Find the general solution of each of the following exact differential equations.

1. $x\frac{\mathrm{d}y}{\mathrm{d}x}+y=\mathrm{e}^x$

$$\frac{\mathrm{d}}{\mathrm{d}x}(xy)=y+x\frac{\mathrm{d}y}{\mathrm{d}x}$$
$$xy=\int \mathrm{e}^x\mathrm{d}x$$
$$xy=\mathrm{e}^x+c$$

2. $\cos x\frac{\mathrm{d}y}{\mathrm{d}x}-y\sin x=x^2$

$$\frac{\mathrm{d}}{\mathrm{d}x}y\cos x=\frac{\mathrm{d}y}{\mathrm{d}x}\cos x-y\sin x$$
$$y\cos x=\frac{1}{3}x^3+c$$

3. $\frac{x}{y}\frac{\mathrm{d}y}{\mathrm{d}x}+\ln y=x+1$

$$\frac{\mathrm{d}}{\mathrm{d}x}x\ln y=\ln y+\frac{x}{y}$$
$$x\ln y=\frac{x^2}{2}+x+c$$

4. $\frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x}-\frac{y}{x^2}=\sin x$

$$\frac{\mathrm{d}}{\mathrm{d}x}yx^{-1}=\frac{\mathrm{d}y}{\mathrm{d}x}x^{-1}-yx^{-2}=\frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}x}-\frac{y}{x^2}$$
$$\frac{y}{x}=-\cos x+c$$

5. $\mathrm{e}^xy+\mathrm{e}^x\frac{\mathrm{d}y}{\mathrm{d}x}=2$

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^xy=\mathrm{e}^xy+\mathrm{e}^x\frac{\mathrm{d}y}{\mathrm{d}x}$$
$$\mathrm{e}^xy=2x+c$$

6. $x\mathrm{e}^y\frac{\mathrm{d}y}{\mathrm{d}x}+\mathrm{e}^y=\mathrm{e}^x$

$$\frac{\mathrm{d}}{\mathrm{d}x}\ x\mathrm{e}^y=\mathrm{e}^y+x\mathrm{e}^y\frac{\mathrm{d}y}{\mathrm{d}x}$$
$$x\mathrm{e}^y=\mathrm{e}^x+c$$

7. $\ln x\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{y}{x}=x\ln x$

$$\frac{\mathrm{d}}{\mathrm{d}x}y\ln x=\frac{\mathrm{d}y}{\mathrm{d}x}\ln x+\frac{y}{x}$$
$$y\ln x=\int x\ln x\ \mathrm{d}x$$
$$=\frac{x^2}{2}\ln x-\int\frac{x^2}{2}\frac{1}{x}\mathrm{d}x$$
$$y\ln x=\frac{x^2}{2}\ln x-\frac{x^2}{4}+c$$

8. $(1+x)\frac{\mathrm{d}y}{\mathrm{d}x}+y=x^3$

$$\frac{\mathrm{d}}{\mathrm{d}x}(1+x)y=y+(1+x)\frac{\mathrm{d}y}{\mathrm{d}x}$$
$$(1+x)y=\frac{x^4}{4}+c$$

9. $x\sec^2 y\frac{\mathrm{d}y}{\mathrm{d}x}+\tan y=\tan x$

$$\frac{\mathrm{d}}{\mathrm{d}x}x\tan y=\tan y+x\sec^2 x\frac{\mathrm{d}y}{\mathrm{d}x}$$
$$x\tan y=\ln\sec x+c$$

10. $\mathrm{e}^x\mathrm{e}^y\frac{\mathrm{d}y}{\mathrm{d}x}+\mathrm{e}^x\mathrm{e}^y=\mathrm{e}^{2x}$

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^x\mathrm{e}^y=\mathrm{e}^x\mathrm{e}^y+\mathrm{e}^x\mathrm{e}^y\frac{\mathrm{d}y}{\mathrm{d}x}$$
$$\mathrm{e}^x\mathrm{e}^y=\frac{1}{2}\mathrm{e}^{2x}+c$$

By using integrating factors, solve the following differential equations.

11. $\frac{\mathrm{d}y}{\mathrm{d}x}+3y=\mathrm{e}^{-3x}$

$$\mathrm{e}^{\int 3\mathrm{d}x}=\mathrm{e}^{3x}$$
$$\mathrm{e}^{3x}y=\int \mathrm{e}^{3x}\mathrm{e}^{-3x}\mathrm{d}x$$
$$\mathrm{e}^{3x}y=x+c$$

12. $\frac{\mathrm{d}y}{\mathrm{d}x}+y\cot x=\operatorname{cosec} x$

$$\mathrm{e}^{\int\frac{\cos x}{\sin x}\mathrm{d}x}=\mathrm{e}^{\ln\sin x}=\sin x$$
$$y\sin x=\int \sin x\ \operatorname{cosec}\ x\ \mathrm{d}x$$
$$y\sin x=x+c$$

13. $x^2\frac{\mathrm{d}y}{\mathrm{d}x}+xy=x+1$

$$\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{1}{x}y=\frac{x+1}{x^2}$$
$$\mathrm{e}^{\int\frac{1}{x}\mathrm{d}x}=\mathrm{e}^{\ln x}=x$$
$$xy=\int\frac{x+1}{x}\mathrm{d}x$$
$$xy=\int\left(1+\frac{1}{x}\right)\mathrm{d}x$$
$$xy=x+\ln x+c$$

14. $\dfrac{\mathrm{d}y}{\mathrm{d}x} - \dfrac{3y}{x+1} = (x+1)^4$

$$\mathrm{e}^{-3\int \frac{1}{x+1}\mathrm{d}x} = \mathrm{e}^{-3\ln(x+1)} = \frac{1}{(x+1)^3}$$

$$\frac{1}{(x+1)^3}y = \int \frac{1}{(x+1)^3}(x+1)^4 \mathrm{d}x$$

$$\frac{1}{(x+1)^3}y = \frac{x^2}{2} + x + c$$

15. $\tan x \dfrac{\mathrm{d}y}{\mathrm{d}x} + y = \mathrm{e}^x \tan x$

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\cos x}{\sin x}y = \mathrm{e}^x$$

$$\mathrm{e}^{\int \frac{\cos x}{\sin x}\mathrm{d}x} = \mathrm{e}^{\ln \sin x} = \sin x$$

$$y \sin x = \int \mathrm{e}^x \sin x \, \mathrm{d}x$$

$$\begin{aligned} I &= \int \mathrm{e}^x \sin x \, \mathrm{d}x \\ &= \mathrm{e}^x \sin x - \int \mathrm{e}^x \cos x \, \mathrm{d}x \\ &= \mathrm{e}^x \sin x - \left[\mathrm{e}^x \cos x - \int \mathrm{e}^x (-\sin x) \, \mathrm{d}x\right] \\ &= \mathrm{e}^x \sin x - \mathrm{e}^x \cos x - \int \mathrm{e}^x \sin x \, \mathrm{d}x \\ 2I &= \mathrm{e}^x \sin x - \mathrm{e}^x \cos x \\ I &= \frac{\mathrm{e}^x}{2}(\sin x - \cos x) \\ y \sin x &= \frac{\mathrm{e}^x}{2}(\sin x - \cos x) \end{aligned}$$

16. $\dfrac{\mathrm{d}v}{\mathrm{d}t} = t - 2vt$

$$\frac{\mathrm{d}v}{\mathrm{d}t} + 2tv = t$$

$$\mathrm{e}^{\int 2t \ \mathrm{d}t} = \mathrm{e}^{t^2}$$

$$\mathrm{e}^{t^2} t = \int t\mathrm{e}^{t^2} \mathrm{d}t$$

$$\mathrm{e}^{t^2} t = \frac{1}{2}\mathrm{e}^{t^2} + c$$

17. $x\dfrac{\mathrm{d}y}{\mathrm{d}x} + 2y = \dfrac{\sin x}{x}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{2}{x}y = \frac{\sin x}{x^2}$$

$$\mathrm{e}^{\int \frac{2}{x}\mathrm{d}x} = \mathrm{e}^{2\ln x} = x^2$$

$$x^2 y = \int \sin x \, \mathrm{d}x$$

$$x^2 y = -\cos x \ + c$$

18. $x\dfrac{\mathrm{d}y}{\mathrm{d}x}=y-x^2\mathrm{e}^{-x}$

$$\frac{\mathrm{d}y}{\mathrm{d}x}-\frac{1}{x}y=-x\mathrm{e}^{-x}$$
$$\mathrm{e}^{-\int\frac{1}{x}\mathrm{d}x}=\mathrm{e}^{-\ln x}=\frac{1}{x}$$
$$\frac{y}{x}=\int-\mathrm{e}^{-x}\mathrm{d}x$$
$$\frac{y}{x}=\mathrm{e}^{x}+c$$

19. $\dfrac{\mathrm{d}r}{\mathrm{d}\theta}+r\cot\theta=\sin\theta$

$$\mathrm{e}^{\int\frac{\cos\theta}{\sin\theta}\mathrm{d}\theta}=\sin\theta$$
$$r\sin\theta=\int\sin^2\theta\,\mathrm{d}\theta$$
$$r\sin\theta=\int\left(\frac{1}{2}-\frac{1}{2}\cos2\theta\right)\mathrm{d}\theta$$
$$r\sin\theta=\frac{\theta}{2}-\frac{1}{4}\sin2\theta+c$$

20. $y+x(x-1)\dfrac{\mathrm{d}y}{\mathrm{d}x}=x^3\mathrm{e}^{-x^2}$

$$\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{1}{x(x-1)}y=\frac{x^3}{x(x-1)}\mathrm{e}^{-x^2}$$
$$\frac{1}{x(x-1)}=\frac{A}{x}+\frac{B}{x-1}$$
$$1=A(x-1)+Bx$$
$$x=0$$
$$A=-1$$
$$x=1$$
$$B=1$$
$$\mathrm{e}^{\int\frac{1}{x(x-1)}\mathrm{d}x}=\mathrm{e}^{\int\left(\frac{1}{x-1}-\frac{1}{x}\right)\mathrm{d}x}=\mathrm{e}^{\ln(x-1)-\ln x}=\frac{x-1}{x}$$
$$\frac{x-1}{x}y=\int\frac{x-1}{x}\times\frac{x^3}{x(x-1)}\mathrm{e}^{-x^2}\mathrm{d}x$$
$$\frac{x-1}{x}y=\int x\mathrm{e}^{-x^2}\,\mathrm{d}x$$
$$\frac{x-1}{x}y=-\frac{1}{2}\int(-2x)\left(\mathrm{e}^{-x^2}\right)\mathrm{d}x$$
$$\frac{x-1}{x}y=-\frac{1}{2}\ln\left(\mathrm{e}^{-x^2}\right)+c$$
$$\frac{x-1}{x}y=\frac{1}{2}x^2+c$$

21. $x\dfrac{\mathrm{d}y}{\mathrm{d}x} = y + x^2(\sin x + \cos x)$ and $y = 0$ when $x = \dfrac{\pi}{2}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{1}{x}y = x(\sin x + \cos x)$$

$$\mathrm{e}^{-\int \frac{1}{x}\mathrm{d}x} = \mathrm{e}^{-\ln x} = \frac{1}{x}$$

$$\frac{y}{x} = \int (\sin x + \cos x)\mathrm{d}x$$

$$\frac{y}{x} = -\cos x + \sin x + c$$

$$0 = -0 + 1 + c$$

$$\frac{y}{x} = -\cos x + \sin x - 1$$

22. $\dfrac{\mathrm{d}y}{\mathrm{d}x} = x - xy$ and $y = 0$ when $x = 0$

$$\frac{\mathrm{d}y}{\mathrm{d}x} + xy = x$$

$$\mathrm{e}^{\int x\ \mathrm{d}x} = \mathrm{e}^{\frac{x^2}{2}}$$

$$\mathrm{e}^{\frac{x^2}{2}}y = \int x\mathrm{e}^{\frac{x^2}{2}}$$

$$\mathrm{e}^{\frac{x^2}{2}}y = \ln\left(\mathrm{e}^{\frac{x^2}{2}}\right) + c$$

$$\mathrm{e}^{\frac{x^2}{2}}y = \frac{x^2}{2} + c$$

$$0 = 0 + c$$

$$\mathrm{e}^{\frac{x^2}{2}}y = \frac{x^2}{2}$$