Linear Algebra MAT313 Fall 2022
Professor Sormani
Lesson 20
Linear Combinations, Spans, Linear Independence, and Basis of a Subspace of Euclidean Space

You will cut and paste the photos of your notes and completed classwork and a selfie taken holding up the first page of your work in a googledoc entitled:

MAT313F22-lesson20-lastname-firstname

and share editing of that document with me sormanic@gmail.com.

If you have a question, type QUESTION in your googledoc next to the point in your notes that has a question and email me with the subject MAT313 QUESTION. I will answer your question by inserting a photo into your googledoc or making an extra video.

This lesson has four parts with homework for each part and everyone must learn all four parts, but you may choose to do them on different dates. Instead of playlists, the videos can be found right next to the notes. Alternatively you can watch the full 313F21-10 Playlist.

Classwork is the notes for the lesson.

Part 1: Linear Combinations and Spans
including solutions sets of homogeneous systems

Watch Video 313F20-10-1

Note this lesson was Lesson 10 in the past.

3:21 AM Mon Sep 14

Linear Algebra

Abstract_Algebr... | Linear Algebra | Proofs and Domi... | Unti

Lesson 10 Part I

Linear Combinations and Spans

Recall:

Defn Scalar Multiplication

$R \in \mathbb{R}$ and $\vec{v} \in \mathbb{R}^m$ then

$$R\vec{v} = R\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{pmatrix} = \begin{pmatrix} Rv_1 \\ Rv_2 \\ \vdots \\ Rv_m \end{pmatrix}$$

i^{th} entry $[R\vec{v}]_i = R[\vec{v}]_i$

Defn Addition of vectors in $\mathbb{R}^m$

$\vec{v}, \vec{w} \in \mathbb{R}^m$

$$\vec{v} + \vec{w} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{pmatrix} = \begin{pmatrix} v_1+w_1 \\ v_2+w_2 \\ \vdots \\ v_m+w_m \end{pmatrix}$$

i^{th} entry $[\vec{v}+\vec{w}]_i = \vec{v}_i + \vec{w}_i$

Linear Algebra

Linear Algebra | Untitled Notebook | Abstract_Algebra_1-5

New Defn:

A linear combination of two vectors $\vec{v}$ and $\vec{w}$ by two real numbers a, b is $a\vec{v} + b\vec{w}$

$$a\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{pmatrix} + b\begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{pmatrix} = \begin{pmatrix} av_1+bw_1 \\ av_2+bw_2 \\ \vdots \\ av_m+bw_m \end{pmatrix}$$

Show by combining scalar mult + vector +

$$[a\vec{v}+b\vec{w}]_i = a[\vec{v}]_i + b[\vec{w}]_i$$

New Defn:

A linear combination of two vectors $\vec{v}_1$ and $\vec{v}_2$ by two real numbers a_1, a_2 is $a_1\vec{v}_1 + a_2\vec{v}_2$

$$a_1\begin{pmatrix} v_{11} \\ v_{22} \\ \vdots \\ v_{m1} \end{pmatrix} + a_2\begin{pmatrix} v_{21} \\ v_{22} \\ \vdots \\ v_{2m} \end{pmatrix} = \begin{pmatrix} av_1 + bw_1 \\ av_2 + bw_2 \\ \vdots \\ av_m + bw_m \end{pmatrix}$$

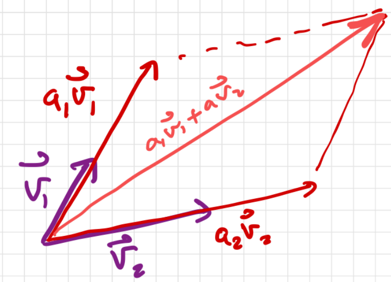
Show by combining scalar mult & vector +

$$[a_1\vec{v}_1 + a_2\vec{v}_2]_i = a_1[\vec{v}_1]_i + a_2[\vec{v}_2]_i$$

New Defn: Given a collection of vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3 \dots \vec{v}_k \in \mathbb{R}^m$ a linear combination

$$\sum_{i=1}^{k} a_i\vec{v}_i = a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_k\vec{v}_k$$

where $a_1, a_2 \dots a_k \in \mathbb{R}$



New Defn: Given a collection of vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3 \dots \vec{v}_k \in \mathbb{R}^m$ a linear combination

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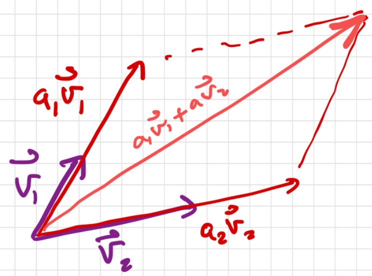
where $a_1, a_2 \dots a_k \in \mathbb{R}$



Recall our solution sets of homogeneous systems

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} = x_2 \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{pmatrix} + x_3 \begin{pmatrix} w_1 \\ \vdots \\ w_m \end{pmatrix} + x_5 \begin{pmatrix} z_1 \\ \vdots \\ z_m \end{pmatrix} : x_2, x_3, x_5 \in \mathbb{R} \right\}$$

free

summed up
free variables times directions

Linear Combo of the Directions.

New Defn Given a collection of vectors $\vec{v}_1, \vec{v}_2 \dots \vec{v}_k \in \mathbb{R}^m$ the span $\langle \vec{v}_1, \vec{v}_2, \dots, \vec{v}_k \rangle$ is the set of all linear combinations:

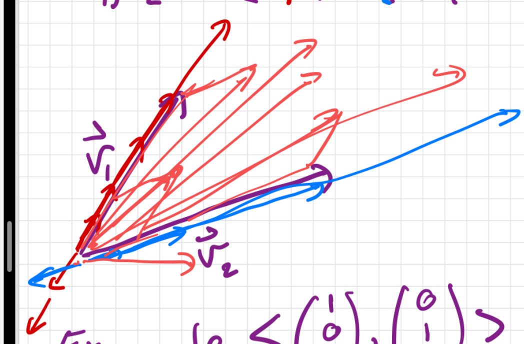
$$= \{ a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_k\vec{v}_k : a_i \in \mathbb{R} \}$$

free

$$= \left\{ \sum_{i=1}^{k} a_i \vec{v}_i : a_i \in \mathbb{R} \right\}$$

run free through $\mathbb{R}$

$$\langle \vec{v}_1, \vec{v}_2 \rangle = \{ a_1\vec{v}_1 + a_2\vec{v}_2 \mid a_i \in \mathbb{R} \}$$

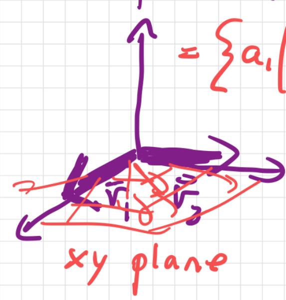


Example $\left\langle \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix} \right\rangle$

$$= \left\{ a_1\begin{pmatrix}1\\0\\0\end{pmatrix} + a_2\begin{pmatrix}0\\1\\0\end{pmatrix} \;\middle|\; a_i \in \mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix}a_1\\a_2\\0\end{pmatrix} \;\middle|\; a_i \in \mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix}x\\y\\0\end{pmatrix} \;\middle|\; x, y \in \mathbb{R} \right\}$$



For HW hints see [Video 313F20-10-2]

$\boxed{HW1}$ Is $\begin{pmatrix}1\\2\\3\end{pmatrix} \in \left\langle \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix} \right\rangle$?

xy plane

$\boxed{HW2}$ Is $\begin{pmatrix}3\\2\\0\end{pmatrix} \in \left\langle \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix} \right\rangle$?

$\boxed{HW3}$ Is $\begin{pmatrix}1\\2\\3\end{pmatrix} \in \left\langle \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix} \right\rangle$?

$\boxed{HW4}$ Is $\begin{pmatrix}1\\4\\4\end{pmatrix} \in \left\langle \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix} \right\rangle$?

$\boxed{HW5}$ Is $\begin{pmatrix}1\\2\\3\\4\\5\end{pmatrix} \in \left\langle \begin{pmatrix}1\\0\\1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}-1\\0\\0\\2\\2\end{pmatrix} \right\rangle$?

Do HW1-5 before Part 2. Note that to solve HW5 there are five equations and three unknowns: you need to solve a system for a,b, and c. Then reduce that system to Echelon form. You do not need to find a,b, and c but must check if that system has a solution by checking any row of zeroes in Echelon form ends in a zero.

Part 2: Subspaces of m dimensional Euclidean Space

Watch [Video 313F20-10-3] for the definition of a subspace:

Lesson 10 Part 2

Subspaces

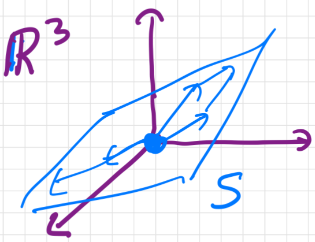
Recall $\vec{v} \in \mathbb{R}^m$ vectors

$\mathbb{R}^m$ Euclidean Space of dimension m

$\mathbb{R}^1$ = real line (dim = 1)

$\mathbb{R}^2$ Euclidean plane (dim = 2)

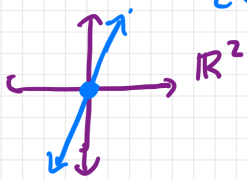
$\mathbb{R}^3$



Euclidean Space of dim 3

A **subspace** $S \subset \mathbb{R}^m$ is a collection of vectors that includes $\vec{0}$ and is "closed under addition" ($\forall \vec{v}, \vec{w} \in S \quad \vec{v}+\vec{w} \in S$) and "closed under scalar mult" ($\forall R \in \mathbb{R} \quad \vec{v} \in S, \quad R\vec{v} \in S$)

Example $S=\left\{\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x\\mx\end{pmatrix}: x\in\mathbb{R}\right\}\subset\mathbb{R}^2$



Check this is a subspace

$\vec{0}\in S$? yes at $x=0$ $\begin{pmatrix}0\\m\cdot 0\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$

Closed under addition?

$\vec{v},\vec{w}\in S$ $\quad \vec{v}=\begin{pmatrix}x_1\\mx_1\end{pmatrix}$ $\quad \vec{w}=\begin{pmatrix}x_2\\mx_2\end{pmatrix}$

$$\vec{v}+\vec{w}=\begin{pmatrix}x_1\\mx_1\end{pmatrix}+\begin{pmatrix}x_2\\mx_2\end{pmatrix}=\begin{pmatrix}x_1+x_2\\mx_1+mx_2\end{pmatrix}$$

$$=\begin{pmatrix}x_1+x_2\\m(x_1+x_2)\end{pmatrix}\in S$$

Yes because $x_1+x_2\in\mathbb{R}$

Closed under scalar?

$R\in\mathbb{R}$ $\quad \vec{v}\in S$ $\quad \vec{v}=\begin{pmatrix}x\\mx\end{pmatrix}$

$$R\vec{v}=R\begin{pmatrix}x\\mx\end{pmatrix}=\begin{pmatrix}Rx\\Rmx\end{pmatrix}=\begin{pmatrix}Rx\\m(Rx)\end{pmatrix}$$

$\in S$ because $Rx\in\mathbb{R}$.

Yes

So S is a subspace of $\mathbb{R}^2$.

See Video 313F20-10-3not for examples of spaces that are not subspaces as in the following three photos. You may choose to only watch the parabola.

An example of a set $S \in \mathbb{R}^2$ which is not a subspace.

Check $\vec{0} \in S = \left\{ \binom{x}{y} : y = x^2 \right\}$

$\vec{0} = \binom{0}{0}$ $\quad 0 = 0^2$ ✓

Check S is closed under add.

Given $\vec{v}, \vec{w} \in S$ Show $\vec{v} + \vec{w} \in S$

$\vec{v} = \binom{v_1}{v_2}$ where $v_2 = v_1^2$

$\vec{w} = \binom{w_1}{w_2}$ where $w_2 = w_1^2$

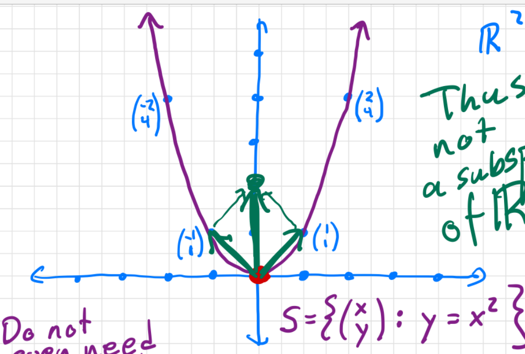
because they are in S

$\vec{v} + \vec{w} = \binom{v_1 + w_1}{v_2 + w_2}$ Check this is in S

$v_2 + w_2 \overset{?}{=} (v_1 + w_1)^2$

$= v_1^2 + 2v_1w_1 + w_1^2$

$v_2 + w_2$ $\quad = v_2 + 2v_1w_1 + w_2$



Thus not a subspace of $\mathbb{R}^2$

Do not even need to check closed under scalar because it already fails

Fails closed under addition when $2v_1w_1 \neq 0$

Counter example: Not closed under addition

$\vec{v} = \binom{1}{1}$ $\quad \vec{w} = \binom{-1}{1}$

$\vec{v} + \vec{w} = \binom{1-1}{1+1} = \binom{0}{2} \notin S$ because $2 \neq 0^2$

$$S = \left\{ \begin{pmatrix} \cos\theta \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \sin\theta \end{pmatrix} : \theta \in \mathbb{R} \right\}$$

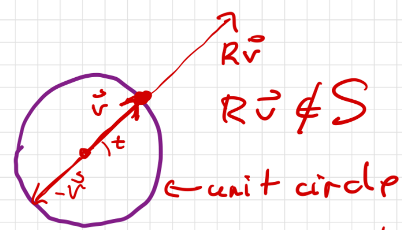
Is this closed under scalar mult?

Take $\vec{v} \in S$ $\quad \vec{v} = \begin{pmatrix} \cos t \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \sin t \end{pmatrix}$

So $\vec{v} = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$

Check $R\vec{v} \in S$:

$$R\vec{v} = R\begin{pmatrix} \cos t \\ \sin t \end{pmatrix} = \begin{pmatrix} R\cos t \\ R\sin t \end{pmatrix}$$



$R\vec{v} \notin S$

Is $R\vec{v} \in S$? <u>Fails</u>

$$\begin{pmatrix} R\cos t \\ R\sin t \end{pmatrix} = \begin{pmatrix} \cos\theta \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \sin\theta \end{pmatrix}$$

try to solve for θ

$$\boxed{\begin{aligned} R\cos t &= \cos\theta \\ R\sin t &= \sin\theta \end{aligned}}\ ?$$

How to solve ?

No solution unless $R = \pm 1$

No solution.

Notice $\underbrace{(R\cos t)^2 + (R\sin t)^2}_{R^2} = \underbrace{(\cos\theta)^2 + (\sin\theta)^2}_{=1}$

11:04 PM Wed Sep 30

Linear Algebra

$S = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| \, y = 5x + 2 \right\}$

Show S fails to have $\vec{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$0 \stackrel{?}{=} 5 \cdot 0 + 2 = 2$

Fails

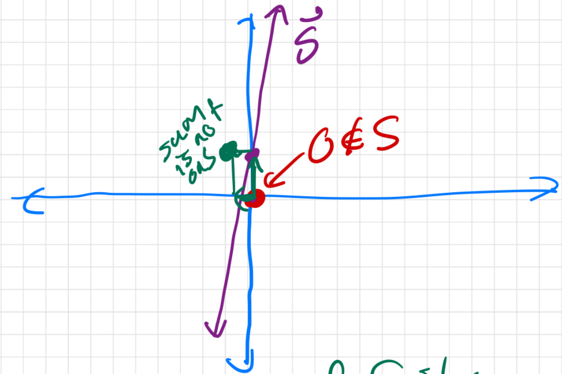
Show S fails closed under addition

$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ $v_2 = 5v_1 + 2$

$\vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ $w_2 = 5w_1 + 2$

$\vec{v} + \vec{w} = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \end{pmatrix}$ Checks $v_2 + w_2 \stackrel{?}{=} 5(v_1 + w_1) + 2$ this will fail

Linear Algebra



The proof fails

$v_2 + w_2 \stackrel{?}{=} 5(v_1 + w_1) + 2$

$(\cancel{5v_1} + 2) + (\cancel{5w_1} + 2) \stackrel{?}{=} \cancel{5v_1} + \cancel{5w_1} + \cancel{2}$

$2 \neq 0$?!

Counter examp le

- Chose any two vectors on S and add them. show the sum is not on S

Linear Algebra

$$S = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| \; y = 5x + 2 \right\}$$

Show S fails to have $\vec{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$0 \stackrel{?}{=} 5 \cdot 0 + 2 = 2$$

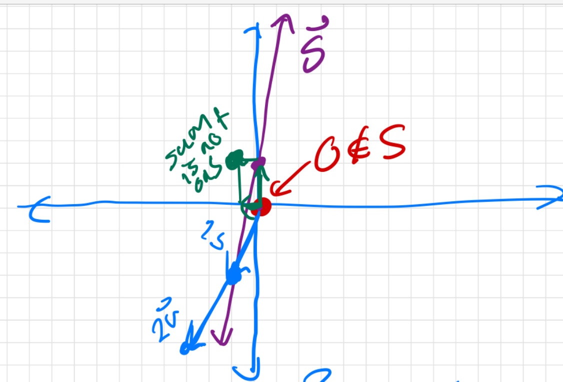
Fails

Show closed under scalar fails.

$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ $\quad v_2 = 5v_1 + 2$

$R\vec{v} = \begin{pmatrix} Rv_1 \\ Rv_2 \end{pmatrix}$ Check $Rv_2 \stackrel{?}{=} 5(Rv_1) + 2$

$$R(5v_1 + 2) \stackrel{?}{=} 5Rv_1 + 2$$

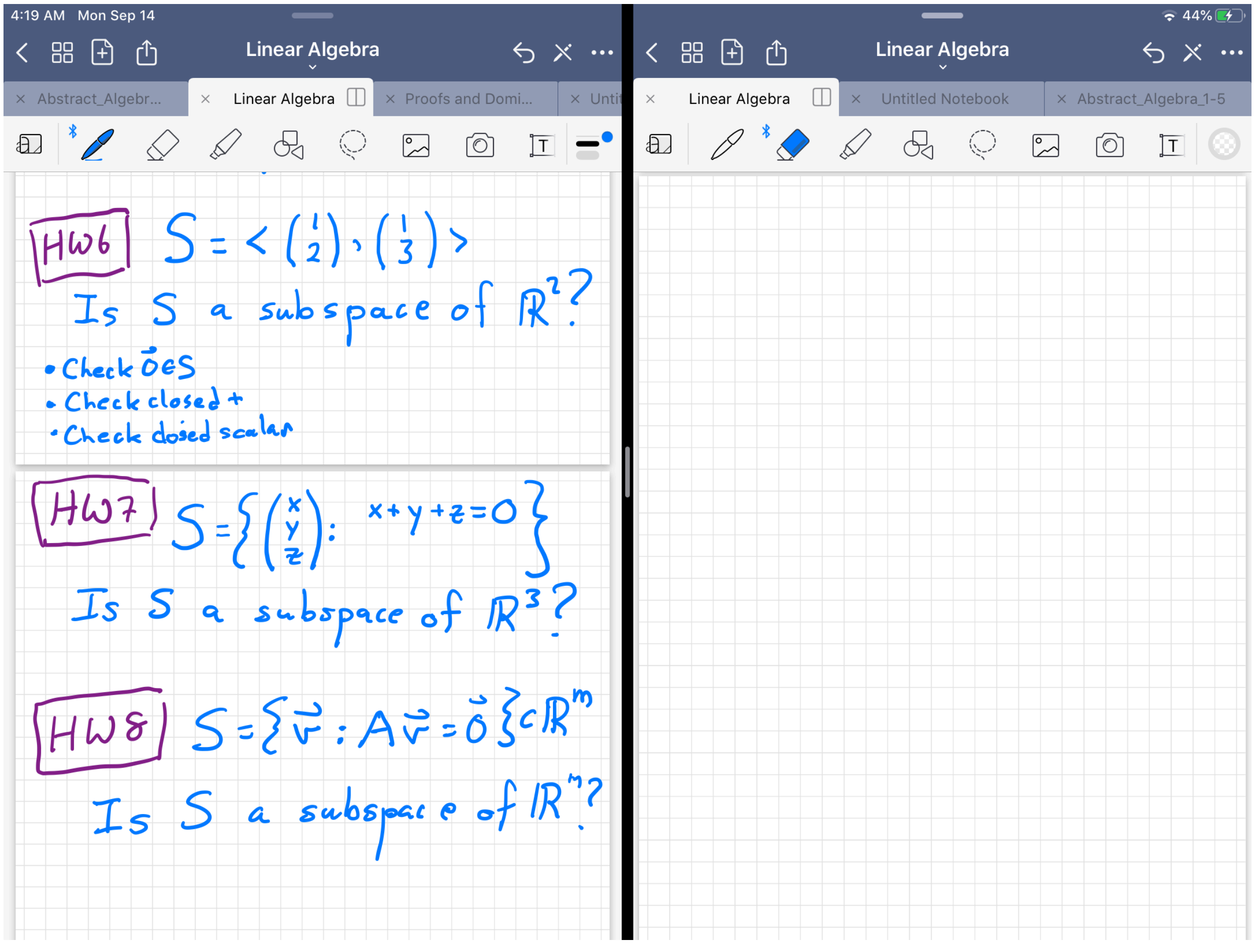


$$\cancel{R}5v_1 + R2 \stackrel{?}{=} 5\cancel{R}v_1 + 2$$

$R \cdot 2 = 2$ fails unless $R = 1$.

Not closed under scalar

Counter example: take any v on the line and mult by any $R \neq 1$



HW6 $S = \left\langle \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right\rangle$

Is S a subspace of $\mathbb{R}^2$?

- Check $\vec{0} \in S$
- Check closed $+$
- Check closed scalar

HW7 $S = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x + y + z = 0 \right\}$

Is S a subspace of $\mathbb{R}^3$?

HW8 $S = \{ \vec{v} : A\vec{v} = \vec{0} \} \subset \mathbb{R}^m$

Is S a subspace of $\mathbb{R}^m$?

Complete the above HW before starting Part 3.

For HW6 -HW8 be sure to check the 0 vector is in the set, and that it is closed under addition and closed under scalar. For HW hints see Video 313F20-10-4 and if necessary more hints in Video 313F20-10-4more if you are having trouble.

Be sure to check these HW hints videos above before submitting your work.

Part 3: linearly independent vectors

Watch Video 313F20-10-5a for the definition and then Video 313F20-10-5b for classwork and hw hints:

Linearly Independent Vectors

Defn: A collection of vectors, $\vec{v}_1, \vec{v}_2, \dots \vec{v}_k$, is linearly independent if

$$\underbrace{\sum_{j=1}^{k} t_j \vec{v}_j}_{\text{linear combination}} = \vec{0} \iff \text{all the } t_j = 0 \text{ for } j = 1, 2 \dots k$$

$$\underbrace{t_1 \vec{v}_1 + t_2 \vec{v}_2 + \dots + t_k \vec{v}_k}_{\text{solve for } t_1, t_2 \dots t_k} = \vec{0}$$

to this homogeneous system.

Watch Video 313F20-10-5b for the following classwork and HW9-10 hints:

Classwork:

Are $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}$ and $\begin{pmatrix}2\\3\\0\end{pmatrix}$ lin. indep?

$$t_1\begin{pmatrix}1\\0\\0\end{pmatrix} + t_2\begin{pmatrix}0\\1\\0\end{pmatrix} + t_3\begin{pmatrix}2\\3\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}1t_1 + 0t_2 + 2t_3\\ 0t_1 + 0t_2 + 3t_3\\ 0t_1 + 0t_2 + 0t_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

no leaders in a column so a free variable

$$\left[\begin{array}{ccc|c}\boxed{1} & 0 & 2 & 0\\ 0 & 0 & \boxed{3} & 0\\ 0 & 0 & 0 & 0\end{array}\right] \xrightarrow{\rho_2 \to \frac{1}{3}\rho_2} \left[\begin{array}{ccc|c}\boxed{1} & 0 & 2 & 0\\ 0 & 0 & \boxed{1} & 0\\ 0 & 0 & 0 & 0\end{array}\right]$$

2 leaders and one free variable

The solution set is not just $\{\vec{0}\}$

There are t_1, t_2, t_3 not all zero such that

$$t_1\begin{pmatrix}1\\0\\0\end{pmatrix} + t_2\begin{pmatrix}0\\1\\0\end{pmatrix} + t_3\begin{pmatrix}2\\3\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

Thus $\begin{pmatrix}1\\0\\0\end{pmatrix} \begin{pmatrix}0\\1\\0\end{pmatrix}$ and $\begin{pmatrix}2\\3\\0\end{pmatrix}$ is not lin indep.

$\boxed{\text{HW9}}$ Are $\begin{pmatrix}1\\0\\1\end{pmatrix}$ $\begin{pmatrix}0\\1\\2\end{pmatrix}$ $\begin{pmatrix}2\\1\\3\end{pmatrix}$ linearly independent?

Hint solve $t_1\begin{pmatrix}1\\0\\1\end{pmatrix} + t_2\begin{pmatrix}0\\1\\2\end{pmatrix} + t_3\begin{pmatrix}2\\1\\3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$

$\boxed{\text{HW10}}$ Are $\begin{pmatrix}1\\0\\1\\0\end{pmatrix}$ $\begin{pmatrix}2\\1\\2\\1\end{pmatrix}$ and $\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$ linearly independent?

Show all work!

Hint: no free variables $\Leftrightarrow \begin{pmatrix}t_1\\ \vdots\\ t_k\end{pmatrix} = \begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}$

then your vectors are lin. indep

However if there are free variables then vectors are not lin. indep.

Hint: You do not need to write the solution set.

Complete the above HW before starting Part 4

Part 4: Basis of a subspace

Watch [Video 313F20-10-6better] which includes homework hints:

4:24 AM Thu Oct 1

Linear Algebra

Abstract_Algebra_1-5 | Linear Algebra

Lesson 10 Part IV

Basis of a Subspace

Defn

Suppose $S \subset \mathbb{R}^m$ is a subspace we say $\vec{v}_1, \vec{v}_2 \dots \vec{v}_k$ is a "basis" for S if

$$S = \langle \vec{v}_1, \vec{v}_2, \dots \vec{v}_k \rangle$$

and

$\vec{v}_1 \dots \vec{v}_k$ are linearly independent

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Linear Algebra

Linear Algebra | Modern Algebra & Number Systems

Example:

$$S = \left\{ \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} : x, y \in \mathbb{R} \right\}$$

$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ are a basis.

To see this we check

$$S = \langle \vec{v}_1, \vec{v}_2 \rangle$$

proof:

$$\langle \vec{v}_1, \vec{v}_2 \rangle = \{ t_1 \vec{v}_1 + t_2 \vec{v}_2 \mid t_1, t_2 \in \mathbb{R} \}$$

$$= \left\{ t_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \;\middle|\; t_1, t_2 \in \mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix} t_1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ t_2 \\ 0 \end{pmatrix} \;\middle|\; t_1, t_2 \in \mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix} t_1 \\ t_2 \\ 0 \end{pmatrix} \;\middle|\; \underbrace{t_1, t_2 \in \mathbb{R}}_{\text{free variable}} \right\} = S$$

free variable $t_1 = x$ $t_2 = y$

Also check linearly indep

$t_1 v_1 + t_2 v_2 = 0 \iff t_1 = 0$ and $t_2 = 0$.

4:25 AM Thu Oct 1

Linear Algebra

Abstract_Algebra_1-5 | Linear Algebra

Lesson 10 Part IV

Basis of a Subspace

Defn

Suppose $S \subset \mathbb{R}^m$ is a subspace we say $\vec{v}_1, \vec{v}_2 \ldots \vec{v}_k$ is a "basis" for S if

$$S = \langle \vec{v}_1, \vec{v}_2, \ldots \vec{v}_k \rangle$$

and

$\vec{v}_1 \ldots \vec{v}_k$ are linearly independent

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Linear Algebra

Linear Algebra | Modern Algebra & Number Systems

$$t_1\begin{pmatrix}1\\0\\0\end{pmatrix} + t_2\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}t_1\\0\\0\end{pmatrix} + \begin{pmatrix}0\\t_2\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}t_1\\t_2\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$t_1 = 0$ and $t_2 = 0$

So they are lin indep

Span + Lin Indep

Thus they are a basis.

Null(A) is a subspace

We know how to find it

$\{ \vec{v} = \text{free variables times directions} \}$

$= \langle$ span of the directions $\rangle$

To find the basis of a null space

- Find the direction vectors
- Check they are lin indep.

$\boxed{\text{Solution to a homogeneous system}}$

Linear Algebra

Example

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{Find } \text{Null}(A) = ?$$

Find it and find its basis

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

already Echelon Form

$1x + 0y + 0z = 0$ $\quad x = 0$

$0x + 1y + 0z = 0$ $\quad y = 0$

$0 = 0$ $\quad z = z$ (free)

$$\text{Null}(A) = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} = z \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{\text{direction}} \;\middle|\; z \in \mathbb{R} \right\}$$

$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is a basis

$\text{Null}(A) = \left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle$

and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is lin indep $\quad t\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Downarrow t = 0.$

4:38 AM Thu Oct 1

Linear Algebra

Abstract_Algebra_1-5 | Linear Algebra

Example:

$$\begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = A$$

Find basis for Null(A)

Find Null(A)

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

already red echelon

$\boxed{x_1} + 0x_2 + 0x_3 + 2x_4 = 0$

$\boxed{x_3} = 0$

$x_1 = -2x_4$

$x_2 = x_2$ (free)

$x_3 = 0$

$x_4 = x_4$ (free)

$$Null(A) = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \end{pmatrix} \;\middle|\; x_2, x_4 \in \mathbb{R} \right\}$$

$$= \left\langle \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle$$ span Null(A)

Next check the directions are linearly indep.

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$$t_1 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ t_1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -2t_2 \\ 0 \\ 0 \\ t_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2t_2 \\ t_1 \\ 0 \\ t_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$-2t_2 = 0$

$t_1 = 0 \longrightarrow t_1 = 0$

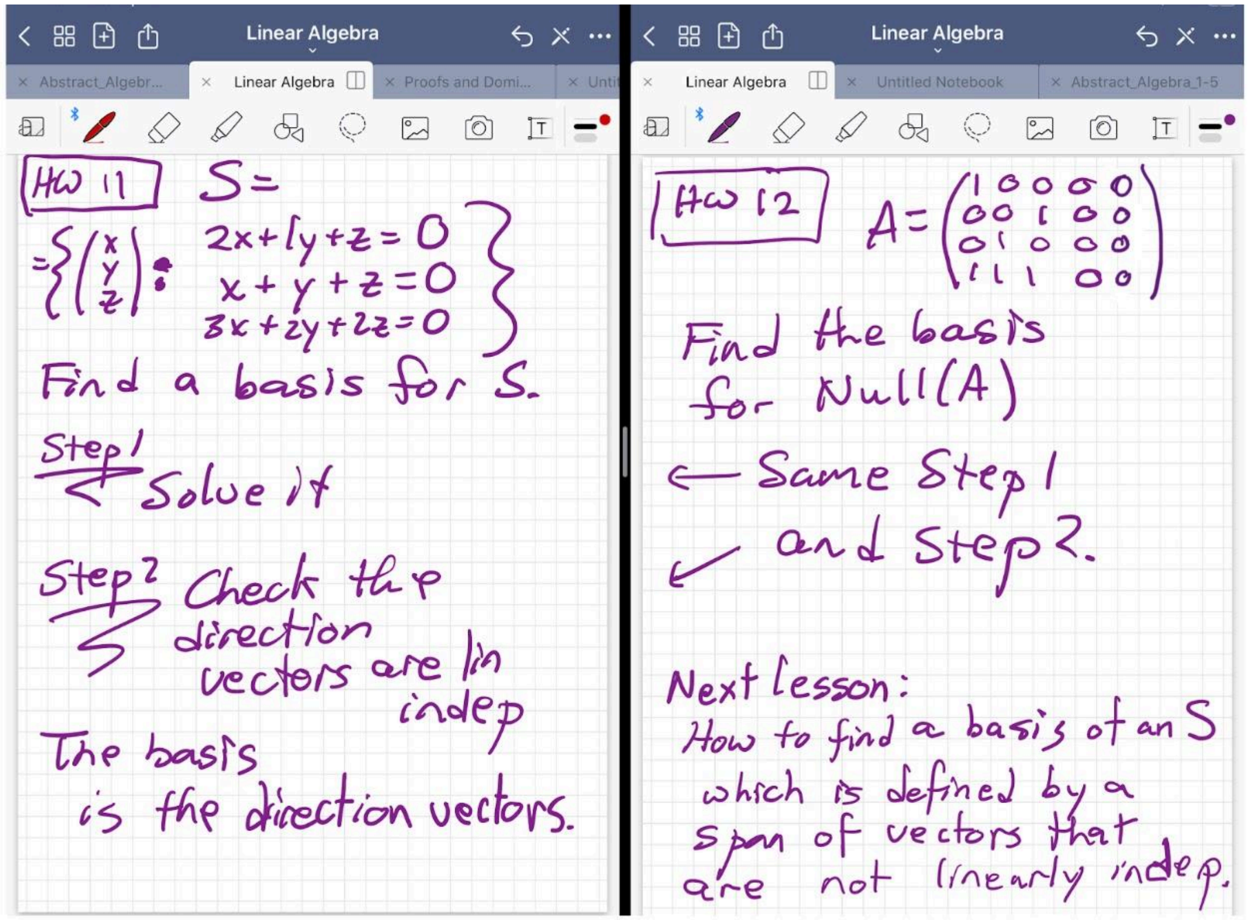
$0 = 0$

$t_2 = 0 \longrightarrow t_2 = 0$

Therefore $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ are linearly indep

Since they also span null(A)

They are a basis for null(A)



There are 12 homework problems above. Here's the full playlist for the lesson to check if you watched everything..

Don't forget to help your team with extra credit. Just search for team in your googledocs.

Check your homework using these solutions. Email me if your answers are different because they might also be correct.

Share with Professor Sormani at sormanic@gmail.com.
Please email questions to Professor Sormani.

Note this lesson was Lesson 10 in the past.