplus and the tariff revenue. This maximizes the total surplus of the consumers in the importing country N1 in case the tariff revenue is redistributed to the consumers.

Observation: In the real world, it is quite possible that the tariff revenue is not completely redistributed to the consumers. In such cases, it is quite possible that the consumers do not benefit from the tariff, even if the total surplus is maximized via the tariff. The optimal import volume is found in (27). With the selected parameters, we get

$$\bar{Q}^{*} = \frac{16\ 4}{2+2} = \frac{12}{4} = 3 \qquad (66)$$

The optimal price at home, *H*, in N1, is (67). Compare equation (36).

$$P_{1}^{**} = a - Q^{*} = \; - \; = \qquad (67)$$

$8\ 3\ 5\ _H$

$b\ b$

The optimal price in the World market is found in (37). With the selected parameter values, we get (68).

$^{**}\ 1\ 1\ \ 2\ 3\ 3.5$

$g\ \ \ ( )$

$P\ Q$

$= + = + = | \ | \ ( ) \quad (68)\ _W$

$h\ h\ \ \ 2$

The optimal tariff is presented in (47). The parameter values from equation (58) give the numerical value found in equation (69).

$16\ 4\ 12\ \ 1.5$

$\bar{T}^*$

$= = =$

$+ \quad (69)$

$4\ 2\ 8^2$

The general quadratic function of the consumer surplus is found in (63). With the optimal import volume from (66), the consumer surplus becomes (70), which can be calculated to be the numerical value found in (71).

$S\ Q\ Q = - 6 \quad (70)$

2

$S = \times - = 6\ 3\ 3\ 9 \quad (71)$

$( )^2$

Figure 11 shows the optimal tariff solution corresponding to the concrete case based on the parameters (58). An alternative solution with the same parameter values and total surplus, via a consumer cartel, without tariffs, is illustrated in Figure 12.

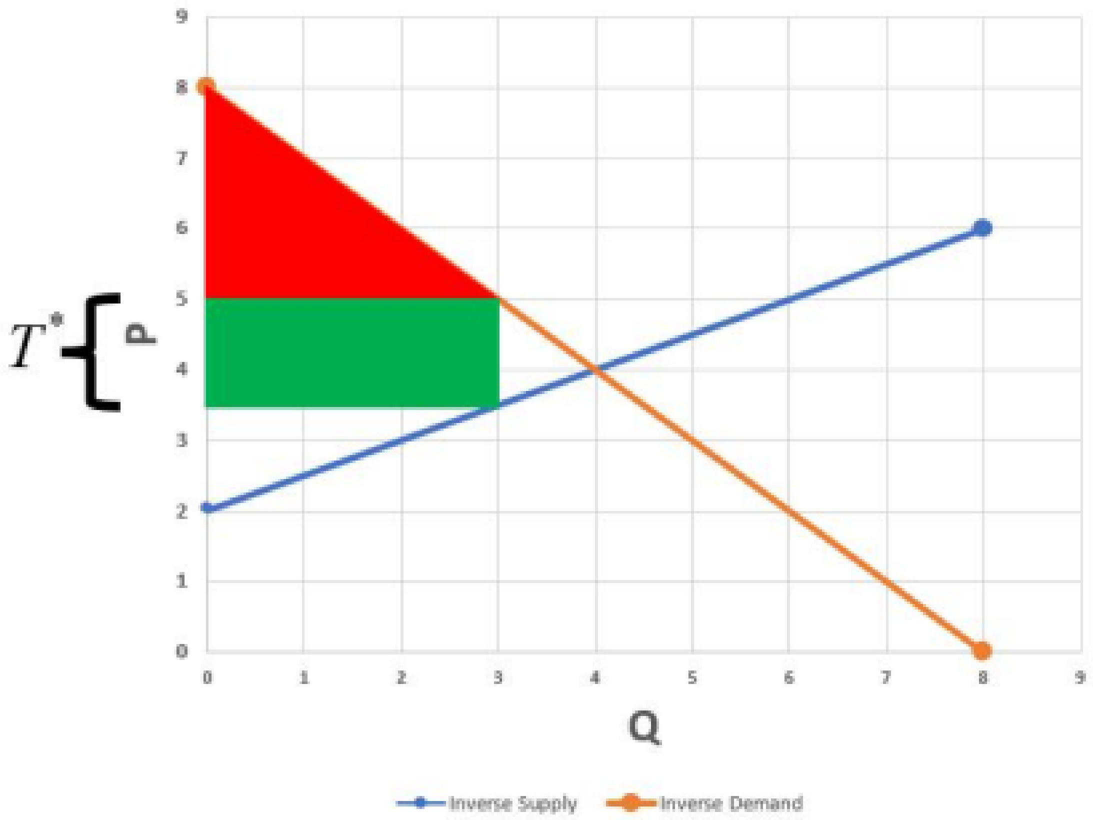


Figure 11. The concrete optimal tariff solution: The consumer surplus is the red area and the tariff gain is the green area. Consumer Surplus = Red Area = (3*3)/2 = 4.5; Tariff Gain = Green Area = (3*1.5) = 4.5; Total Surplus = 9.

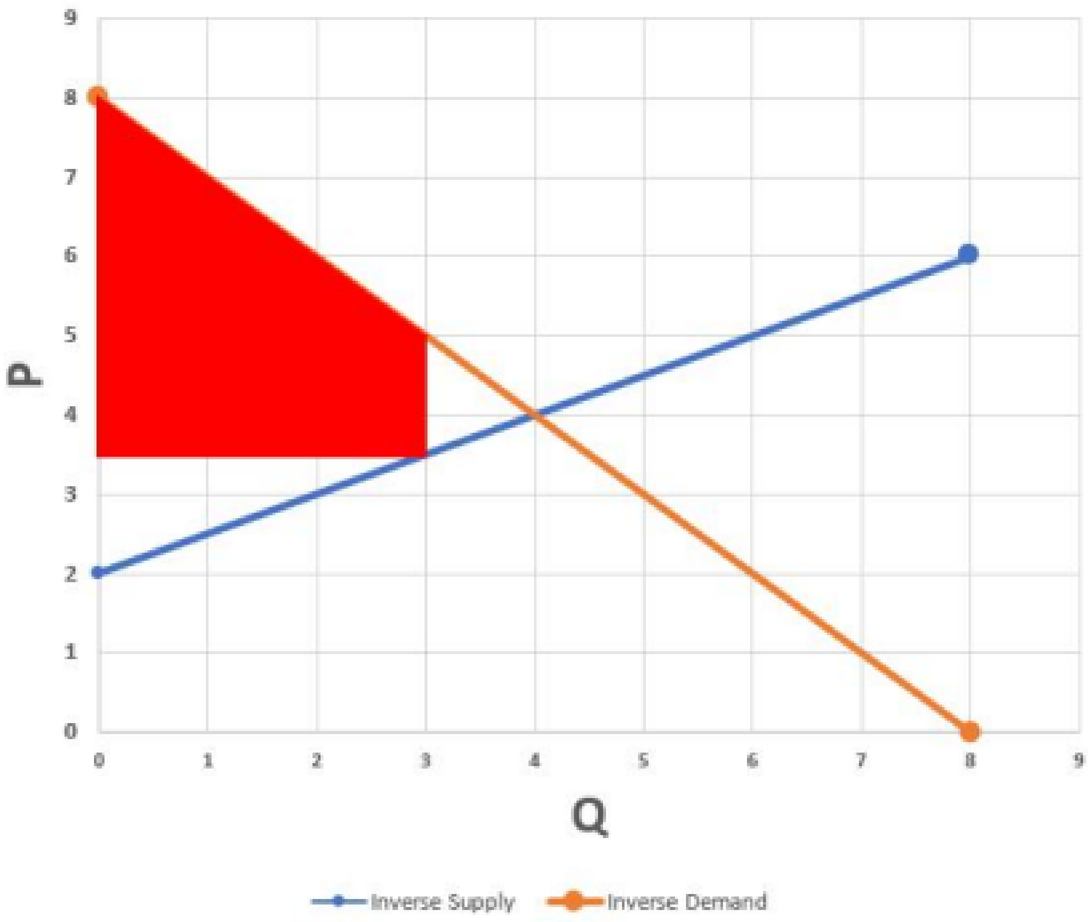


Figure 12 .

The concrete optimal consumer surplus solution: The consumer surplus (red area) = 9.

Alternative interpretation:

The optimal tariff solution derived in the earlier part of this paper, may be compared to the following classical approach from economics. Compare equations (72) to (78), Figure 12 and Figure 13.

The cost per unit is determined in (72).

$$P = \frac{C}{Q} = 2 + 0.5Q \quad (72)$$

From the cost per unit (72), the cost is determined in (73).

$$C = 2Q + 0.5Q^2 \quad (73)$$

The marginal cost is found in (74).

$$\frac{dC}{dQ} = \frac{Q}{2} + \qquad (74)$$

The total revenue is the function in equation (75).

$$\Phi(Q) \qquad (75)$$

Maximization of surplus gives (76).

$$\max_{Q} \Phi(Q) - C(Q) \qquad (76)$$

The first order optimum condition is (77).

$$\frac{d\Phi}{dQ} - \frac{dC}{dQ} = 0 \qquad (77)$$

The first order optimum condition determines that the marginal revenue equals marginal cost, as seen in (78).

$$\frac{d\Phi}{dQ} = \frac{dC}{dQ} \qquad (78)$$

This is also illustrated in Figure 13, and gives the same optimal import volume as Figures 11 and 12.

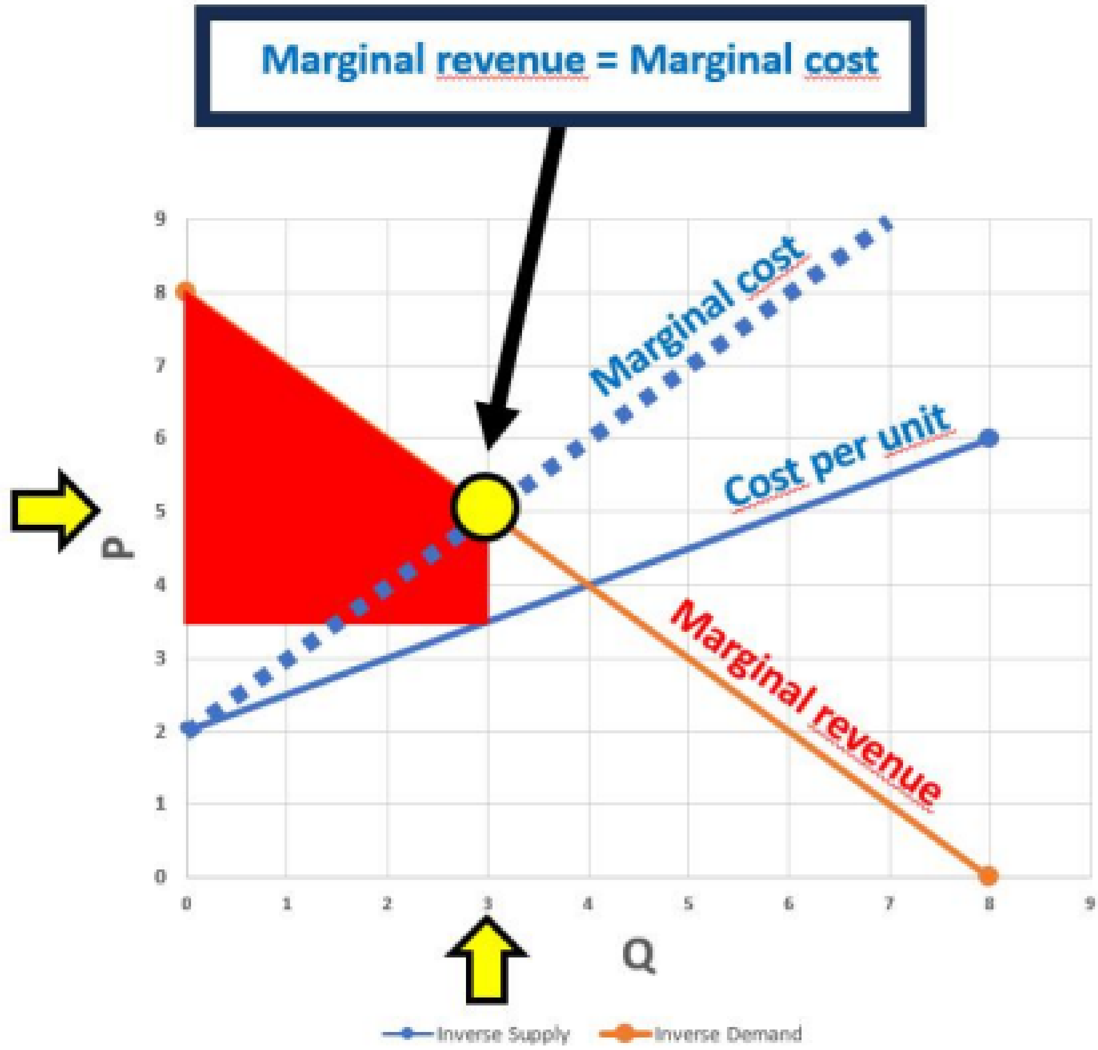


Figure 13.

The alternative solution.

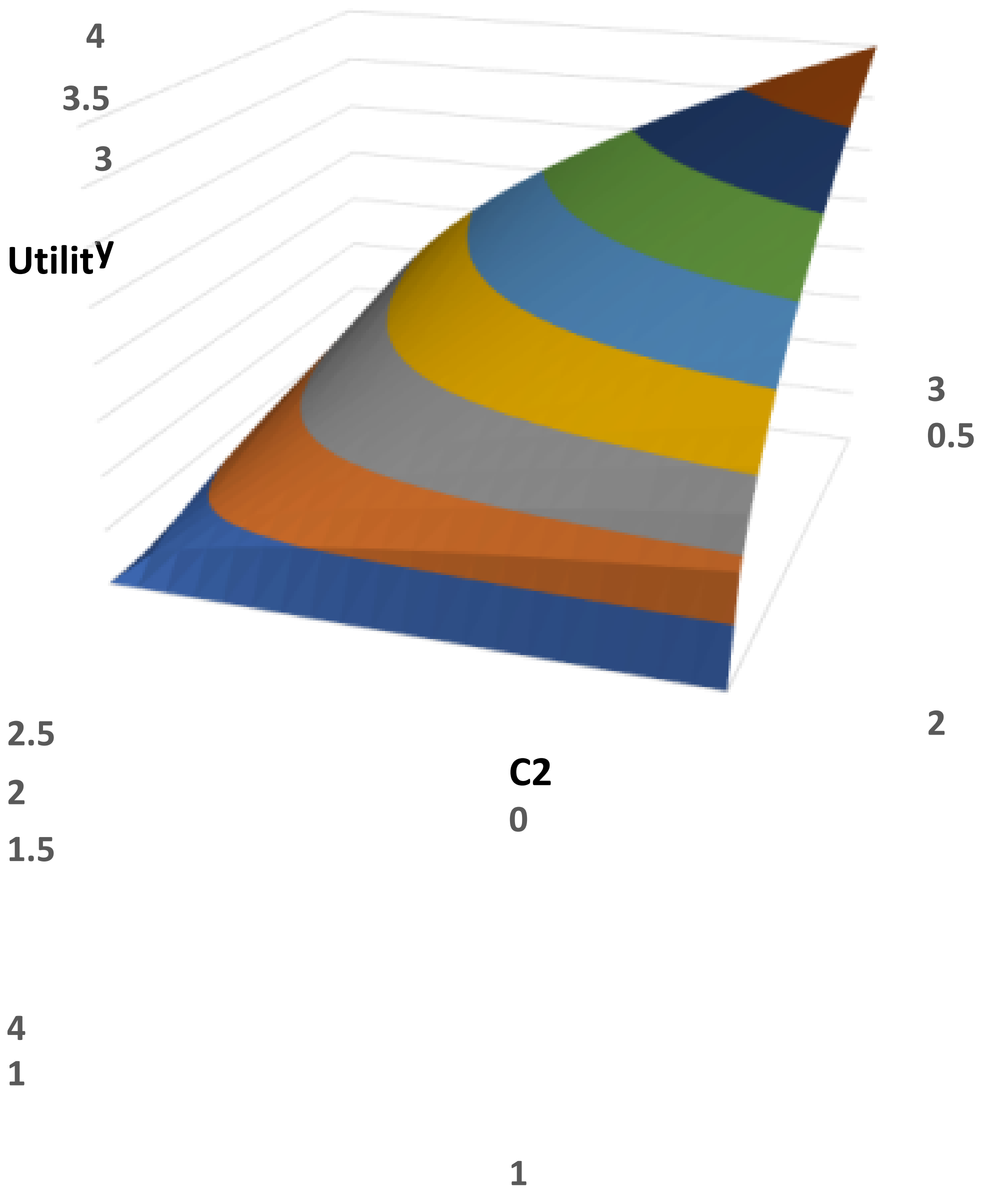
Some observations from the concrete example:

The total surplus in N1 increases from 8 to 9, when the free trade solution is replaced by the optimized solution. The consumer surplus is reduced from 8 to 4.5 and the tariff revenue increases from 0 to 4.5. In the real world, it is quite possible that the tariff revenue is not completely redistributed to the consumers. In such cases, it is quite possible that the consumers do not benefit from the tariff, even if the total surplus is maximized via the tariff. The free trade equilibrium price is 4, the optimal tariff is 1.5, The world price in the new equilibrium when the optimal tariff is applied is 3.5 and the price in N1 when the optimal tariff is applied is 5. The optimal tariff is 37.5 % of the free trade equilibrium price and the optimal tariff is 42.9 % of the world price in new equilibrium when the optimal tariff is applied.

3. General equilibrium analysis.

Now, the derivations are generalized, in the sense that nonlinearities are introduced and that the prices are endogenously determined. Furthermore, more than one nation can make decisions that affect the solutions and results for all involved parties. First, the free trade solution is derived. This represents the international trade equilibrium based on optimized production and optimized consumption.

Figure 14 illustrates a typical utility function in one nation. The utility is a function of the consumption levels, c1 and c2, of two kinds of products, x1 and x2. Figure 15 shows the level curves of the utility function.



0 0.4 0.8 1.2 1.6 2 2.4 2.8

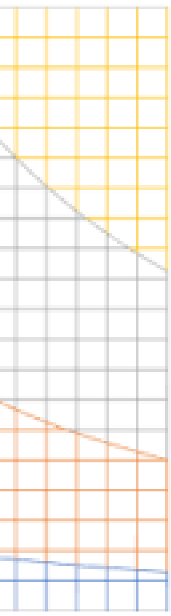
3.2

3.6 4

0

0-0.5 0.5-1 1-1.5 1.5-2 2-2.5 2.5-3 3-3.5 3.5-4

Figure 14. Typical utility function. $U = c_1^{\alpha_1} c_2^{\alpha_2}$. In this illustration, $\alpha_1 = \alpha_2 = \frac{1}{2}$.



4

3.6

3.2

2.8

2.4

C2

2

1.6

1.2

0.8

0.4

0

0 0.4 0.8 1.2 1.6 2 2.4 2.8 3.2 3.6 4

C1

0-1 1-2 2-3 3-4

Figure 15.

Level curves of the utility function. (Iso utility

curves.) *Nonlinear analysis:*

The utility function $_{,1}\,_{,},\ldots,\,_{i\,i}c\;c\;_{\eta}.$ selection of the consumption levels,

$U_i$of nation *i*, (79) is maximized via the

$_{\alpha\,\alpha}{}^{\eta} = = \times \times$

$_{\eta\,\eta}\kappa^{(79)}\;(\;)^{,1\,,}\max\;,.,\;\ldots\;_{,1\,,\,,1\,,}$

$_{i}{}^{i}U\;U\;c\;c\;c\;c\;_{i\,i\,i\,i\,i\,i}$

The consumption budget must be satisfied. This means that the total cost of consumption must not exceed the budget, B_i. $p$denotes price of product *j*. $_j$

$_{1\,,1\,,}\ldots\;p\;c\;p\;c\;B\;_{i\,i\,i} + + \le\;_{\eta\,\eta}(80)\;1\;\kappa_i$

= .

In the illustrated model, the sum of the exponents is 1, compare (81),

and η

$\sum{}^{=}(81)\;\alpha$

$1\;_{i\,j}$

$_{,}$

$j \quad = \quad 1$

The Lagrange function of consumption optimization is found in (82).

$_{\alpha\,\alpha}{}^{\eta} = \times \times + - - - \kappa\;\lambda\;_{\eta\,\eta\,\eta}(82)\;_{i}{}^{i}L\;c\;c\;B\;p\;c\;p\;c\;_{i\,i\,i\,i\,i\,i\,i}$

$^{,1\,,}$

$(\;)$

$_{,1\,,\,1\,,1\,,}\cdots\;\cdots$

The first order derivatives are (83) to (85).

dL $B\ p\ c\ p\ c$

i

$\lambda = - - -$ (83)

$i\ i\ i$ $1\ ,1\ , \cdots$

d $\eta\ \eta$

i

α

$dL\ U$

$i\ i\ i$

$,1$

$= - \lambda$(84)

$i\ ,$

η

$\ldots$

$dc\ c$ α p

$i\ i$ 1

$,1\ ,1$ i

$dL\ U\ i\ i$

$= - \lambda$(85) p

$dc\ c\ i\ i$ $,\ ,$ $\eta\ \eta$ η i

The first order optimum conditions of an interior solution are reported in (86).

dL $B\ p\ c\ p\ c$

$= - - - - =$

d i $\lambda\ i$ $\ldots\ 0$ $i\ i\ i\ 1\ ,1\ ,$ $\eta\ \eta$

$dL\ U\ \alpha$

$= - =\ i\ i\ i$

$,1$

$dc\ c\ i\ i$

$,1\ ,1$

$dL\ U\ \alpha$

1

p λ_i (86)

0

$= - = |$

$i\ i_i$

$,$

η λ_i

$dc\ c_{i\,i,,\,\eta\eta}$ p

η

0

Determination of the optimal solution and the demand functions:

From (86) we get (87) and (88).

α

$dL\ U$

$1,...,\ ,\ 1,...,\ {}^{i\,j\,i}{}_{i}$

$,$

$= - \forall \in \in \lambda\ \eta_{(87)}\ p\ i\ I\ j$

$dc\ c_{i\,j\,i\,j,,}$ $j\ i$ $\{\ \}\ \{\ \}$

$(\ )\ (\ )\ |\ ||\ |= \Rightarrow - = |\ ||\ |(\ )\ (\ )^{(88)}$

α

$dL\ U$

$0\ 0\ {}^{i\,j\,i}{}_{i}$

$,$ λ

p

$dc\ c_{i\,j\,i\,j,,}$ $j\ i$

Equation (88) leads to (89) and (90).

U

α

$i\ j\ i$

$,$

$= \lambda_{(89)}$

$p_{j\,i}$

c

$i\ j$

$,$

α U

$i\ j\ i$

$,$

c

$=_{(90)}\ {}^{i\,j}$

$,$ p λ

$j\ i$

Equation (86) gives (91).

$$\eta \quad = \sum \quad (91)$$

$$B_{ij} \, p_{ij} \, c \quad , \quad j = 1$$

Equations (90) and (91) lead to (92).

$$\left(\right) \quad {}^{\eta}\alpha \quad U \quad = \Big| \Big|\Big| \Big| \left(\right) \sum \quad (92)$$

$$i\,j\,i \quad , \quad B_{ij}\, p \; p \quad \lambda_{j\,j\,i} = \quad 1$$

Equation (92) leads to (96) via the steps in (94) and (95).

$$U \quad {}^{\eta}\alpha \quad = \sum \quad (93)$$

$$i\,j\,i \quad , \quad B_i \quad \lambda_{j\,i} = \quad 1$$

$$U \quad \eta \quad = \sum \quad (94)$$

$$i\,i\,j \quad B \quad i \quad \alpha \quad \lambda_{=\,j\,i} \quad , \; 1 \quad \left(\right) \quad - \; 1$$

$$= = \; = \Big| \Big| \; \eta\,\eta \; \left(\right) \sum \; \sum \quad (95) \quad U_{B\;B\;for}$$

$$\lambda_i \quad \alpha\;\alpha_{j\,j\;i\,=\,=} \quad 1 \quad , \, , \; i\,i\,j\,i\,i\,j \quad 1\;1$$

$$i\,i\,j \quad B \quad \alpha$$

$$c_{i,j} = \frac{\alpha_{i,j} B_i}{p_j} \quad \forall i,j \tag{96}$$

<u>Observation:</u>

The optimal demand (or consumption) function, in nation i, for product j, is proportional to the consumption budget and to the respective utility function exponent. It is inversely proportional to the product price.

Let us consider one of the nations and two products. Compare (96). Here, in (97), the notation is simplified.

$$c_i = \frac{\alpha_i B}{p_i} \tag{97}$$

Consider the optimal consumption levels of two particular products, 1 and 2, as seen in (98) and (99).

$$c_1 = \frac{\alpha_1 B}{p_1} \tag{98}$$

$$c_2 = \frac{\alpha_2 B}{p_2} \tag{99}$$

In (99) we find that the optimal consumption ratio is a function of the relative prices. The optimal consumption ratio may also be considered as a function of the ratio of the exponents in the utility function.

$$\frac{c_2}{c_1} = \left(\frac{\alpha_2}{\alpha_1}\right)\left(\frac{p_1}{p_2}\right) \tag{100}$$

Optimal production:

The analysis should be as general as possible. However, it is necessary to specify some function form that easily captures the essential properties of the problems. The type of function used to define the production possibility frontier (101) has properties that correspond to classical models of international trade theory.

In a nation, the production possibility frontier is:

$$(m_1 x_1)^2 + (m_2 x_2)^2 \le 1, \quad x_1 \ge 0, \quad x_2 \ge 0 \tag{101}$$

The parameters m_1 and m_2 may be different in different nations. The production volumes of products 1 and 2, produced in N1, are x_{11} and x_{12}. The production volumes of products 1 and 2, produced in N2, are x_{21} and x_{22}.

The production possibility frontier in N1 is (102).

$$(m_{11} x_{11})^2 + (m_{12} x_{12})^2 \le 1, \quad x_{11} \ge 0, \quad x_{12} \ge 0 \tag{102}$$

The production possibility frontier in N2 is (103).

$$(m_{21} x_{21})^2 + (m_{22} x_{22})^2 \le 1, \quad x_{21} \ge 0, \quad x_{22} \ge 0 \tag{103}$$

Production possibility frontiers:

If trade should be of interest to the different countries, they should have production possibility frontiers with different properties. Figure 16 and Figure 17 shows how this can be obtained via the functional forms present in equations (102) and (103). In Nation N1, we have the parameters (m_{11}, m_{12}) are (1/16, 1/4), and in Nation N2, the parameters (m_{21}, m_{22}) have the values (1/4, 1/16).

3. 1

5 0.

3 5

2. 0

5 0.1

x2 2

4.

1.

5

5

4

0.7 1.1 1.5 1.9

0.3 0.5 0.9 1.3 1.7

X1

Figure 16.

Production possibility frontier in N1.

2.5

2

5

X2 1

1

.

0.5 0 1 2 3 4 5 X1

0

Figure 17.

Production possibility frontier in N2.

Production optimization:

The consumption budgets that can be used for consumption optimization, are optimized in the different nations. This is done via optimization of the production levels. The problem defined in (104) and (105), is the profit maximization problem in an arbitrary nation. The objective function π denotes the profit, which later is used as the consumption budget. In the later parts of this paper, the prices (p_1, p_2) can be defined as world market prices or prices affected by the different tariffs.

$$\max_{x_1, x_2} \pi = p_1 x_1 + p_2 x_2 \tag{104}$$

$$s.t. \quad m_1 x_1^2 + m_2 x_2^2 \le 1, \quad x_1 \ge 0, \quad x_2 \ge 0 \tag{105}$$

The Lagrange function of the production optimization problem is (106).

$$L = p_1 x_1 + p_2 x_2 + \lambda\left(1 - m_1 x_1^2 - m_2 x_2^2\right) \tag{106}$$

The Karush- Kuhn- Tucker conditions are applied. With full production according to the production possibility frontier, and strictly positive production of all kinds of products, the equations (107) are satisfied.

$$\begin{cases} \dfrac{dL}{d\lambda} = 1 - m_1 x_1^2 - m_2 x_2^2 = 0 \\ \dfrac{dL}{dx_1} = p_1 - 2\lambda m_1 x_1 = 0 \\ \dfrac{dL}{dx_2} = p_2 - 2\lambda m_2 x_2 = 0 \end{cases} \tag{107}$$

λ

dx ^{2} $^{2\,2\,2}$

From (107) we get (108), which leads to (109) and (110).

x p_1

$=$

$\frac{1}{2}$ m_1 λ (108)

$= x$ p_2

$\frac{2}{2}$ m_2 λ

p_2

$x\,m$ $_{2}{}^{2}$

2 λ

$=$ (109)

$x\,p$ $_{1}\,_{1}$

2 m_1 λ

$x\,m\,p$

$=$ (110)

$_{2\,1\,2}$

$x\,m\,p$

$_{1\,2\,1}$

In (111), the relative price is defined.

$=$ (111) ^{1}

Let R
p

p ^{2}

The optimal production ratio is a function of the relative prices, as shown in

(112). $x\,m$ R

$_{2\,1}{}^{1}$

$$= (112)\; x\; m \quad {}_{1\,2}$$

Optimal production ratios in the two nations:

Now, the concrete numerical values of the production possibility functions in nations N1 and N2 are used. Compare Figures 16 and 17.

In Nation N1, we have equation (113) and obtain the function (114).

$$1\ 1\ \max\ ,\max\ 2,\ 4\ ,\ ,\quad x\ x\ m\ m \quad (\ ) \quad = \;\Rightarrow\; = |\ | \ (\ ) \quad (113)$$

$$(\ {}_{1\,2\,1\,2}\ )\ (\ )\ (\ ) \quad 4\ 16 \quad x \quad {}_{2}{}^{1}$$

$$= \quad x_1 \quad 4 \quad R \quad (114)$$

In Nation N2, we have equation (115) and obtain the function

$$(116).\ 1\ 1\ \max\ ,\max\ 4,\ 2\ ,\ ,\quad x\ x\ m\ m \quad (\ ) \quad = \;\Rightarrow\; = |\ | \ (\ ) \quad (115)$$

$$(\ {}_{1\,2\,1\,2}\ )\ (\ )\ (\ ) \quad 16\ 4 \quad x \quad 1 \quad {}_{2}{}^{1}$$

$$= \quad x^{1} \quad 4 \quad R \quad (116)$$

Figure 18 shows how the optimal consumption ratio function and the optimal production ratio functions can be combined to determine the optimal combinations of relative prices and relative production volumes in the two nations without trade.

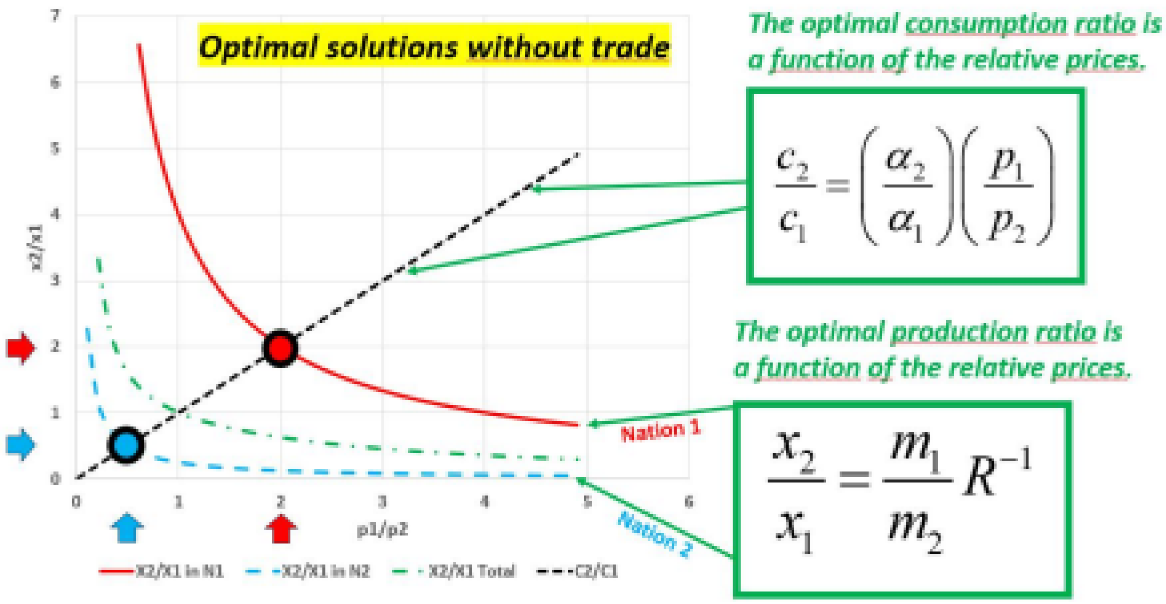


Figure 18.

In the absence of international trade, the optimal combinations of production and consumption ratios, and the relative prices, can be determined via the graph.

Figure 19 shows how the optimal consumption ratio function and the optimal production ratio functions can be combined to determine the optimal combinations of relative prices and relative production volumes in the two nations with trade.

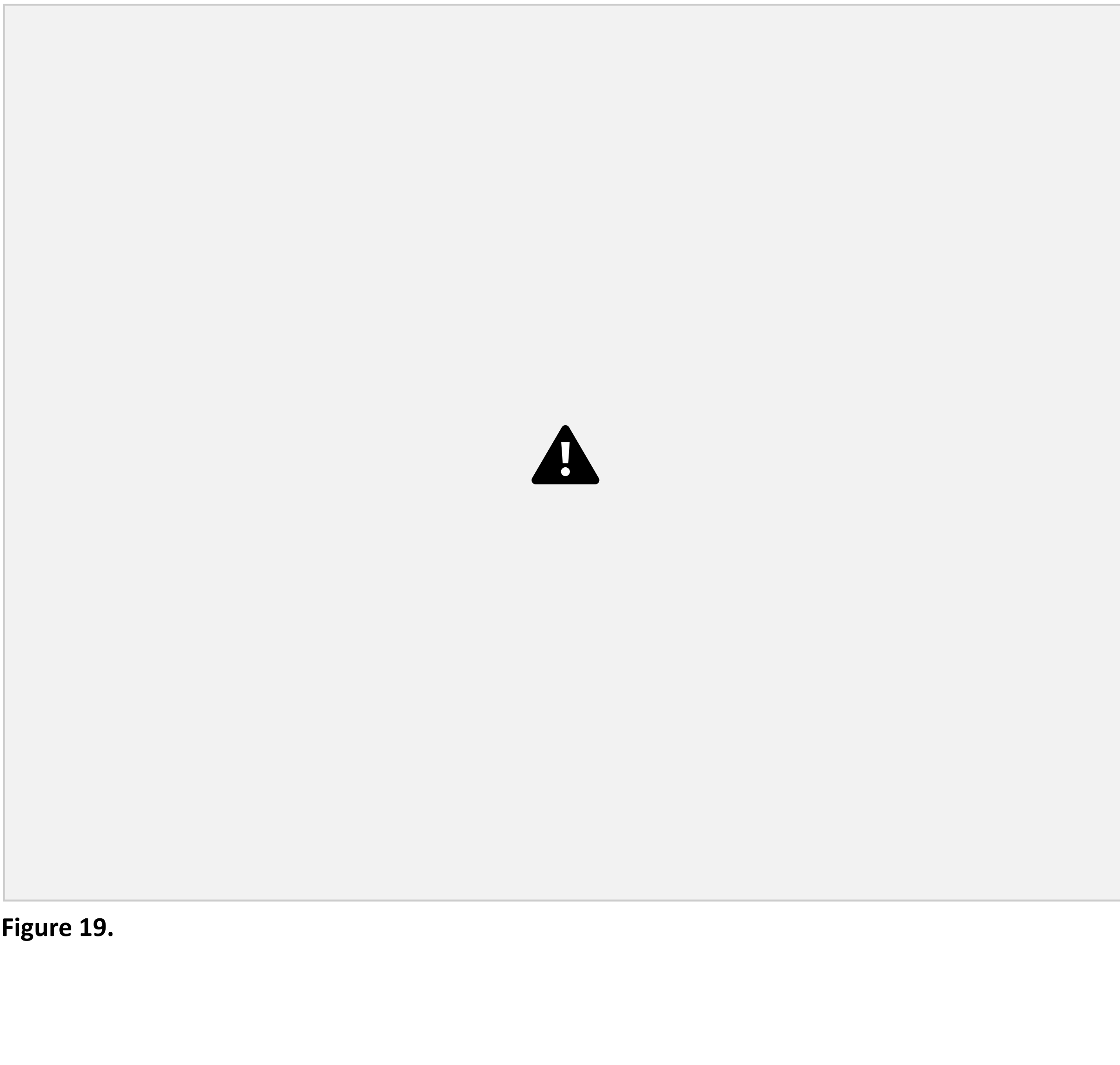


Figure 19.

The equilibrium relative price in the world, with free trade, is found via the point denoted ”World Total”. This equilibrium relative price, which is ”1” in the example, determines the optimal relative production levels in the different nations. ”x2/x1” increases in Nation 1 and decreases in Nation 2 when free trade is introduced.

If free trade is introduced, the relative prices in the two countries converge to the same value. This is illustrated in Figure 20.

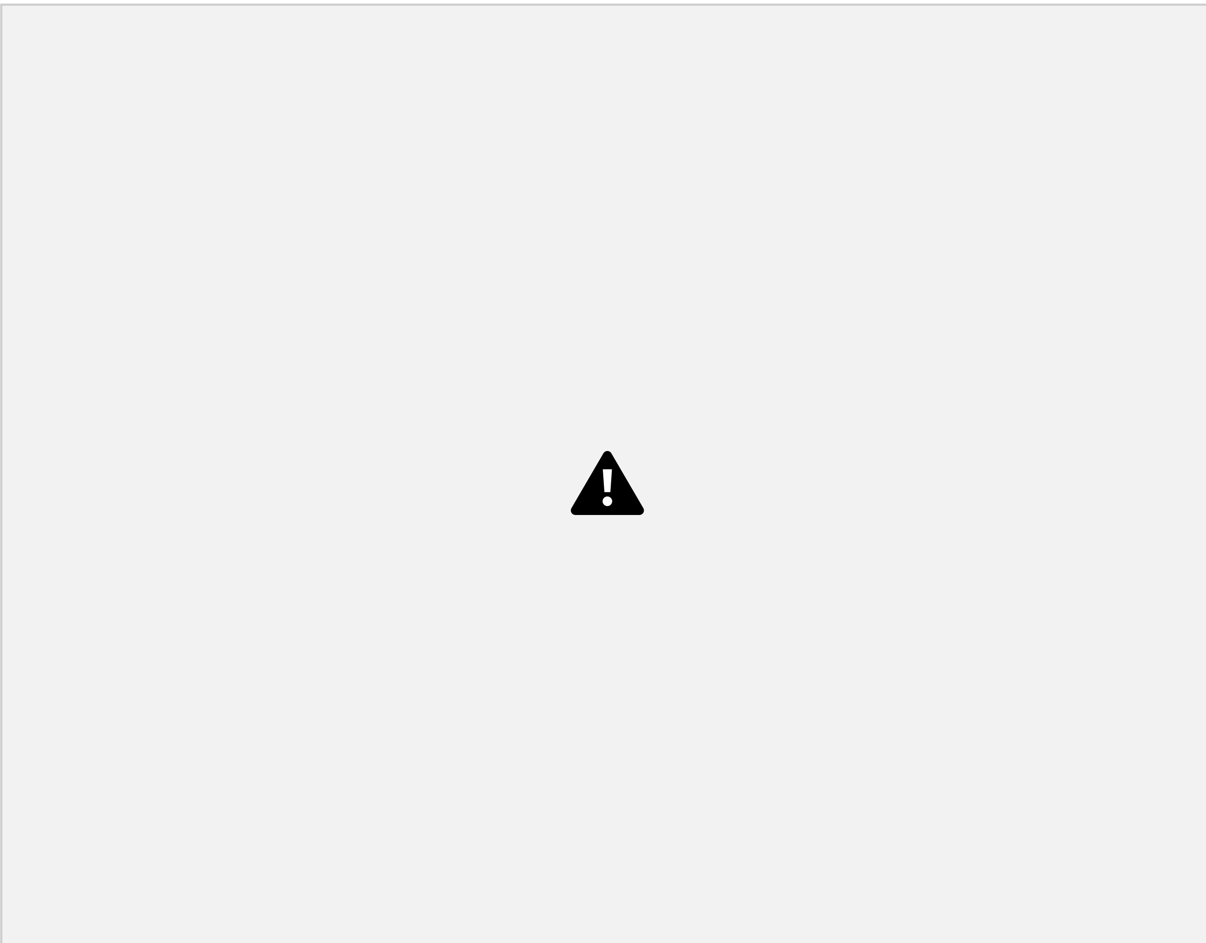


Figure 20..

With free trade, we obtain product price equalization, which also leads to a determination of the relative product ratio at the international level.

The optimal absolute production levels, with and without free trade, in the different nations, are graphically determined in Figures 21 and 22, based on the optimal relative production volumes and the production possibility frontiers.

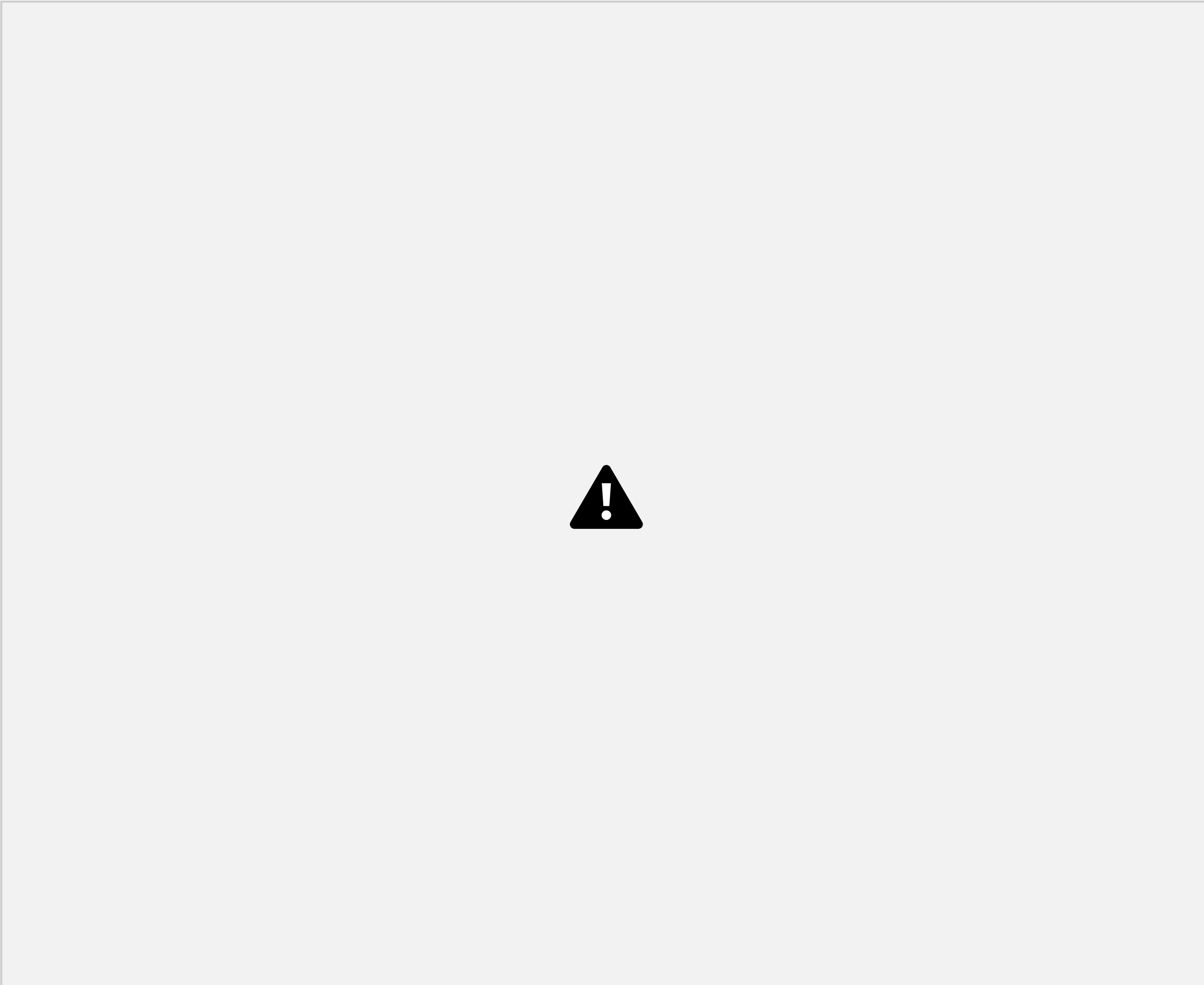


Figure 21.

Nation N1: Free trade leads to increased specialization.

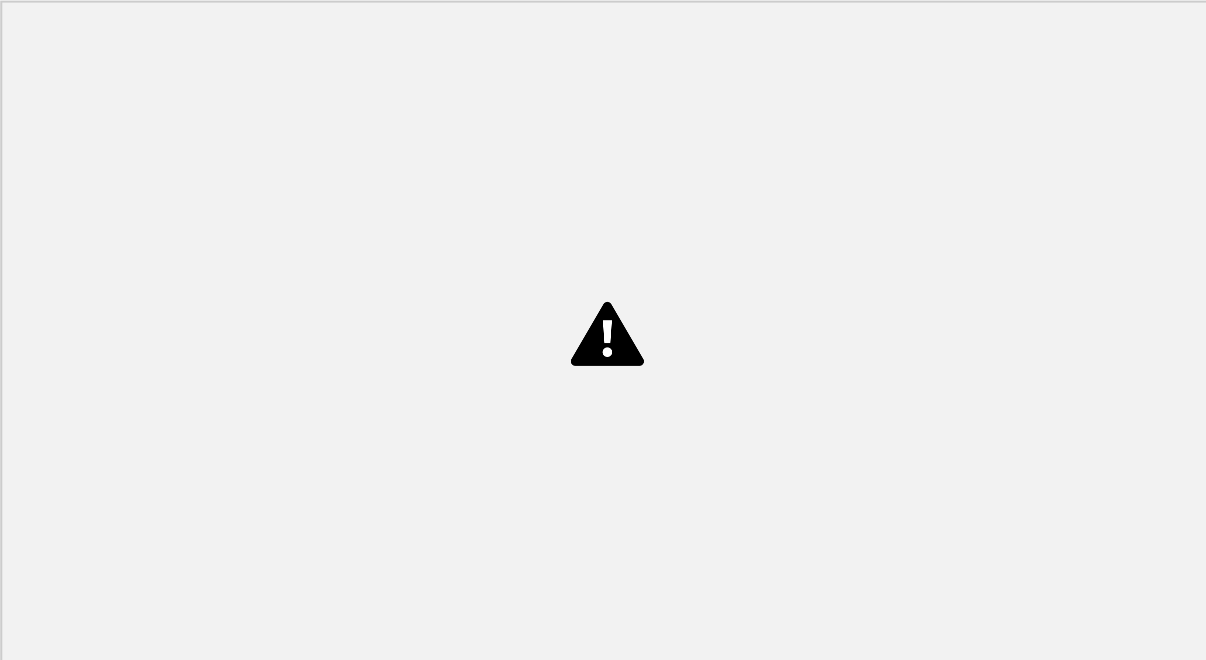


Figure 22.

Nation N2: Free trade leads to increased specialization.

Mathematical determination of the optimal absolute production volumes in the different countries:

In N1, the optimal production volumes of the different products are determined in (117) to (131).

The production possibility frontier is (117) and the optimal product ratio is (118). $m\ x\ m\ x\ _{1\ 1\ 2\ 2}$ +

=1 (117)

$2\ 2$

( ) ( )

$x\ m$

$k\ R$

$2\ 1\ ^{1}$

$-$

= =(118) $x\ m$

$1\ 2$

(118) gives (119).

$_{2\ 1}x\ k\ x$ =(119) (117) and (119) give (120).

$^{2\ 2}m\ x\ m\ k\ x\ _{1\ 1\ 2\ 1}$ + =1 (120)

( ) ( )

Equations (121) to (123) follow from (120).

$_{1\ 2\ 1}m\ m\ k\ x$ + =1 (121) ( )( )$^{2}\ _{2}$

1

x
$m\ m\ k$

= + (122)

$^{1}\ _{2}$

( )
$_{1\ 2}$

1

x
$m\ m\ k$

= + (123)

$^{1}\ _{2}$

( )
$_{1\ 2}$

The optimal absolute production volume of product 1 in Nation 1 is (124).

( )^{1}

$_{1\ 1\ 2}x\ m\ m^{2}\ _{2}$
k

= + (124) Equation (119) gives (125).

= (125) $_{1\ 2}x\ k\ x$

1

(117) and (125) lead to (126).

+ = (126)

$^{2\ 2}\ _{1}$

( ) ( )

$_{1\ 2\ 2\ 2}m\ k\ x\ m\ x\ 1$

(126) is transformed to (127), (128), (129) and (130).

+ = (127) $_{2\ 2}m\ k\ x\ m\ x\ _{1\ 2\ 2\ 2}1$

( ) ( ) $_{2}$

+ = (128) ( )( )$^{2}\ _{2}$

$_{1\ 2\ 2}m\ k\ m\ x\ 1$

x ^{2} 1
$m\ k\ m$ $^{-}$

$=_{+}$ (129)

^{2} $_{2}$

$(\ )$
$_{1\ 2}$

1
x
$m\ k\ m$ $^{-}$

$=_{+}$ (130)

^{2} $_{2}$

$(\ )$
$_{1\ 2}$

The optimal absolute production volume of product 2 in Nation 1 is (131).

$(\ )^{1}$

$(\ )$

^{2} $_{2}$

$_{2\ 1\ 2}\ x\ m\ k\ m$

$= +$ (131) In nation N2, the optimal production volumes of the different products are determined via the same logic as according to (117) to (131).

Figures 23, 24 and 25 illustrate how the optimal production in the two nations are affected by the relative price. They also show how the equilibrium prices without and with trade can be graphically determined.

mal

Opti prod

uctio N1
5
n in

4

3

2

1

0

X1 X2

Figure 23.
0 1 2 3 4 5 6 p1/p2

The optimal production levels in N1 as a function of the relative price. Without trade, the equilibrium relative price is approximately 4.

Opti n in
5 4 3

mal
N2
prod 2 1

uctio

0

X1 X2

Figure 24.
0 1 2 3 4 5 6 p1/p2

The optimal production levels in N2 as a function of the relative price. Without trade, the equilibrium relative price is approximately 0.25 .

5

Total optimal production (in N1 4

3

+

2

N2)

1

0

0 1 2 3 4 5 6 p1/p2

X2_Tot X1_Tot

Figure 25.

7

6

The optimal total production levels in N1 and N2 as a function of the relative price. With trade, the equilibrium relative price is approximately 1.

Figures 26 and 27 show how the total production in the two countries are affected by the relative price, and where optima may be found.

5

Total Production (X1 + X2) Average per Nation

4.5

1

4

0.5

3.5

0

3

2.5

Figure 26.

2

0 1 2 3 4 5 6 P1/P2

1.5

The average per nation, optimal total ($x_1 + x_2$) production levels in N1 and N2, as a function of the relative price. This average is maximized when the relative price is approximately 1.

Total

Production (X1

+ X2)
5

4.5 4

3.5 3

2.5 2

1.5 1

0.5 0

0 2 4 6 P1/P2

TotProd Average per Nation

TotProd in N1

TotProd in N2

Figure 27.

The total production levels per nation, and the average per nation, optimal total ($x_1 + x_2$) production levels in N1 and N2, as a function of the relative price. These functions are all maximized when the relative price is approximately 1.

Table 1 shows the optimized consumption levels in the different nations, with free trade and without trade.

Table 1.

Consumption levels in the optimized system in equilibrium:

	Nation 1 No trade	Nation 1 Free trade	Nation 2 No trade	Nation 2 Free trade
c1	1.414	2.236	2.828	2.236
c2	2.828	2.236	1.414	2.236

Table 2 shows the optimized consumption utility levels in the different nations, with free trade and without trade. In both nations, the utility levels are strictly higher with free trade than without trade. Observation: If the utility function exponents are high (2.0), then the ratio of the value of the utility function with free trade divided by the value of the utility function without trade, is higher than if the utility function exponents are low (0.5).

Table 2.

Utility levels in the optimized system in equilibrium:

	Nations 1 and 2, No trade Utility function exponents = (0.5, 0.5)	Nations 1 and 2, Free trade Utility function exponents = (0.5, 0.5)	Nations 1 and 2, No trade Utility function exponents = (2.0, 2.0)	Nations 1 and 2, Free trade Utility function exponents = (2.0, 2.0)
u1	2.000	2.236	16	25
u2	2.000	2.236	16	25

Observations of the utility levels and trade:

The utility levels in the optimized system equilibrium increase when free trade is introduced. The relative utility increase is sensitive to the values of the exponents in the utility functions. If these exponents are low, (= 0.5), then the utility levels increase by 11.80 % when free trade is introduced. If these exponents are high, (= 2.0), then the utility levels increase by 56.25 % when free trade is introduced.

4. The optimal tariffs without and with cooperation in general equilibrium

In this section, the trade solution with tariffs is studied. The trade equilibrium is determined based on optimized production and optimized consumption. One or both nations introduce tariffs on imports. Dynamic, nonlinear, non-zero sum, game theory is applied. The Nash equilibrium and the cooperative
solution are determined. The optimal tariffs and utility changes in both nations, with and without cooperation, are derived.

In several equations, the characters and explicit notation used in the numerical appendix are applied, which makes it easier to follow the logic of the appendix. The following variables are introduced: T1 is the relative tariff in Nation 1 on imports of product 2 and T2 is the relative tariff in Nation 2 on imports of product 1. RELP is the relative price p2/p1, where p1 and p2 are the world market prices of the products 1 and 2. p3 is the price of product 1 inside Nation 1 as a function of the world market price p1 and Tariff_1 in Nation 1 on imported product 1. p4 is the price of product 2 inside Nation 2 as a function of the world market price p2 and Tariff_2 in Nation 2 on imported product 2.

Equations (132) to (135) show how the introduced variables are connected.

Tariff_1 = p1 * T1 (132)

p3 = p1 + Tariff_1 (133) Tariff_2 = p2 * T2 (134) p4 = p2 + Tariff_2 (135)

With tariffs, the production and consumption optimization problems, inside the borders of the different nations N1 and N2, are based on different relative prices. The relevant relative prices in N1 and N2 are defined in (136) and (137).

RELP_N1 = p2 / p3 (136) RELP_N2 = p4 / p1 (137)

The optimized production decisions in the different nations, when prices may be affected by tariffs in the different nations, are reported in equations (138) to (141). These four equations are consistent with the earlier derived equations (124) and (131). x11, x12, x21 and x22 are the optimal production levels in N1 and in N2.

x11 = (1 / (m11 * (m11 / m12 * RELP_N1 ^ (2) + 1))) ^ .5 (138) x12 = (m12 ^ (-1) – 1 / (m11 * RELP_N1 ^ (2) + m12)) ^ .5 (139) x21 = (1 / (m21 * (m21 / m22 * RELP_N2 ^ (2) + 1))) ^ .5 (140) x22 = (m22 ^ (-1) – 1 / (m21 * RELP_N2 ^ (2) + m22)) ^ .5 (141)

In the numerical model found in the appendix, the trade equilibrium is determined for large numbers of combinations of tariffs in the different countries.

For each relative tariff combination (T1, T2):

The consumption levels are functions of the consumption budgets in N1 and N2. The consumption budgets in N1 and N2 are functions of the tariff revenues. The tariff revenues are functions of the import volumes. The import volumes are functions of the consumption levels. For this reason, fix point iteration is used to determine the equilibrium. This is done via iteration in the i-loop, found in the appendix. Iterations with 100 steps give solutions very close to the equilibria.

For each combination (T1, T2):

In one loop, A, the relative prices are adjusted, until the total excess supply is zero (for every product).

In a second loop, B, within A, for each relative price level, fix point iteration is used to determine approximate solutions to all production levels, consumption levels, all consumption budgets, imported volumes and exported volumes. Furthermore, the utility levels in N1 and N2 are determined as functions of the optimized consumption levels. W1 and W2 are transformed utility functions in N1 and N2. The transformed utility functions make it easy to investigate how utilities change when tariffs are adjusted.

In Tables 3 and 4, the values of W1 and W2 are reported, as functions of the tariff combinations (T1, T2). You get the value of W1, if you take the utility U1 in N1 for the tariff combination (T1, T2) minus the utility U1 in N1 for zero tariffs, and multiply this difference by 10000. W2 corresponds to W1, but for nation N2.

Table 3.

Numerical approximation of the transformed utility function W1 in N1.

W1(T1,T2) = (U1(T1,T2)-U1(0,0))*10000

	T2(%)	0	20	40	60	80	100
T1 (%)							
0		0	-579	-1017	-1351	-1602	-1797
20		504	-158	-661	-1056	-1358	-1605
40		750	18	-551	-991	-1346	-1631
60		818	22	-599	-1086	-1475	-1793
80		765	-89	-751	-1280	-1704	-2050
100		629	-278	-982	-1542	-1989	-2361

Table 4.

Numerical approximation of the transformed utility function W2 in N2.

W2(T1,T2) = (U2(T1,T2)-U2(0,0))*10000

	T2(%)	0	20	40	60	80	100
T1 (%)							
0		0	501	748	825	766	626
20		-581	-158	13	23	-93	-275
40		-1018	-666	-551	-602	-754	-979
60		-1348	-1055	-993	-1086	-1283	-1543
80		-1602	-1360	-1348	-1477	-1704	-1990
100		-1798	-1603	-1629	-1794	-2051	-2361

Determination of an approximation W1(T1,T2):

Hypothesis: The function can be approximated by a multivariate Taylor polynomial of the second order. (A quadratic polynomial can be used.) This is found in (142).

$$W_1(T_1,T_2) = k_1T_1 + k_{11}T_1^2 + k_2T_2 + k_{12}T_1T_2 + k_{22}T_2^2 \quad (142)$$

Regression analysis based on the numerically determined values of W1(T1, T2):

When the hypothesis (142) and the observations reported in Table 3 are used, the estimated function (143) is obtained:

$$W_1 \approx 2399T_1 - 1820T_1^2 - 3102T_2 - 1279T_1T_2 + 1349T_2^2 \quad (143)$$

With computer characters, (144) corresponds to (143). This function is included in the numerical model in the Appendix.

W1 = 2399 * T1 – 1820*T1_2 – 3102*T2 – 1279*T1T2 + 1349*T2_2 (144)

More details about the regression analysis are reported in Tables 5 and 6. In Table 5 we note that the value of R is very close to 1. Table 6 shows that all the P-values are extremely low. For these reasons, we know that the estimated function (143) almost perfectly replicates the data in Table 3. This is very

important, since the continued derivations are based on the approximating function (143) and the numerically specified version (144).

Table 5.

Regression statistics part 1

Multiple R	0.999251068
R Square	0.998502696
Adjusted R Square	0.966051431
Standard Error	50.31183919
Observations	36

Table 6.

P-value

Regression statistics part 2

Coefficients Standard Error t Stat

Intercept 0 #N/A #N/A #N/A T1 2399.057995 72.00241976 33.31913015 7.68387E-26 T1_2 -1820.216135 78.26211912 -23.25794593 3.47836E-21 T2 -3101.7426 72.00241976 -43.07831057 3.15284E-29 T1T2 -1279.316822 58.32762742 -21.93329094 1.93636E-20 T2_2 1349.203508 78.26211912 17.23954734 1.94472E-17

Figure 28 shows the transformed utility function W1 in nation N1. In the yellow region, in particular if T2 is zero, N1 benefits from considerable tariffs. For large values of T2, W1 is strictly negative, for all values of T1. It is also clear that, for every possible value of T2, there is an optimal value of T1. As T2 increases, the optimal value of T1 decreases. We may imagine the W1 function as a mountain, with a mountain ridge. One part of the ridge is located above the ocean level, this is where T2 is low. At the

ocean level, the value of W1 is zero. For high values of T2, the mountain ridge is located below the determined by the value of T2.
surface of the ocean. The optimal value of T1 is always found on the mountain ridge. The location is

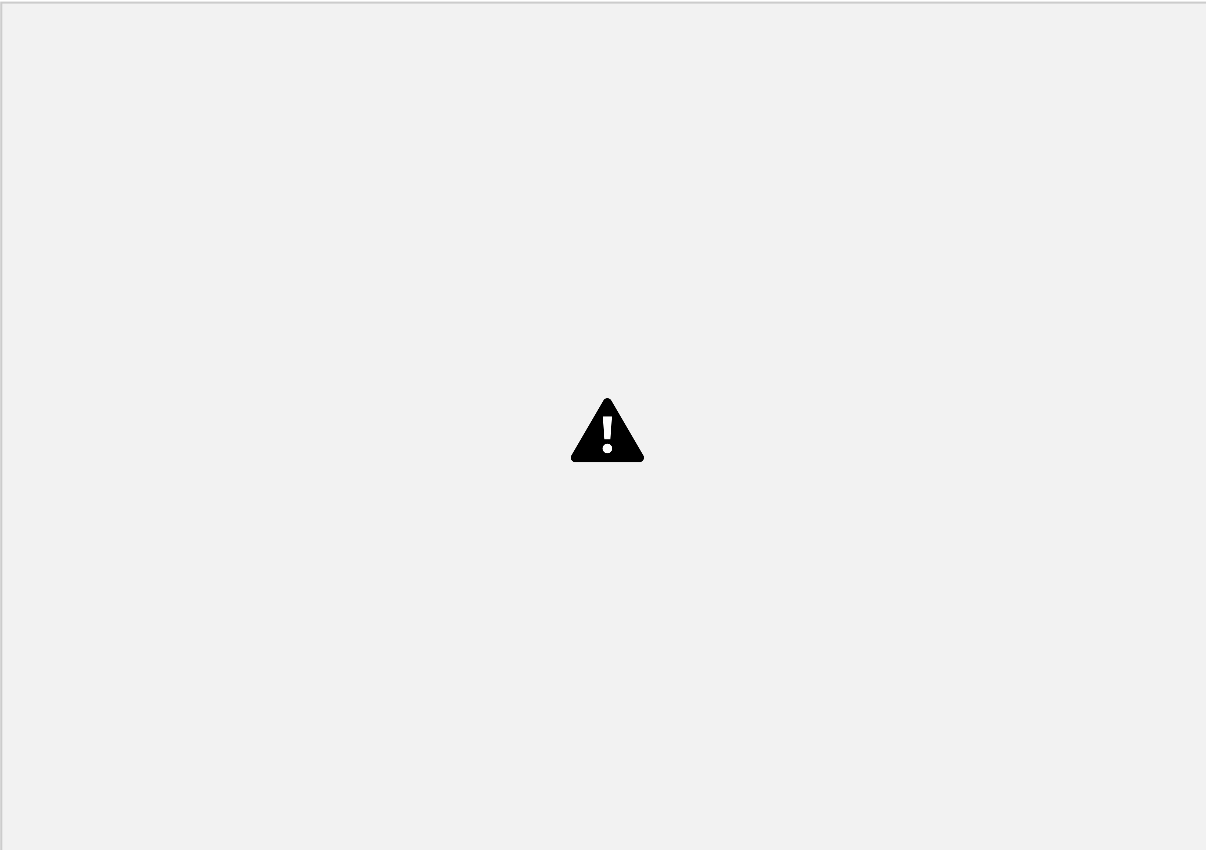


Figure 28.

Illustration of W1, the transformed utility function in N1, via the approximated function (143).

The approximated function W1 is now assumed to be exact (145).

$_{2}{}^{2}\, W\, T\, T\, T\, TT\, T\, _{1\,1\,1\,2\,1\,2\,2} = - - - + 2399\ 1820\ 3102\ 1279\ 1349$ (145) The first derivative of (145) with respect to T1 is (146).

$2399\ 3640\ 12_{79}\ \frac{dW}{}\ T\ T$

$_{1}$

$= - -$ (146)

$_{1\,2}$

dT

$_{1}$

Since the analytical background to the problem is exactly known, and symmetry is present, (145) leads to (147) and (148).

$$ {}^{22} W_{2}\; T\; T\; T\; TT\; T_{2221121} = - - - + 2399\; 1820\; 3102\; 1279\; 1349 \text{(147)}\; 2399\; 3640\; 1_{279}\; \frac{dW}{}\; T\; T $$

$$ = - - \text{(148)} $$

$$ {}_{21} $$

$$ dT_{2} $$

The first derivative of W1 with respect to T1 as a function of T1 and T2 is illustrated in Figure 29 in three dimensions. Compare (146). The graph clearly shows the region of T1 and T2 where it is optimal for nation N1 to increase the value of T1 and where it is optimal to decrease T1. Figure 30 shows the line where equation (146) equals zero, and where there is no reason for nation N1 to change the level of T1.

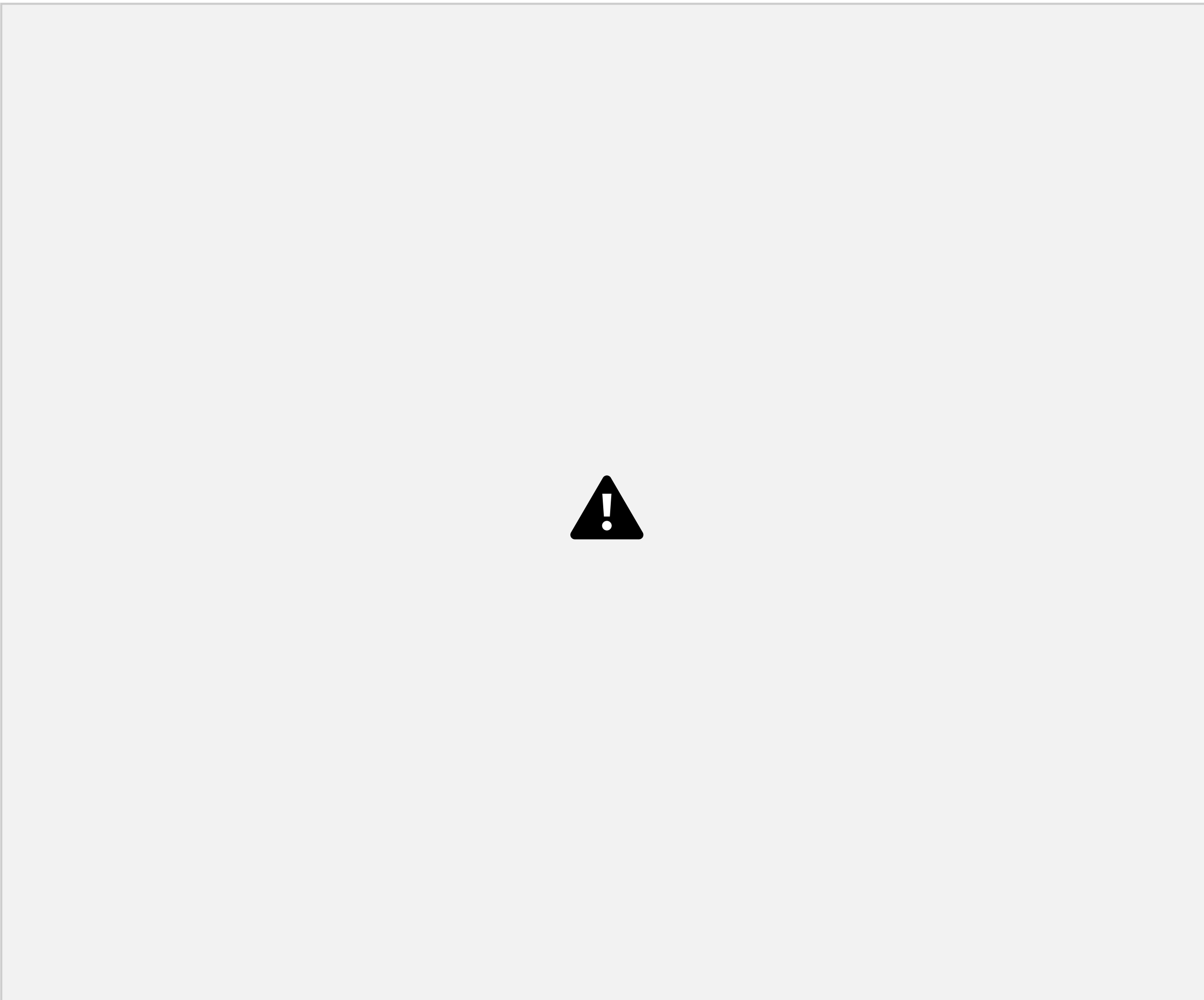


Figure 29.

Illustration of the first derivative of W1 with respect to T1 as a function of T1 and T2 in three dimensions. Compare (146).

Figure 30.

Illustration of the first derivative of W1 with respect to T1 as a function of T1 and T2 in two dimensions. Compare Figure 29 and equation (146).

Basic principles:

N1 maximizes W1 via the selection of T1. If the derivative (146) exceeds 0, it is rational to increase T1. If T2 = 0, it is always rational for N1 to increase T1 from zero. Then, the optimal value of T1 is determined according to equation (149).

$$\left(\frac{dW_1}{dT_1}\right) = 2399 - 3640T_1 = 0 \Rightarrow T_1 \approx 66\% \tag{149}$$

Hence, N1 finds it optimal to introduce a 66% tariff and deviate from free trade.

N2 maximizes W2 via the selection of T2. If the derivative (148) exceeds 0, it is rational to increase T2. If T1 = 0, it is always rational for N2 to increase T2 from zero. Then, the optimal value of T2 is determined according to equation (150).

$$\left(\frac{dW_2}{dT_2}\right) = 2399 - 3640T_2 = 0 \Rightarrow T_2 \approx 66\% \tag{150}$$

Here, N2 finds it optimal to introduce a 66% tariff and deviate from free trade.

Nash equilibrium:

N1 maximizes W1 via T1, conditional on the fact that N2 maximizes W2 via T2. See (151).

$$\frac{dW_1}{dT_1} = 2399 - 3640T_1 - 1279T_2 = 0 \tag{151}$$

N2 maximizes W2 via T2, conditional on the fact that N1 maximizes W1 via T1. Compare

(152).

$$\frac{dW_2}{dT_2} = 2399 - 1279T_1 - 3640T_2 = 0 \tag{152}$$

The equation system (153) is satisfied in the unique Nash equilibrium:

$$\begin{bmatrix} 3640 & 1279 \\ 1279 & 3640 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} 2399 \\ 2399 \end{bmatrix} \tag{153}$$

Cramer's rule and (153) gives the explicit Nash equilibrium tariff solutions (154) and (155).

$$T_1 = \frac{\begin{vmatrix} 2399 & 1279 \\ 2399 & 3640 \end{vmatrix}}{\begin{vmatrix} 3640 & 1279 \\ 1279 & 3640 \end{vmatrix}} = \frac{5664039}{11613759} \approx 0.48770 \approx 49\% \tag{154}$$

$$T_2 = \frac{\begin{vmatrix} 3640 & 2399 \\ 1279 & 2399 \end{vmatrix}}{\begin{vmatrix} 3640 & 1279 \\ 1279 & 3640 \end{vmatrix}} = \frac{5664039}{11613759} \approx 0.48770 \approx 49\% \tag{155}$$

Figure 31 illustrates the derivations that lead to the Nash equilibrium solution.

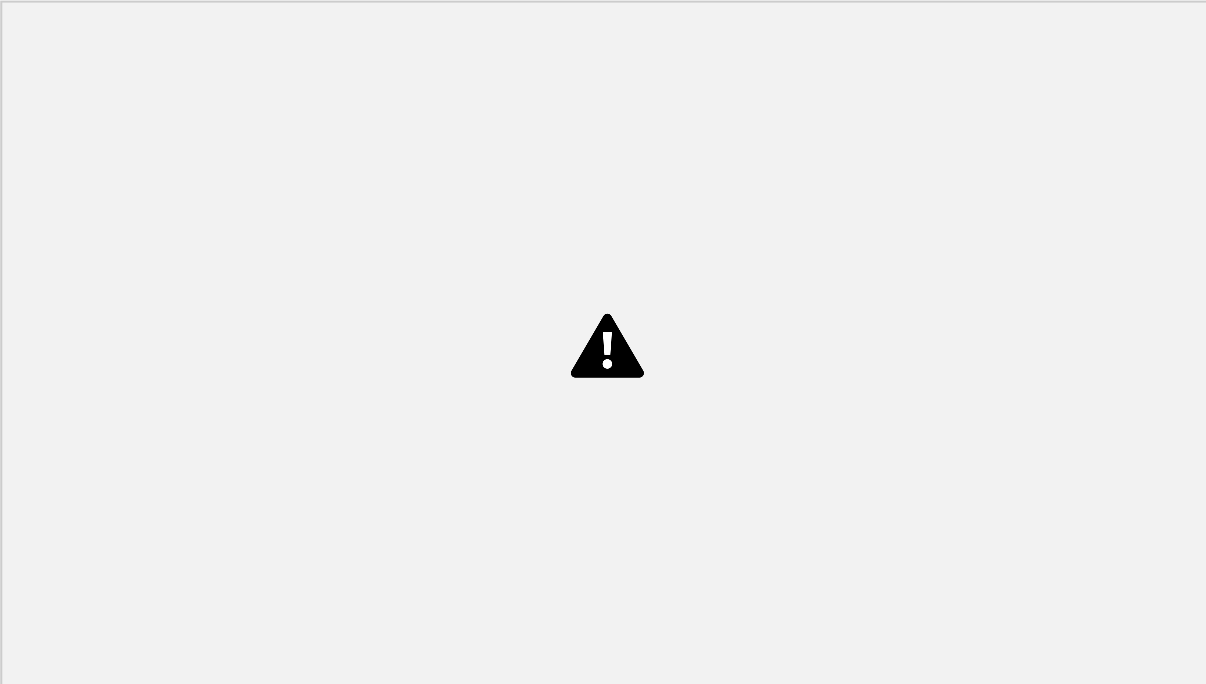


Figure 31.

Illustration of the Nash solution.

Figure 32 shows how the nations N1 and N2 should adjust the values of T1 and T2 respectively, to optimize the respective objective functions W1 and W2, in different situations.

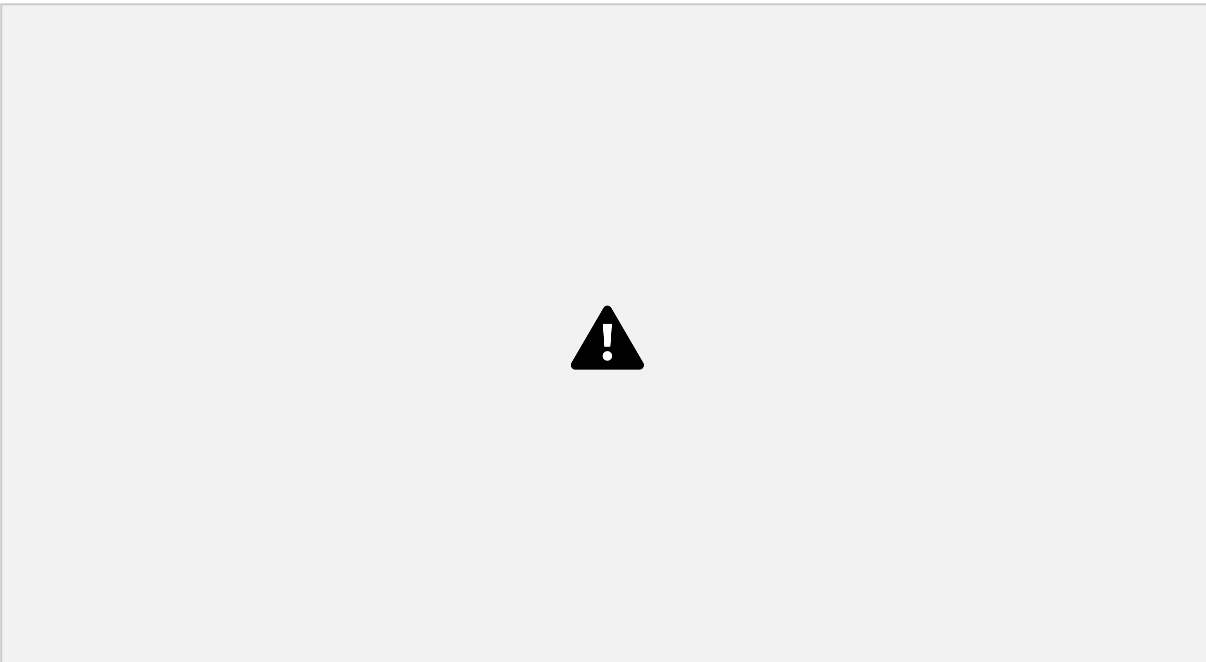


Figure 32.

Directions of optimal tariff changes. Blue arrows correspond to optimal changes of T1 and red arrows correspond to optimal changes of T2.

Optimal dynamic changes of the tariffs are illustrated in Figures 33 to 37, provided that the system starts with free trade without any tariffs. The first move is taken by nation N1, as shown in Figure 33

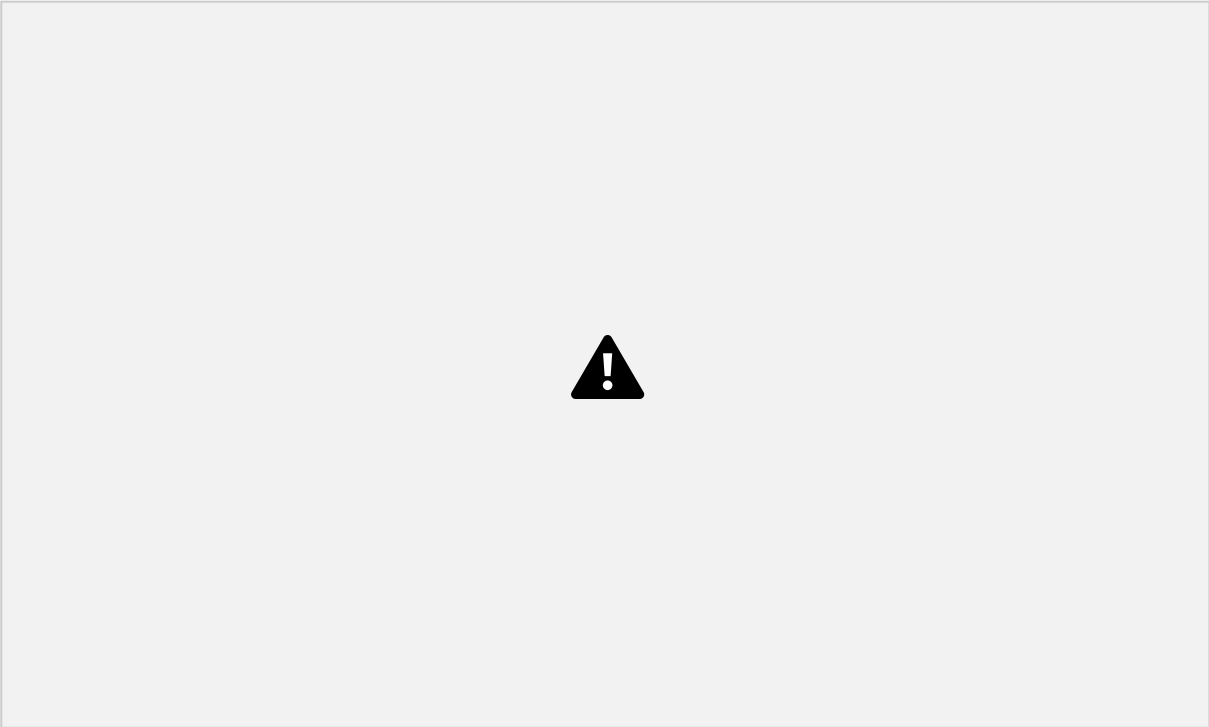


Figure 33.

Optimal dynamic tariff adjustment 1.

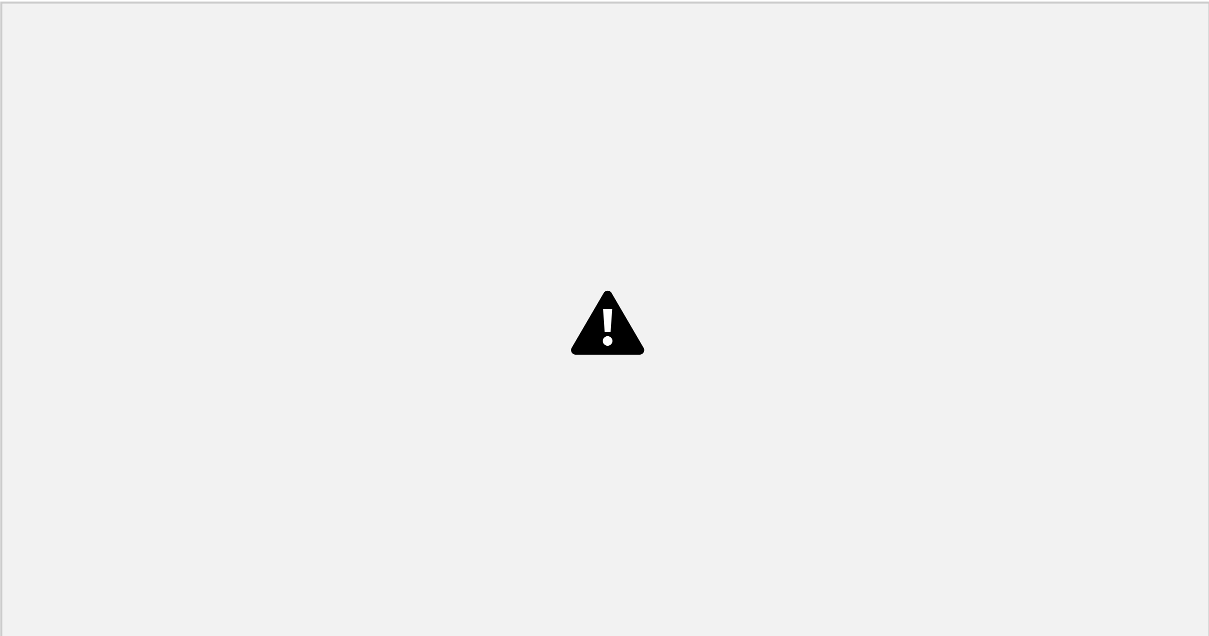


Figure 34.

Optimal dynamic tariff adjustment 2.

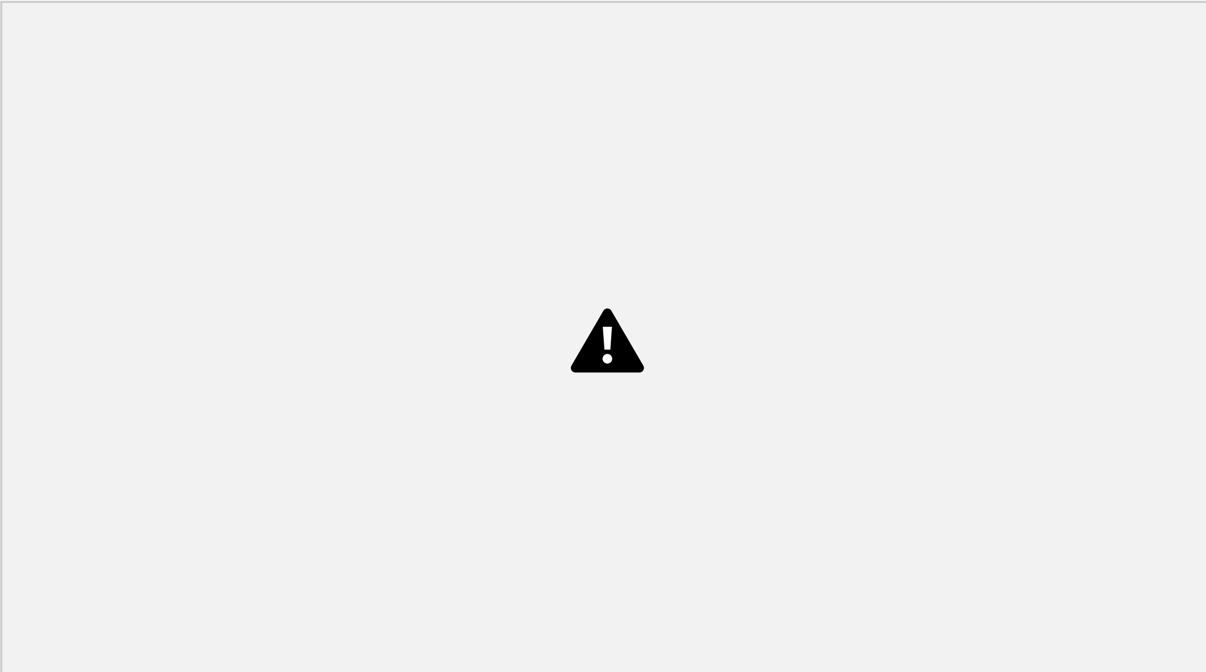


Figure 35.

Optimal dynamic tariff adjustment 3.

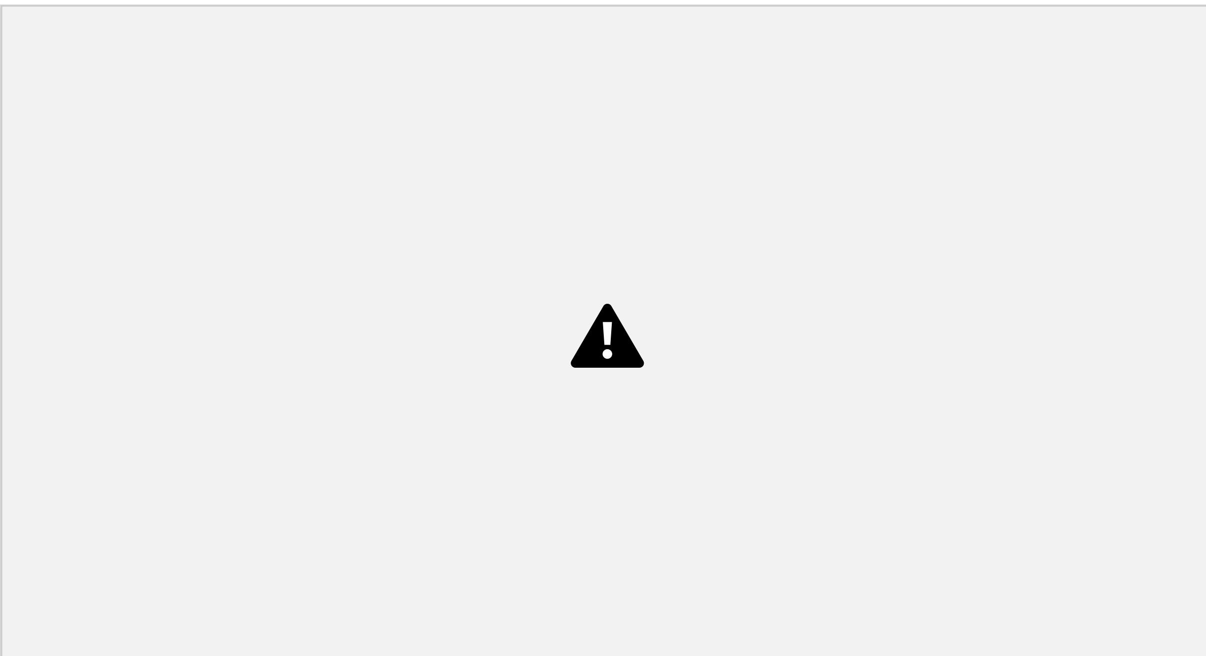


Figure 36.
Optimal dynamic tariff adjustment 4.

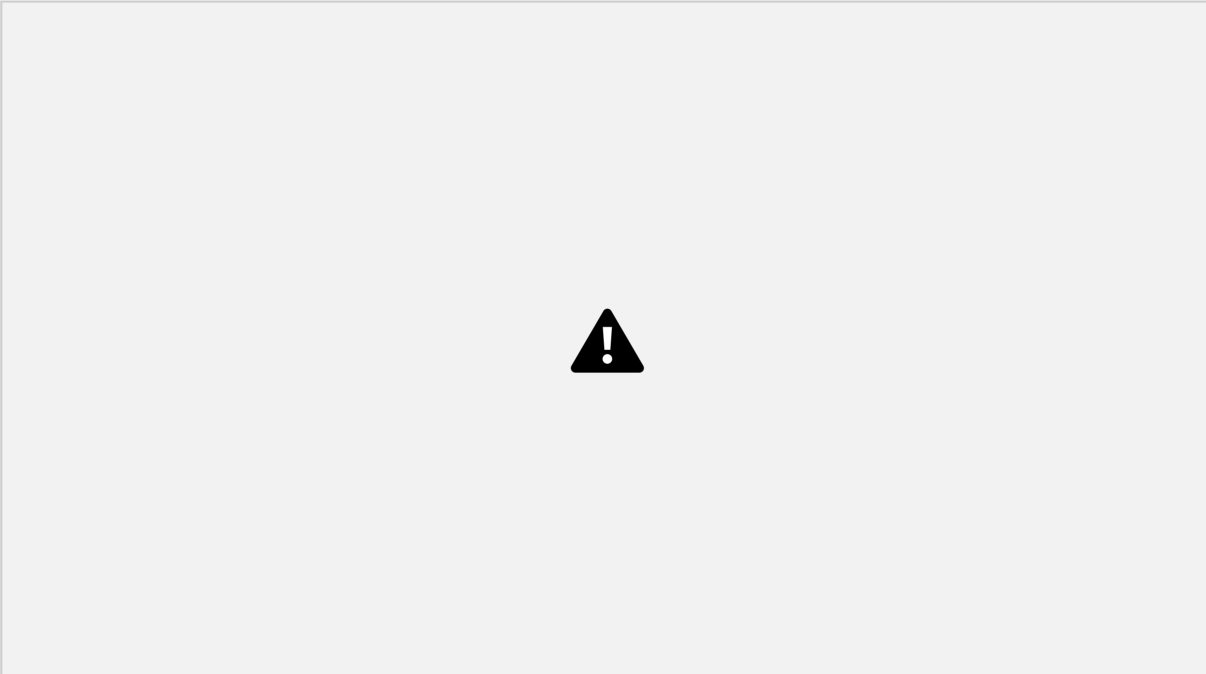


Figure 37.

The optimal tariff adjustments converge to the Nash equilibrium.

The Nash equilibrium tariff combination is determined from the equation system in (156).

$$\begin{bmatrix} 3640 & 1279 \\ 1279 & 3640 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} 2399 \\ 2399 \end{bmatrix} \quad (156)$$

The system is simplified in (157) where all elements from (156) are divided by 2399.

$$\begin{bmatrix} 1.517 & 0.533 \\ 0.533 & 1.517 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (157)$$

The system (157) is generalized to (158).

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \end{bmatrix} \begin{bmatrix} T_1 \\ \end{bmatrix} = \begin{bmatrix} 1 \\ \end{bmatrix}$$

= | | | | | | ⌊ ⌋ ⌊ ⌋ ⌊ ⌋ (158)

21 22 2

α α T 1

The Nash equilibrium values of the tariffs are derived via the general functions and the approximate parameters, in equations (159) and (160).

α

1 1 1.517 0.533 0.488

α α α

1222 22 12

– –

= = ≈ ≈

– – (159)

T

α α α α α α (1.517) (0.533)

1 2 2 11 12 11 22 12 21

α α

21 22

α

1 1 1.517 0.533 0.488

α α α

1121 11 21

– –

= = ≈ ≈

T

α α α α α α

– – (160)

(1.517) (0.533)

2 2 2

11 12 11 22 12 21

α α

21 22

It is important to know if the tariff levels will converge to the Nash equilibrium. In the investigation of the dynamics of the tariff system, the notation is simplified according to (161).

$$(T_1, T_2) = (m, n) \quad (161)$$

From (158) and (161), the system (162) is obtained.

$$\begin{bmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{bmatrix}\begin{bmatrix} m \\ n\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix} \quad (162)$$

Tariff Time Loop:

It is assumed that the system starts in free trade, where all tariffs are zero. Nation 1, N1, starts the deviation from free trade by increasing T1, denoted m, to the optimal value, conditional on the value of T2, denoted n.

m(t) is calculated based on n(t), as shown in (163), and n(t+1) is calculated based on m(t), via (164).

$$\alpha_{11} m_t + \alpha_{12} n_t = 1 \Rightarrow m_t = \frac{1-\alpha_{12} n_t}{\alpha_{11}} \quad (163)$$

$$\alpha_{21} m_t + \alpha_{22} n_{t+1} = 1 \Rightarrow n_{t+1} = \frac{1-\alpha_{21} m_t}{\alpha_{22}} \quad (164)$$

With a step in time, (164) gives (165).

$$n_t = \frac{1-\alpha_{21} m_{t-1}}{\alpha_{22}} \quad (165)$$

Equations (163) and (165) give (166), where n has been eliminated.

$$m_t = \frac{1 - \alpha_{12}\left(\dfrac{1-\alpha_{21} m_{t-1}}{\alpha_{22}}\right)}{\alpha_{11}} \quad (166)$$

Equations (167), (168) and (169), lead to a presentation of m as a first order autoregressive process with two parameters, K and L.

( ) ( )

1 1

$1\ m\ m_{t\ t}$

α_{21}

= − − | | | | | | ( ) ( ) (167)

$\alpha_{12\ 1}$

−

$\alpha\ \alpha\ \alpha_{11\ 22\ 22}$

( ) α α α

1

12 12 21

$1\ m\ m_{t\ t}$

= − + | | ( ) (168)

−

1

$\alpha\ \alpha\ \alpha$

11 22 22

( ) | | −

α

12

( ) = + (169)

1 12 21

$\alpha\ \alpha\ \alpha^{22}$

$m\ m_{t\ t}$ -

1

$\alpha\ \alpha\ \alpha$

11 11 22

( ) | | −

α

1 12

( ) = + = = (170)

$\alpha\ \alpha\ \alpha$

22 12 21

$, , m\ K\ L\ m\ K\ L_{t\ t}$

−

1 11 11 22

$\alpha\ \alpha\ \alpha$

The parameters K and L in the first order autoregressive process of m, representing T1, determine the dynamics of the optimal tariff T1, and indirectly of the dynamically linked optimal tariff T2.

According to (170), the parameters K and L are known functions of the parameters in the equation system (162), which means that the dynamic properties, such as convergence, are known functions of the parameters in (162). The tariff system converges to the Nash equilibrium if $K > 0$ and $0 < L < 1$. Figure 38 illustrates the dynamics of the system (170).

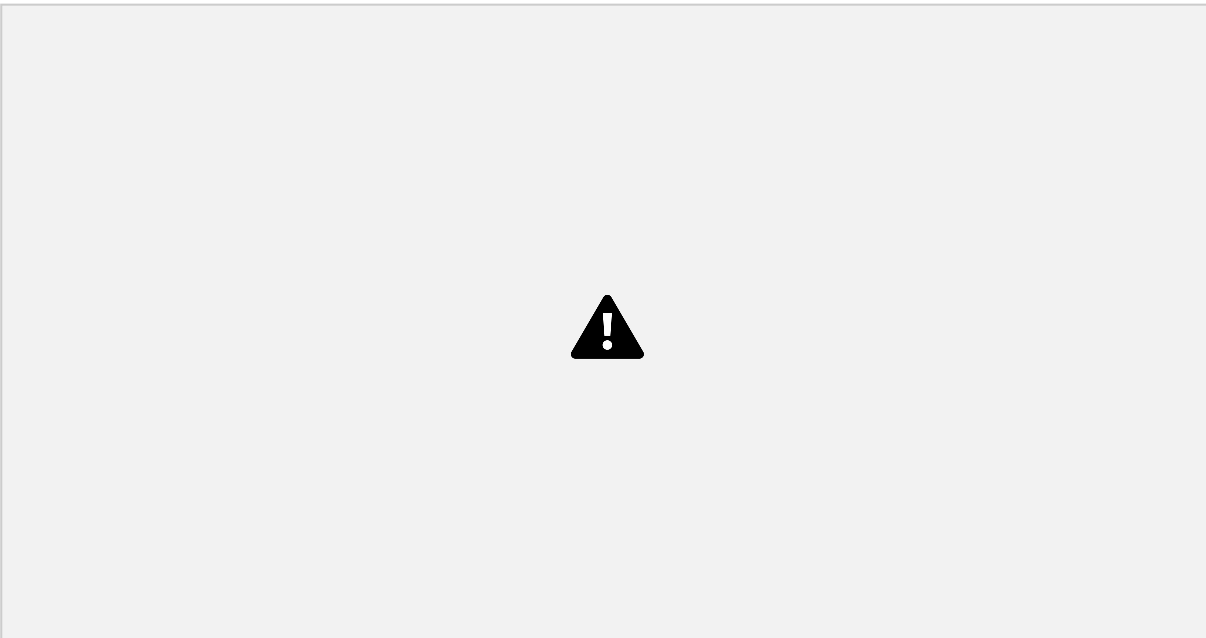


Figure 38.

The tariff system converges to the Nash equilibrium if $K > 0$ and $0 < L < 1$.

The tariff system converges to the Nash equilibrium if the conditions in (171) are

satisfied. ( ) | | −

α

1 12

( ) = > ∧ < = < (171)

α α α

22 12 21

K L 0 0 0

α α α

11 11 22

It is possible to investigate if the equation system based on the parameters in (157) leads to convergence. Equations (172) and (173) show that the parameters from (157) imply that K and L obtain values such that convergence to the Nash equilibrium is obtained.

( ) ( ) | | − | | −

α

0.533 1 11.517 0.428 0

12

( ) ( ) = = ≈ > (172) α

K

22 11 1.517

α

α α

0.533 0.533 0 0.123 1

12 21

×

< = = ≈ <

L

α α 11 22 × (173) 1.517 1.517

The conclusion based on (172) and (173) is that the tariff system converges to the Nash equilibrium. This is also confirmed via the illustrations in Figures 33 to 37, and in Figure 38.

The Nash equilibrium in general form via the general iteration function:

From (169), we get (174).

$$\left(\;\right) \left|\;\right| - \quad \alpha \quad 1 \quad {}^{12} \quad \left(\;\right) = + \quad \alpha\ \alpha\ \alpha \quad {}_{22}\ {}_{12\,21} \quad m\ m_{t\ t} - \quad {}^{1} \quad \alpha\ \alpha\ \alpha \quad {}_{11\ 11\ 22} \tag{174}$$

In dynamic equilibrium, the value of the process will not change over time. This is found in (175).

$$\left(\;\right) \left|\;\right| - \quad \alpha \quad 1 \quad {}^{12} \quad \left(\;\right) = + \quad \alpha\ \alpha\ \alpha \quad {}_{22}\ {}_{12\,21} \quad {}_{NE}{}^{NE} m\ m \quad \alpha\ \alpha\ \alpha \quad {}_{11\ 11\ 22} \tag{175}$$

Equation (175) is transformed to (176). This leads to (177) and (178).

$$\left(\;\right) \left|\;\right| - \quad \alpha \quad 1 \quad {}_{12} \quad \left(\;\right)\left(\;\right) \left|\;\right| - = \quad \alpha\ \alpha\ \alpha \quad {}_{12\,21}\,{}^{22} \quad {}^{NE} m \quad 1 \quad \alpha\ \alpha\ \alpha \quad \left(\;\right) \quad {}_{11\,22\,11} \left(\;\right) \left(\;\right) \left|\;\right| \left|\;\right| - \tag{176}$$

$$\alpha \quad 1 \quad {}^{12} \quad \alpha \quad \alpha\ \alpha \quad {}_{22} \quad \left|\;\right| \alpha \quad {}^{NE} m \quad {}_{11} \quad \left|\;\right| \left(\;\right) \quad \left|\;\right| \left(\;\right) = \left(\;\right) \left|\;\right| - \tag{177}$$

$$\alpha_{12}\alpha_{21}$$

$$1$$

$$(\ )_{11\ 22}$$

$$\alpha$$

$$m^{NE}(\ )_{22} = \left(\ \right) 1 - \alpha_{12}\alpha\ \alpha \qquad \alpha_{12}\alpha_{21}\ \alpha_{11} \ (\ )_{22} \tag{178}$$

The dynamic equilibrium value of m is found in (179) as a function of the system parameters. This is identical to the equilibrium value of T1 according to the earlier calculation. Compare equation (159).

$$m^{NE} = \frac{\left(\alpha_{22} - \alpha_{12}\right)}{\left(\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}\right)} \tag{179}$$

Optimal cooperation:

The Tables 3 and 4 show that, starting with free trade, both nations prefer to increase the respective tariffs, provided that the other nations do not increase the tariffs.

In case the different nations adjust the tariff levels independently and optimally, based on the tariffs presented by the other nation(s), the tariffs converge to the Nash tariff equilibrium. However, it is not necessarily the case that the Nash equilibrium is a better solution for the different nations, than the original solution, with free trade and no tariffs.

Table 7 clearly shows that the sum of the transformed utility functions decreases if T1, T2 or both T1 and T2 increase.

Table 7.

Numerical approximation of the sum of the transformed utility functions in all nations, WSUM(T1, T2) = W1(T1, T2) + W2(T1, T2). Compare Tables 3 and 4.

Observations: The maximum of the total utility is obtained when all tariffs are zero. The total utility decreases if some tariff(s) increase(s).

	T2(%)	**0**	**20**	**40**	**60**	**80**	**100**
T1 (%)							
0		0	-78	-269	-527	-836	-1171
20		-77	-316	-648	-1033	-1451	-1880
40		-268	-647	-1101	-1593	-2100	-2610
60		-530	-1033	-1592	-2173	-2758	-3336
80		-837	-1450	-2099	-2757	-3407	-4041
100		-1169	-1881	-2612	-3336	-4041	-4721

The derived Nash equilibrium from (154) and (155) gives (180).

$(T\,T_{1\,2},\ 0.48770,\ 0.48770\ ) = (\ )$ (180) The transformed utility functions are found in (181) and (182).

${}_2{}^2\ W\ T\ T\ T\ TT\ T_{1\,1\,1\,2\,1\,2\,2} = - - - + 2399\ 1820\ 3102\ 1279\ 1349$ (181) ${}_2{}^2\ W\ T\ T\ T\ TT\ T_{2\,2\,2\,1\,1\,2\,1} =$

$- - - + 2399\ 1820\ 3102\ 1279\ 1349$ (182)

Based on the Nash equilibrium tariffs (180), and the transformed utility functions (181) and (182), equation (183) is derived. Clearly, both nations get a worse situation in the Nash equilibrium, than with free trade, without tariffs.

$$W\ W_{1\,2} = \approx -759 \quad (183)$$

Observation:

It is better for N1 and for N2 to select free trade, without any tariffs, than to fight a tariff war, and select the best possible tariff levels, based on the tariffs in the other nations. If both nations play optimally according to the Nash equilibrium, they both gain if they cooperate and reduce all tariffs to zero. Then, free trade is obtained. This free trade is however unstable. Some nation may stop the cooperation, increase tariffs, and assume or hope that the other nations will become too scared to respond with the optimal tariff changes. If the other nations respond optimally, the Nash solution will be found. Then, again, all rational nations will find it rational to cooperate and move back to the free trade solution without tariffs.

5. Conclusions

Optimal tariff strategies, with and without optimal responses, are derived, based on a linear partial equilibrium trade model, A, and a nonlinear general equilibrium trade model with two nations, B.

Under free trade, models A and B are applied to prove that it is always profitable for one nation, N1, to introduce a strictly positive tariff on imports, T1, as long as other nations do not introduce tariffs.

With model B, it is proved that it is rational also for nation N2, to introduce a tariff, T2, if N1, introduces a tariff, T1. The Nash equilibrium tariff combination is unique and stable. The economic results, in both nations, are however higher with free trade, than in the Nash equilibrium tariff solution.

If both nations know that the other nations will respond to increasing tariffs, and select the optimal tariff, then it is not rational for any nation to leave free trade and introduce tariffs. Free trade agreements are rational for the participants.

It may be rational for a dictator, to introduce a tariff, even if this is not rational for the consumers in the same nation. The dictator may keep the tariff revenues and use these for other purposes. In such a case, the tariff reduces the consumer surpluses in the exporting and the importing nations.

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References

1. Krugman, P.R.; Obstfeld, M.; *International Economics*, Theory and Policy; Addison-Wesley Publishing Company; Reading, Massachusetts, USA, 770 p, 2000. ISBN 0-321-07727-X.

2. Dávila, E.; Rodríguez-Clare, A.; Schaab, A.; Tan, S.Y.; A Dynamic Theory of optimal tariffs, *COWLES FOUNDATION DISCUSSION PAPER NO. 2444, YALE UNIVERSITY*, May 2025. https://cowles.yale.edu/sites/default/files/2025-06/d2444.pdf

3. Chacholiades, M.; *International Trade Theory and Policy*; McGraw-Hill Inc.; Tosho Printing Co. Ltd., Tokyo, Japan, 614 p, 1978.

4. Irwin, D.A.; TARIFF INCIDENCE: EVIDENCE FROM U.S. SUGAR DUTIES, 1890-1930; *NATIONAL BUREAU OF ECONOMIC RESEARCH*; Working Paper 20635, October 2014; 1050 Massachusetts Avenue, Cambridge, MA 02138. www.nber.org/papers/w20635

5. Bowden, L.; Rethinking the Optimal Tariff Theory; *Chicago Policy Review*; January 15, 2015, (Updated September 15, 2020). https://chicagopolicyreview.org/2015/01/15/rethinking-the-optimal tariff-theory/

6. Broda, Christian, Nuno Limao, and David E. Weinstein; Optimal Tariffs and Market Power: The Evidence; *American Economic Review* 98 (5) 2008: 2032–65. DOI: 10.1257/aer.98.5.2032. https://www.aeaweb.org/articles?id=10.1257/aer.98.5.2032

7. Lohmander, P.; Applications and Mathematical Modeling in Operations Research, In: Cao BY. (ed) Fuzzy Information and Engineering and Decision. IWDS 2016. *Advances in Intelligent Systems and Computing*, vol 646. Springer, Cham, 2018 Print ISBN 978-3-319-66513-9, Online ISBN 978-3-319-66514-6, eBook Package: Engineering, https://doi.org/10.1007/978-3-319-66514-6_5

8. Nash, J.; Non-Cooperative Games; *The Annals of Mathematics*, Second Series, Volume 54, Issue 2, Sep. 1951, 286-295. https://dn720001.ca.archive.org/0/items/non-cooperative-games-nash/Non cooperative%20games%20-%20nash_text.pdf

9. Nash, J.; Equilibrium Points in n-Person Games; *Proceedings of the National Academy of Science*, 36, 1950, 48-49. https://doi.org/10.1073/pnas.36.1.48

10. Rasmusen, E.; *Games and Information*; Basil Blackwell Ltd, 3 Cambridge Center, Cambridge, Massachusetts 02142, USA, 352 p, Reprinted with corrections 1990.

11. Isaacs, R.; *Differential Games, A Mathematical Theory with Applications to Warfare and Pursuit, Control and Optimization*; John Wiley and Sons, Inc.; New York, USA, 384 p., 1965.

12. Washburn, A.R.; *Two-Person Zero-Sum Games*; Topics in Operations Research Series, Military Applications Society, INFORMS, Institute for Operations Research and the Management Sciences; Linthicum, MD 21090, USA, 125 p., 2003.

13. Lohmander, P.; Optimal decisions and expected values in two player zero sum games with

diagonal game matrixes-explicit functions, general proofs and effects of parameter estimation errors; *International Robotics and Automation Journal*, 5(5), 2019:186-198. https://medcraveonline.com/IRATJ/IRATJ-05-00193.pdf

14. Lohmander, P.; Fyra centrala militära beslutsproblem, Generella metoder och lösningar (Eng. Four central military decision problems, General methods and solutions); *The Royal Swedish Academy of War Sciences; Proceedings and Journal*, No. 2/2019, pages 119-134. https://kkrva.se/hot/2019:2/09- lohmander-fyra-centrala-militara-beslutsproblem.pdf

15. Washburn, A., Kress, M.; *Combat Modeling*, International Series in Operations Research and Management Science, Springer Science + Business Media, New York, USA, 277 p., 2009.

16. Lohmander, P., Optimal Dynamic Control of Proxy War Arms Support. *Automation*2023, 4, 31-56. https://doi.org/10.3390/automation4010004

17. Lubik, Thomas A.; When Are Tariffs Optimal; *Federal Reserve Bank of Richmond Economic Brief*, No. 25-21, May 2025. https://www.richmondfed.org/publications/research/economic_brief/2025/eb_25-21

18. Chu, B., Wainwright, D., Leake, P.; How much cash is the US raising from tariffs?, *BBC Verify*, 8 August 2025; https://bbc.com/news/articles/cr5rm7v5166o

Appendix

The computer code can be used to derive the numerical results reported in Tables 3, 4 and 7 in the main text. The code is developed in QB64. The algorithm is described in the main text. The output sections of the code have been excluded from this appendix.

```
Rem Tariff Game 250820
Rem Peter Lohmander

Screen _NewImage(1800, 2000, 256)
Open "C:\Users\Peter\OneDrive\Desktop\Tariff_Game_Out\Tariff_Game_Out.txt" For Output As #2

DefDbl A-Z

Dim W1(10, 10), W2(10, 10), Wsum(10, 10)
Dim Mx11(10, 10), Mx12(10, 10), Mx21(10, 10), Mx22(10, 10)
Dim Mc11(10, 10), Mc12(10, 10), Mc21(10, 10), Mc22(10, 10)
Dim Ms11(10, 10), Ms12(10, 10), Ms21(10, 10), Ms22(10, 10)
Dim M_relp_p2_p1(10, 10), M_relp_p2_p3(10, 10), M_relp_p4_p1(10, 10)

For T1_index = 0 To 10
 For T2_index = 0 To 10
 W1(T1_index, T2_index) = 0
 W2(T1_index, T2_index) = 0
 Next T2_index
Next T1_index
```

Rem Production possibility frontier parameters.
Rem The maximum production levels of (x11, x12, x21, x22) are (mx11, mx12, mx21, mx22) = (2, 4, 4, 2).
Rem mij = (mxij)^(-2) for all i,j.

m11 = (1 / 2) ^ 2
m12 = (1 / 4) ^ 2

m21 = (1 / 4) ^ 2
m22 = (1 / 2) ^ 2

Rem a1 and a2 are the parameters of the utility functions u1 and u2.

a1 = 1 / 2
a2 = 1 / 2

Rem T1 is the relative tariff in Nation 1 on imports of product 2.
For T1_index = 0 To 10 Step 1
T1 = 0.2 * T1_index

Rem T2 is the relative tariff in Nation 2 on imports of product 1.
For T2_index = 0 To 10 Step 1
T2 = 0.2 * T2_index

abss1_opt = 10

Rem RELP is the relative price p2/p1, where p1 and p2 are the world market prices of the products 1 and 2. Rem p3 is the price of product 1 inside Nation 1 as a result of the world market price p1 and Tariff_1 in Nation 1 on imported product 1.
Rem p4 is the price of product 2 inside Nation 2 as a result of the world market price p2 and Tariff_2 in Nation 2 on imported product 2.

For RELP = 0.2 To 5 Step 0.001

Rem Initial guess of the export of product 1 from Nation 1. (A negative value indicates import.) s11 = -1.5
s22 = -1.5

p1 = 100
p2 = p1 * RELP
Tariff_1 = p1 * T1
p3 = p1 + Tariff_1
Tariff_2 = p2 * T2
p4 = p2 + Tariff_2

Rem Relative prices in N1 and in N2.

RELP_N1 = p2 / p3
RELP_N2 = p4 / p1

Rem x11, x12, x21 and x22 are the optimal production levels in N1 and in N2.

x11 = (1 / (m11 * (m11 / m12 * RELP_N1 ^ (2) + 1))) ^ .5
x12 = (m12 ^ (-1) - 1 / (m11 * RELP_N1 ^ (2) + m12)) ^ .5

x21 = (1 / (m21 * (m21 / m22 * RELP_N2 ^ (2) + 1))) ^ .5
x22 = (m22 ^ (-1) - 1 / (m21 * RELP_N2 ^ (2) + m22)) ^ .5

Rem The consumption levels are functions of the consumption budgets in N1 and N2. Rem The consumption budgets in N1 and N2 are functions of the tariff revenues. Rem The tariff revenues are functions of the import volumes.
Rem The import volumes are functions of the consumption levels.
Rem For this reason, fix point iteration is used to determine the equilibrium.
Rem This is done via the i-loop, found below. The 100 steps give a solution very close to the equilibrium.


```
For i = 1 To 100

Trev_1 = Tariff_1 * (-s11)  Trev_2
= Tariff_2 * (-s22)

B1 = p3 * x11 + p2 * x12 + Trev_1  B2 = p1
* x21 + p4 * x22 + Trev_2

c11 = a1 * B1 / p3
c12 = (1 - a1) * B1 / p2

c21 = a2 * B2 / p1
c22 = (1 - a2) * B2 / p4

s11 = x11 - c11
s12 = x12 - c12
s21 = x21 - c21
s22 = x22 - c22

Next i

s1 = s11 + s21
s2 = s12 + s22

u1 = (c11 ^ a1) * (c12 ^ (1 - a1))  u2 =
(c21 ^ a2) * (c22 ^ (1 - a2))

abss1 = s1
If s1 < 0 Then abss1 = -s1
If abss1 > abss1_opt Then GoTo 100

RELP_eq = RELP
u1_eq = u1
u2_eq = u2

x11_eq = x11
x12_eq = x12
x21_eq = x21
x22_eq = x22

c11_eq = c11
c12_eq = c12
c21_eq = c21
c22_eq = c22

s11_eq = s11
s12_eq = s12
s21_eq = s21
```

```
s22_eq = s22

s1_eq = s1
s2_eq = s2

rel_p2_p1_eq = p2 / p1
rel_p2_p3_eq = p2 / p3
rel_p4_p1_eq = p4 / p1
```



```
abss1_opt = abss1

100 Rem
Next RELP

u1_0 = 2.23606
u2_0 = 2.23606
W1(T1_index, T2_index) = (u1_eq - u1_0) * 10000
W2(T1_index, T2_index) = (u2_eq - u2_0) * 10000
Wsum(T1_index, T2_index) = W1(T1_index, T2_index) + W2(T1_index, T2_index)
Mx11(T1_index, T2_index) = x11_eq
Mx12(T1_index, T2_index) = x12_eq
Mx21(T1_index, T2_index) = x21_eq
Mx22(T1_index, T2_index) = x22_eq

Mc11(T1_index, T2_index) = c11_eq
Mc12(T1_index, T2_index) = c12_eq
Mc21(T1_index, T2_index) = c21_eq
Mc22(T1_index, T2_index) = c22_eq

Ms11(T1_index, T2_index) = s11_eq
Ms12(T1_index, T2_index) = s12_eq
Ms21(T1_index, T2_index) = s21_eq
Ms22(T1_index, T2_index) = s22_eq

M_relp_p2_p1(T1_index, T2_index) = rel_p2_p1_eq
M_relp_p2_p3(T1_index, T2_index) = rel_p2_p3_eq
M_relp_p4_p1(T1_index, T2_index) = rel_p4_p1_eq

Next T2_index
Next T1_index

Close #2
End
```