⌘ Course Content

- Derivative of inverse trigonometric, exponential and logarithmic function by definition, relationship between continuity and differentiability, rules for differentiating hyperbolic function and inverse hyperbolic function, L'Hospital's rule (0/0, ∞/∞), differentials, tangent and normal, geometrical interpretation and application of Rolle's theorem and mean value theorem.

⌘ Learning Outcomes

On completion of this unit, students will be able to:

- find the derivatives of inverse trigonometric, exponential and logarithmic functions by definition.
- establish the relationship between continuity and differentiability.
- differentiate the hyperbolic function and inverse hyperbolic function
- evaluate the limits by L'Hospital's rule (for ,).
- find the tangent and normal by using derivatives.
- interpret geometrically and verify Rolle's theorem and Mean Value theorem.

14.1 Introduction

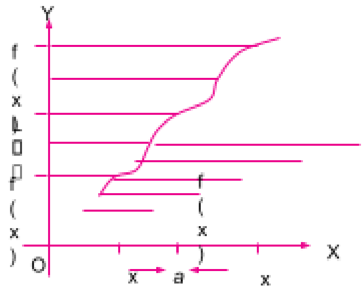
In this chapter, a brief review of limits, continuity, derivatives and applications of derivatives is given as the basic concepts that are given in Grade XI. The derivative of a function is an important tool in mathematics. The motion of an object can be described by the derivative. The velocity of the object is a measure of the rate of change of distance with respect to time. Acceleration is a measure of the rate of change of velocity with respect to time. It also describes the rate of change of current in an electric circuit. In biology, derivate is used to determine the rate of growth of bacteria in a culture. In economics, it is used to describe the problems dealing with profit and loss.

14.2 Limit of a Function

Let 'a' be a fixed point and $f(x)$ be a function defined in the neighborhood of the point $x = a$. The function $f(x)$ is said to have the limit L, L$\in \mathbb{R}$ at $x = a$, if $f(x)$ gets closer and closer to a number L as x gets closer and closer to the number 'a' from both sides. This behavior of $f(x)$ is expressed by

$$f(x) = \text{L}.$$

Geometrically, $f(x) = \text{L}$ means that L is the height of the graph of $y = f(x)$ near a, as in the adjoining figure.



One sided Limits

A function $f(x)$ is said to tend to the limit L_1, ($\text{L}_1 \in \mathbb{R}$) from the right of '$a$' if $f(x)$ approaches L_1, when x approaches a from the numbers greater than a. This is denoted by $f(x) = \text{L}_1$ or $f(a + h) = \text{L}_1$. This is known as one sided limit or right hand limit of $f(x)$ at $x = a$.

Similarly, the function $f(x)$ is said to tend to the limit L_2, ($\text{L}_2 \in \mathbb{R}$) from the left of '$a$' if $f(x)$ approaches L_2 when x approaches 'a' from the numbers smaller than 'a'. This is denoted by

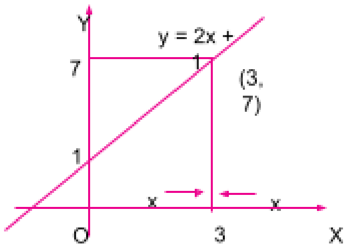
$$f(x) = \text{L}_2 \quad \text{or} \quad f(a - h) = \text{L}_2$$

This is known as one sided limit or left hand limit of $f(x)$ at $x = a$.

Two sided Limit

$f(x) = \text{L}$ is called the **two sided limit** if $f(x) = \text{L} = f(x)$.

Example 1. Consider $f(x) = 2x + 1$ and the point $x = 3$ in the domain of f. The table below shows that the functional values of $f(x)$ at different values of x. The limiting value of $f(x)$ is 7 as x approaches 3, from the left of 3 and from the right of 3. Therefore $f(x) = 7$.



Table

x	2	2.5	2.9	2.99	2.999	→	3	3.5	3.1	3.01	3.001	→	3
$f(x)$	5	6	6.8	6.98	6.998	→	7	8	7.2	7.02	7.002	→	7

14.3 Continuity of a Function at a Point

The function $f(x)$ is said to be continuous at the point $x = a$ if and only if the following three conditions are satisfied:

1. $f(a)$ exists.
2. $f(x)$ exists.
3. $f(x) = f(a)$.

If any one of the above conditions are not satisfied, then the function is said to be discontinuous at $x = a$.

If a function is continuous at $x = a$, then the limit of $f(x)$ at $x = a$ exists. But the converse may not be true. For the converse to be true $f(a)$ must exist.

Equivalently, the function $f(x)$ is continuous at $x = a$ if

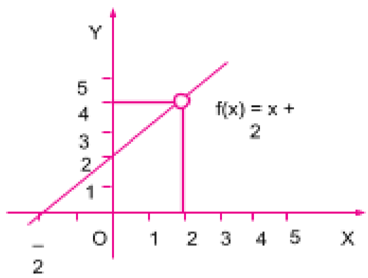
$$f(x) = f(x) = f(a) = L,\ L \in \mathbb{R}.$$

Illustrative Examples

Example 1. The function $f(x) =$ has limiting value 4 at $x = 2$, but $f(2)$ does not exists.

Solution

The graph in the adjoining figure shows that $f(x)$ is discontinuous at $x = 2$. There is a break in the graph at $x = 2$. In fact, $f(2)$ is not defined. The function $f(x)$ can be made continuous if we take $f(2) = 4$.



Example 2. The graph of the function $f(x) =$ has a jump at $x = 2$. Show that $f(x)$ is discontinuous at $x = 2$.

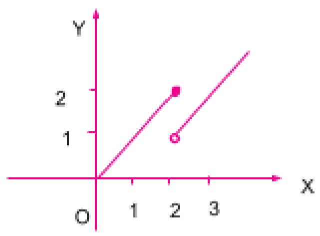
Solution

Here, $f(2) = 2$

$f(x)$ does not exist, since

$f(x) = 2$ and $f(x) = = 1$

$\therefore$ $f(x)$ is discontinuous at $x = 2$.



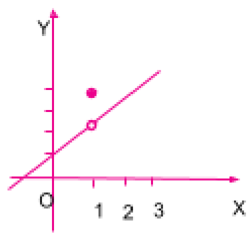
Example 3. Show that the function defined by $f(x) =$ is discontinuous at $x = 1$.

Solution

$f(x) = (2x + 1) = 3$

$f(x) = (2x + 1) = 3$

Thus



$f(x) = 3$.

But $f(1) = 4$.

Therefore, $f(x) \neq f(1)$

Hence, $f(x)$ is discontinuous at $x = 1$

If we take $f(1) = 3$, then $f(x)$ will be continuous at $x = 1$.

Example 4. Let h be defined by $h(x) =$. Show that $h(x)$ is discontinuous at $x = 3$.

Solution

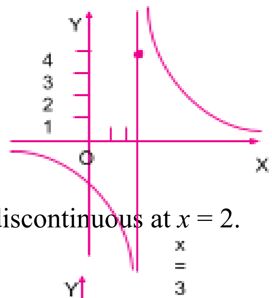
Here, $h(3) = 4$.

$h(x) = \quad = \infty$

and $h(x) = \quad = -\infty$.

Therefore, $h(x)$ does not exist

Hence, $h(x)$ is discontinuous at $x = 3$.



Example 5. Let $f(x)$ be a function defined by $f(x) =$. Show that $f(x)$ is discontinuous at $x = 2$.

Solution

Here, $f(2) = 3$

Since $|x - 2| = x - 2$ if $x > 2$

and $|x - 2| = -(x - 2)$ if $x < 2$

We have

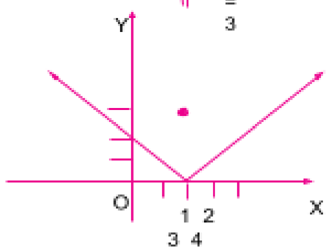
$f(x) = (x - 2) = 0$

and $f(x) = (-x + 2) = 0$

Therefore, $f(x) = 0$.

Here, $f(x) \neq f(2)$.

Hence, $f(x)$ is discontinuous at $x = 2$.



Example 6. Discuss the continuity of the function $f(x) =$, when $x \neq 0,2$, when $x = 0)$) , at $x = 0$.

Solution

Left hand limit of $f(x)$ at $x = 0$ is

$f(x) \quad = f(0 - h)$

$=$

$= \quad =$

$= 2$

Right hand limit of $f(x)$ at $x = 0$ is

$f(x) \quad = f(0 + h)$

$=$

$= 0$

Here, $f(x) \neq f(x)$

Hence, the given function is discontinuous at $x = 0$.

Example 7. A function is defined as $f(x) = e^{1/x}$, $x \neq 0$ and $f(0) = 0$. Examine it for continuity at $x = 0$.

Solution

$f(x)$ $= e^{1/x}$

$=$ . . . . .

$= \infty$

Hence, the given function $f(x)$ is discontinuous at $x = 0$.

Example 8. Discuss the continuity of the function $f(x) =$, at $x = 0$.

Solution

We know that for all real $x \neq 0$, $-1 \leq \sin \leq 1$

This implies that $-x^2 \leq x^2 \sin \leq x^2$

Now, $(-x)^2 = 0$ and $x^2 = 0$.

Hence by Squeeze theorem, we have

$x^2 \sin = 0$

Also we have $f(0) = 0$

Therefore the given function $f(x)$ is continuous at $x = 0$.

Example 9. Discuss the continuity of $f(x) =$ at $x = 8$.

Solution

Right hand limit of $f(x)$ at $x = 8$ is

$f(x)$ $= f(8 + h)$

$=$

$= 1$

Left hand limit of $f(x)$ at $x = 8$ is

$f(x)$ $= f(8 - h)$

$=$

$= -1$

$\therefore$ $f(x) \neq f(x)$

Thus, $f(x)$ is discontinuous at $x = 8$.

Exercise 14.1

1. Discuss the continuity and discontinuity of the following functions at the point specified:

a. $f(x) =$, at $x = 2$.

b. $f(x) =$, at $x = 0$.

c. $f(x) =$, at $x = 0$.

2. For what value of k is the function

a. $f(x) =$ continuous at $x = 2$?

b. $f(x) =$ continuous at $x = 0$?

3. A function $f(x)$ is defined as $f(x) =$. Show that it is continuous at $x = 0$ and $x = 1$.

4. A function $f(x)$ is defined as $f(x) =$. Test the continuity of $f(x)$ at $x = 0$.

5. Examine for continuity at $x = 0$ for the function $f(x)$ defined by $f(x) =$.

Answers

1. a. Discontinuous b. Discontinuous c. Discontinuous

2. a. b. **4.** Discontinuous **5.** Discontinuous

14.4 Derivative of a Function at a Point

The **derivative** of a function $f(x)$ at a point $x = a$, denoted by $f'(a)$, is defined as

$f'(a) =$ . . . (1)

provided the limit exists.

Equivalently, $f'(a) =$ . . . (2)

Taking $a + h = x$ in (1), we get (2).

If $f(x)$ has a derivative at a point, $f(x)$ is said to be **differentiable** or **derivable** at that point. The derivative of a function is also called its **differential coefficient**. The process of finding out the derivative of a function is called **differentiation**.

The differentiable coefficient of $y = f(x)$ is generally written as , $f'(x)$, $(f(x))$, D$f(x)$.

Thus, $f'(x) =$ is the derivative of $f(x)$ at x.

Right Hand and Left Hand Derivatives

The right hand derivative of $f(x)$ at $x = a$, denoted by R$f'(a)$ is defined as

R$f'(a) =$, $h > 0$ . . . (3)

provided the limit exists.

The left hand derivative of $f(x)$ at $x = a$ denoted by L$f'(a)$ is defined as

L$f'(a) =$, $h > 0$ . . . (4)

provided the limit exists. If (3) and (4) are equal to a finite number L, then $f(x)$ is said to have derivative at $x = a$.

If (3) and (4) exist but are different, then R$f'(a)$ is called the **progressive** and L$f'(a)$ is called **regressive** differential coefficient of $f(x)$ at $x = a$.

Relationship between Continuity and Differentiability

Theorem: If a function $f(x)$ is derivable at a point, it is continuous at that point.

Proof: Let $f(x)$ be derivable at $x = a$. Then,

$f'(a) =$ exists and it is finite.

Now

$[f(a + h) - f(a)] = \times h$

or, $[f(a + h) - f(a)] = f'(a) \cdot h = f'(a) \cdot 0$

or, $[f(a + h) - f(a)] = 0$

∴ $f(a + h) = f(a)$

Hence, $f(x)$ is continuous at $x = a$.

The converse of the theorem may not be true. This can be illustrated by the following example.

Example: The function $f(x) = |x|$ is continuous at $x = 0$ but not differentiable at $x = 0$.

We have, $|x| =$.

For the continuity of $f(x)$ at $x = 0$:

Right hand limit of $f(x)$ at $x = 0$

$f(x) = f(0 + h) = |0 + h| = h = 0$

Left hand limit of $f(x)$ at $x = 0$

$f(x) = f(0 - h) = |0 - h| = h = 0$

and $f(0) = 0$

∴ $f(x)$ is continuous at $x = 0$.

For the differentiability of $f(x)$ at $x = 0$:

Right hand derivative of $f(x)$ at $x = 0$

$Rf'(0) = \quad = \quad = 1$

Left hand derivative of $f(x)$ at $x = 0$

$Lf'(0) = \quad = \quad = -1$

∴ $Rf'(0) \neq Lf'(0)$

Hence, $f(x)$ is not differentiable at $x = 0$.

Thus "The continuity of a function at a point is the necessary but not the sufficient condition for the existence of the derivative of the function at that point."

Rules of Differentiation

We first revisit some basic derivatives some of which were derived in Grade XI.

1. a. $(x^n) = nx^{n-1}$ b. $(e^x) = e^x$
   b. $(a^x) = a^x \ln a$ c. $(\ln x) =$
   d. $= \cos x$ e. $= -\sin x$
   f. $(\tan x) = \sec^2 x$ g. $(\sec x) = \sec x \tan x$
   g. $(\cot x) = -\text{cosec}^2 x$ i. $(\text{cosec } x) = -\text{cosec } x \cdot \cot x$
2. a. $(\sin^{-1} x) = \ , (-1 < x < 1)$ b. $(\cos^{-1} x) = \ , (|x| < 1)$
   c. $(\tan^{-1} x) = \ , (x \in \mathbb{R})$ d. $(\cot^{-1} x) = \ , (x \in \mathbb{R})$
   e. $(\sec^{-1} x) = \ , (|x| > 1)$ f. $(\text{cosec}^{-1} x) = \ , (|x| > 1)$
3. a. $= u \ + v$ b. $=$
4. If $x = f(t)$ and $y = \varphi(t)$ then, $=$

We now evaluate the derivative of some functions using first principle method.

Illustrative Examples

Example 1. Find from first principle the derivative of

a. $\cos^{-1} x$ b. a^x c. $e^{\sin x}$

d. $\ln x^x$ e. $\sin(\ln x)$

Solution

a. Let $f(x) = \cos^{-1} x$. Then,

$f(x + h) = \cos^{-1}(x + h)$

From first principle

$f'(x)$ =

=

Let $\cos^{-1} x = z$ and $\cos^{-1} (x + h) = k + z$. Then

$k = \cos^{-1} (x + h) - \cos^{-1} x$, and $k \to 0$ as $h \to 0$

∴ $f'(x)$ =

=

= θθθ

= – =

= [∵ $\cos^{-1} x = z \Rightarrow x = \cos z$]

∴ $f'(x)$ = .

b. Let $f(x) = a^x = e^{\ln a^x} = e^{x \ln a}$. Then

$f(x + h) = e^{(x+h)\ln a}$

From first principle, we have

$f'(x) =$

$=$

$= \ e^{x\ln a}$

$= e^{x\ln a} \cdot \cdot \ln a$

$= e^{x\ln a} \cdot 1 \cdot \ln a$

$= a^x \ln a$

∴ $f'(x) = a^x \ln a$.

c. Let $f(x) = e^{\sin x}$. Then

$f(x + h) = e^{\sin (x + h)}$.

From first principle, we have

$f'(x) \quad =$

$=$

Putting $\sin x = z$ and $\sin(x + h) = z + k$, then $k = \sin(x + h) - \sin x$ and $k \to 0$ as $h \to 0$

∴ $f'(x) \quad =$

$= e^z \ \cdot$

$= e^z \cdot 1 \cdot$

$= e^z$

$= e^z \ \cos \b(x + \f(h,2))$

$= e^z \cdot \cos x \ \cdot 1$

∴ $f'(x) \quad = e^{\sin x} \cos x$.

d. Let $f(x) = \ln x^x = x \ln x$. Then

$f(x + h) = (x + h) \ln(x + h)$

By definition or from first principle, we have

$f'(x) \quad =$

$=$

$=$

$=$

$= \ + \ \ln (x + h)$

$= \ + \ \ln (x + h)$

∴ $f'(x) \quad = 1 + \ln x$.

e. Let $f(x) = \sin(\ln x)$. Then,

$f(x + h) = \sin \ln(x + h)$

By definition or from first principle, we have

$f'(x) \quad =$

$=$

Putting $z = \ln x$ and $z + k = \ln(x + h)$, then $k = \ln(x + h) - \ln x$. When $h \to 0, k \to 0$

= ·

= \f(sin(*z* + *k*) – sin *z*,*k*)) \f(*k*,*h*))

= \f(ln(*x* + *h*) – ln *x*,*h*))

= ·

= cos *z* · · 1

=

∴ $f'(x)$ = .

Example 2. Consider the function $f(x)$ defined as $f(x)$ = . Discuss the continuity and differentiability of $f(x)$ at $x = 0$.

Solution

Right hand limit of $f(x)$ at $x = 0$ is	Left hand limit of $f(x)$ at $x = 0$ is
$f(x) = (0 + h)$ sin	$f(x) = f(0 - h)$
$= h$ sin	$= (0 - h)$ sin
$= 0$	$= h$ sin $= 0$

and $f(0) = 0$

∴ $f(x) = f(0)$

Hence, $f(x)$ is continuous at $x = 0$.

Again

R $f'(0) =$

$= =$ sin , which does not exist

∴ $f'(0)$ does not exist

Hence, $f(x)$ is not differentiable at $x = 0$.

Example 3. A function $f(x)$ is defined as $f(x) = \infty$. Does f' exist?

Solution

The right hand derivative of $f(x)$ at $x =$ is

R f' =

=

= $h = 0$

Left hand derivative of $f(x)$ at $x = \pi$

L f' =

=

=

=

=

$= 1 \cdot 0 = 0$

Here, R f' = L f'

Therefore, f' πexists and is equal to zero.

Example 4. If $f(x) =$, find $f'(0)$.

Solution

The right hand derivative of $f(x)$ at $x = 0$ is

R $f'(0)$ = ,

=

=

$= e$

Putting $k = \cos h - 1$, when $h \to 0$, $k \to 0$

∴ R $f'(0)$ = · ·

$= e$ \f(e^k – 1,k)) \f(cos h – 1,h))

$= e \cdot 1 \cdot$

$= -2e \times$

$= -2e \cdot 1 \cdot 0$

$= 0$

The left hand derivative of $f(x)$ at $x = 0$ is

L f' (0) =

=

=

= – e

= – $e \cdot 0$ [As above]

= 0

Since R f' (0) = L f' (0) = 0, we have $f'(0) = 0$

Example 5. Find the differential coefficient of $\tan^{-1}$.

Solution

Let $x^2 = \cos 2\theta \Rightarrow \theta = \cos^{-1} x^2$ and

$\tan^{-1}$ = y.

Now

y = $\tan^{-1}$

= $\tan^{-1}$

= $\tan^{-1}$

= $\tan^{-1}$)

∴ y = + θ

Thus

=

= × × $2x$ = .

Example 6. If $y = ex \sin x^3 + (\tan x)^x$, find .

Solution

Let $u = ex \sin x^3$ and $v = (\tan x)^x$. Then

$y = u + v$ and = + . . . (1)

Now

= $(ex \sin x^3)$

= $ex \sin x^3$ $(x \sin x^3)$

= $ex \sin x^3$ [1 $\cdot$ $\sin x^3 + x \cdot \cos x^3 \cdot 3x^2$]

= $(\sin x^3 + 3x^3 \cos x^3) ex \sin x^3$

Further, $\ln v = x \ln (\tan x)$

∴ = 1 $\cdot$ $\ln \tan x + x \cdot$ $\sec^2 x$

or, = v [$\ln (\tan x) + x \cot x \sec^2 x$]

∴ = $(\tan x)^x$ [$\ln (\tan x) + x \cot x \sec^2 x$]

Hence from (1),

= $(\sin x^3 + 3x^3 \cos x^3) ex \sin x^3 + (\tan x)^x$ [$\ln (\tan x) + x \cot x \sec^2 x$].

Example 7. If $x^y = e^{y-x}$, show that = .

Solution

We have

$x^y = e^{y-x}$

Taking logarithm on both sides, we get

$y \ln x = (y - x)\ln e = y - x$ [$\because \ln e = 1$]

or, $y - y \ln x = x$

or, $y(1 - \ln x) = x$

$\therefore$ $y =$

Now, differentiating both sides w.r.t. x, we get

=

=

= .

Example 8. If + = $a(x-y)$, show that = .

Solution

Let $x = \sin\theta$, $y = \sin\varphi$, then, we have

+ = $a(\sin\theta - \sin\varphi)$

or, $\cos\theta + \cos\varphi = a(\sin\theta - \sin\varphi)$

or, 2 cos cos = $2a$ cos sin

or, cos = a sin

or, cot = a

or, = $\cot^{-1} a \Rightarrow \theta - \varphi = 2\cot^{-1} a$

or, $\sin^{-1} x - \sin^{-1} y = 2\cot^{-1} a$

Differentiating with respect to x, we get

– · = 0

∴ =

Exercise 14.2

Find, from first principle, the derivatives of the following functions (1 – 5).

1. a. e b. $e^{\cos x}$ c.

2. a. $\ln\cos x$ b. ln sin c. $\ln\sec x$

3. a. $\sin^{-1} x$ b. $\tan^{-1} x$ c. $\ln\cos^{-1} x$

4. a. $x\ln x$ b. x^x c. 2^{x2}

5. a. $\sin x^2$ b. $x^2\tan x$ c.

6. Find the derivatives of the following functions at the points specified:

a. $e^{\cos x}$ at $x = 0$ b. $\ln\cos x$ at $x = 0$

7. If $f(x)$ = , then show that $f(x)$ is continuous at $x = 0$, but $f'(0)$ does not exist.

8. If $f(x)$ = , does f' exist?

9. Find $f'(0)$ for the following functions:

a. $e^{\sin x}$ b. $\ln\cos x$ c. $f(x) = \neq$.

10. Find the derivatives of

a. $(\sin x)^{\cos x}$ b. $(\ln x)^{\tan x}$ c. $(\sin x)^{\ln x}$.

d. ex^x e. xe^x

11. Find , when

a. $x^y y^x = 1$ b. $x^m y^n = (x+y)^{m+n}$ c. $e^{\sin x} + e^{\sin y} = 1$

d. $x^y = y^x$ e. $x^{\sin x} = y^{\sin y}$.

12. If $x^y = e^{x-y}$, prove that = .

13. If $x\cos y = \sin(x+y)$, find .

14. If $\sin y = x\sin(a+y)$, prove that = .

15. a. If $y = (\sin x)^{\tan x} + (\cos x)^{\sec x}$, find .

b. If $y = (\cos x)^{\ln x} + (\ln x)^x$, find .

16. Find of the following functions:

a. $y =\ -\ \ln.$

b. $y = \ + \ \sin^{-1}.$

c. $y = \tan^{-1}.$

Answers

1. a. ,2\r(x)) b. $-\sin x\, e^{\cos x}$ c.
2. a. $-\tan x$ b. c. $\tan x$
3. a. b. c.
4. a. $1 + \ln x$ b. $x^x(1 + \ln x)$ c. $2x \ln 2 \cdot 2^{x2}$
5. a. $2x \cos x^2$ b. $2x \tan x + x^2 \sec^2 x$ c.
6. a. 0 b. 0
8. Yes
9. a. 1 b. 0 c. 0
10. a. $[(\sin x)^{\cos x} (\cos x \cot x - \sin x \ln \sin x]$ b. $(ln x)^{\tan x}$
    c. $(\sin x)^{\ln x}$ d. $ex^x\, x^x\, (1 + \ln x)$ e. $xe^x \,.\, e^x$
11. a. b. c. d. e.
13.
15. a $(\sin x)^{\tan x}\, [1 + \ln \sin x \sec^2 x] + (\cos x)^{\sec x} \cdot \sec x \tan x\, [\ln (\cos x) - 1]$
    b. ·
16. a. b. c.

Multiple Choice Questions

1. Which of the following statement is true?

   a. If a function is differentiable at a point, it is continuous at that point
   b. If a function is continuous at a point, it is differentiable at that point
   c. A function is differentiable if and only if it is continuous
   d. None of them

2. The function $f(x) = |\, x\,|$ is

   a. differentiable at each point
   b. differentiable at $x = 0$ only
   c. not differentiable at $x = 0$
   d. None of them

3. The derivative of $\sin^{-1} 2x$ is

   a.
   b.
   c.
   d.

4. The derivative of $\ln(\sin x)$ is

   a. $\cot x$
   b. $\operatorname{cosec} x$
   c. $\cos x$
   d. $\tan x$

5. The derivative of $e^{\tan-1x}$ is

   a.
   b. $e^{\tan-1x}$
   c.
   d.

6. The derivative of a^x is

   a. a^x
   b. $a^x \ln a$
   c. $\ln a$
   d. None of them

Answers

1	2	3	4	5	6				
a	c	b	a	d	b				

14.5 Hyperbolic Functions

The combination of e^x and e^{-x} give a new function, known as **hyperbolic function.** The hyperbolic sine is defined as

sinh x = . . . (1)

and hyperbolic cosine is defined as

cosh x = . . . (2)

The hyperbolic functions are related to the hyperbola in much the same way that the trigonometric functions are related to the circle.

The functions defined in (1) and (2) have properties similar to the trigonometric functions sin x and cos x .

Therefore

tanh x = = , $x \in$ (

cosech x = = , $x \in$ (

coth x = = , $x \in$ (– {0}

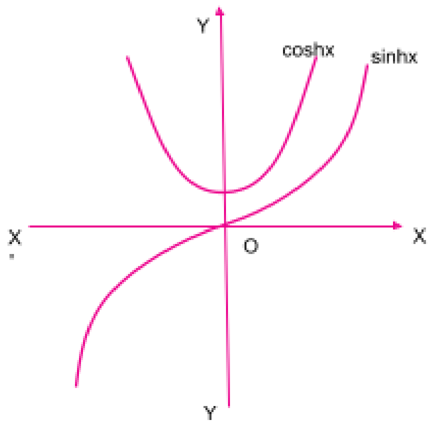


Fig. 1

Also, cosh x + sinh x = e^x

cosh x – sinh x = e^{-x}

It is important to note that

cos x = and sin x =

∴ cosh x = cos ix (∵ i^2 = – 1)

sinh x = – i sin x

Moreover, sinh 0 = 0 and cosh 0 = 1

Also, sinh (–x) = – sinh x, cosh (–x) = cosh x and $\cosh^2 x - \sinh^2 x = 1$

We also have

sin h x = = ∞

sin h x = = – ∞

Similarly, cosh x = ∞, which can easily be seen in the graphs of Figure 1.

Also, tanh 0 = 0 and tanh (–x) = – tanh x

Now

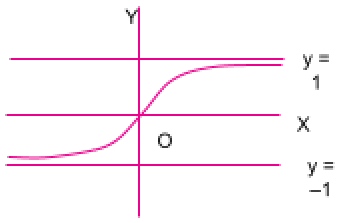
tanh x =

= = 1.

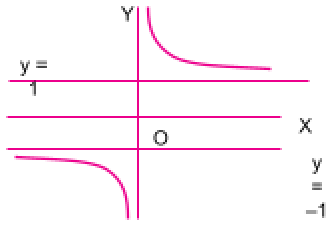
and tanh x =

= = – 1.

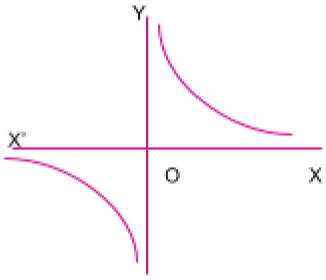
The graph of y = tanh x is shown in fig 2. Similarly other graphs are shown in figs 3, 4 and 5.



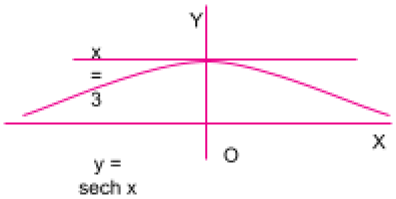
y = tanh x
Fig. 2



y = coth x
Fig. 3



y = cosech x
Fig. 4



y = sech x
Fig. 5

14.6Derivative of Hyperbolic Functions

Derivative of sinh x

Let y = sinh x, then

= (sinh x) = $(e^x - e^{-x})$ = = cosh x

Similarly, (cosh x) = sinh x.

Derivative of tanh x

Let y = tanh x =

∴ = = = $\text{sech}^2 x$

Hence, (tanh x) = $\text{sech}^2 x$

Similarly, (coth x) = – $\text{cosech}^2 x$

(sech x) = – sech x · tanh x

(cosech x) = – cosech x · coth x.

Derivative of Inverse Hyperbolic Functions

Derivative of $\sinh^{-1} x$

Let $y = \sinh^{-1} x$

then $x = \sinh y$ and

$= \cosh y$

$=$

$=$

$\therefore$ $=$

Similarly

$(\cosh^{-1} x) =$, $(x > 1)$

$(\tanh^{-1} x) =$, $(-1 < x < 1)$

$(\coth^{-1} x) =$, $(|x| > 1)$

$(\text{sech}^{-1} x) =$, $(0 < x < 1)$

and $(\text{cosech}^{-1} x) =$, $(x \in$ ⦗ $- \{0\})$.

Illustrative Examples

Example 1. Show that $\sinh^{-1} x = \ln [x +]$. Also, find $[\sinh^{-1} x]$.

Solution

Let $y = \sinh^{-1} x$. Then

$x = \sinh y =$

or, $e^y - 2x - e^{-y} = 0$

or, $e^{2y} - 2xe^y - 1 = 0$, which is quadratic in e^y

$(e^y)^2 - 2xe^y - 1 = 0$

Solving by the quadratic formula, we get

$e^y = \quad = x \pm$

Since $e^y > 0$, but $x - \quad < 0$

So the minus sign is inadmissible and we have

$e^y = x +$

Therefore, $y = \ln (x +)$

Now

$(\sinh^{-1} x) \quad = \quad [\ln (x + \quad)]$

$=$

$=$

$=$

$=$

$\therefore \quad = \ .$

Example 2. If $y = \tanh^{-1} (\sin x)$, then show that $= \sec x$.

Solution

$y = \tanh^{-1} (\sin x)$

Differentiating w.r.t. x, we get

$=$

$= \ \cdot$

$= \ \cos x$

$=$

$=$

$= \sec x$

$\therefore \quad = \sec x.$

Example 3. Find the derivative of 2 $\tanh^{-1}$.

Solution

Let y = 2 $\tanh^{-1}$. Then

=),d \b(tan \f(x,2)))

= 2 · $\sec^2$ ·

= =

= = sec x

∴ = sec x.

Example 4. Find the derivative of $e^{\sinh x}$.

Solution

Let $y = e^{\sinh x}$, then

=

=

= $e^{\sinh x}$ · cosh x

∴ = $e^{\sinh x}$ cosh x.

Example 5. Find the derivative of $x^{\cosh x}$.

Solution

Let $y = x^{\cosh x}$, then ln y = ln $x^{\cosh x}$ = cosh x ln x

Differentiating w.r.t. x, we get

(ln y) = (cosh x ln x)

or, = cosh x + ln x (cosh x)

or, = cosh x · + ln x · sinh x = ·

∴ = .

Example 6. Find the differential coefficient of (sinh x)cosh^{-1} x.

Solution

Let y = (sinh x)cosh^{-1} x, then

ln y = $\cosh^{-1} x$ ln (sinh x)

Differentiating w.r.t. x, we get

= $\cosh^{-1} x$ · + ln (sinh x) ($\cosh^{-1} x$)

or, = $\cosh^{-1} x$ · · cosh x + ln (sinh x) .

or, = $\cosh^{-1} x$ coth x +

or, =

∴ = .

Exercise 14.3

Find the derivative of the followings:

1. ln

2. ln (cosh 3x)

3. $e^{\cosh^{-1} x}$

4. $\text{sech}^{-1} x - \text{cosech}^{-1} x$

5. sech $\tan^{-1} x$

6. $\tan^{-1}$ sinh x

7. $\tanh^{-1}$ 8. $\cosh^{-1}(\sinh x)$ 9. $x \tanh^{-1}$

10. $x^{\cosh}$ 11. 12.

13. $x_{\cosh^2}$ 14.

Answers

1. 2. $3 \tanh 3x$ 3.

4. – 5. 6. $\text{sech } x$

7. 8. 9. $+ \tanh^{-1}$

10. 11.

12. \b\bc\[(\f(x^2,a) coth \f(x,a) + $2x$ ln sinh \f(x,a)) 13.

14. n

Multiple Choice Questions

1. The derivative of $\cosh 3x$ is
   a. $\sinh 3x$ b. $-\sinh 3x$
   c. $3 \sinh 3x$ d. $-3 \sinh 3x$

2. The derivative of $\sinh^{-1}$ is
   a. b.
   c. d.

3. If $y = \tanh^{-1}(\sin x)$, then is
   a. $\sec^2 x$ b. $\sec x$
   c. d.

4. If $y = \ln$, then is
   a. tanh b. tanh
   c. d. – tanh

5. The derivative of $e^{\sinh 2x}$ is
   a. $e^{\sinh 2x}$ b. $e^{\cosh 2x}$
   c. $2 \cosh 2x\, e^{\sinh 2x}$ d. $2\, e^{\sinh 2x}$

Answers

1	2	3	4	5					
c	d	c	a	c					

Application of Derivatives

14.7Differentials and Approximations

If $y = f(x)$, then its derivative with respect to x is

$= f'(x)$ . . . (1)

where is not a fraction. We wish to define the symbols dy and dx so that $f'(x)$ is represented by the ratio of dy to dx. The quantities dx and dy are called **differentials.**

If Δx and Δy are the corresponding increments in x and y, respectively of the function $y = f(x)$, then

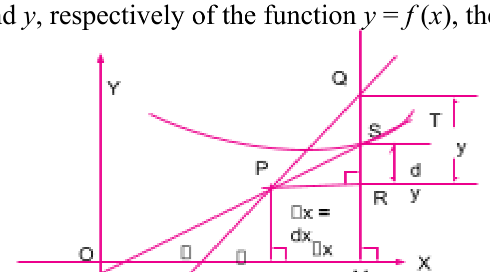
$f'(x) =$

Hence, $= f'(x) + \in$,

with $\in \to 0$ when $\Delta x \to 0$.

Thus

$\Delta y = f'(x)\, \Delta x + \in\Delta x$ . . . (2)



Since $\in$ is numerically small when Δx is small, $f'(x)\, \Delta x$ is called the **differential** of $f(x)$ and is denoted by dy or d$f(x)$ which geometrically represents the amount that the tangent line rises or falls. Thus

$dy = f'(x)\, \Delta x$ . . . (3)

From the figure, tanφ = and slope of the tangent PT is tan $\varphi = f'(x)$,

Thus, $f'(x) =$

$\therefore$ SR $= f'(x)\, \Delta x = dy$

and Δy = QR. The curve in the figure above is concave upward and $dy < \Delta y$. In the case where the graph is concave downward, $dy > \Delta y$.

When $\Delta x \to 0$, dy and dx are almost the same and dy can be calculated as a good estimate of Δy, i.e. dy is the approximate value of Δy. From the figure, it is clear that

$\Delta y - dy = \in\Delta x$ and since $\in \to 0$ as $\Delta x \to 0$

$\Delta y - dy$ is the error

Relative error = $\approx$

and percentage error = 100 × $\approx$ 100 ×

Remarks: Approximate change in y : $dy = f'(x)\Delta x$

Actual change in y : $\Delta y = f(x + \Delta x) - f(x)$

Error in the estimate : $\Delta y - dy$

Percentage error in the estimate : $\Delta \times 100$

Illustrative Examples

Example 1. If $y = x^2 + x$, find dy.

Solution

If $y = f(x) = x^2 + x$, then $f'(x) = 2x + 1$

$\therefore \quad dy = f'(x)\,dx = (2x + 1)\,dx$

Example 2. If $u = 2 \sin \theta$, find du.

Solution

If $u = 2\sin\theta$, then $du = 2\cos\theta\, d\theta$

Example 3. If $y = 3x^2 - 2x + 1$, find dy when $x = 2$ and $\Delta x = 0.01$.

Solution

If $y = f(x) = 3x^2 - 2x + 1$, then $f'(x) = 6x - 2$ and

$$\begin{aligned} dy &= f'(x)\,\Delta x \\ &= (6x - 2)\,\Delta x \\ &= (6 \cdot 2 - 2)\,0.01 \\ &= 0.1 \end{aligned}$$

Example 4. If $y = f(x) = x^2 + 1$, calculate Δy and dy, if $x = 2$ and $\Delta x = dx = 0.01$.

Solution

$$\begin{aligned} \Delta y &= f(x + \Delta x) - f(x) \\ &= [(2.01)^2 + 1] - (2^2 + 1) \\ &= 0.0401 \\ dy &= f'(x)\,dx \\ &= 2x \cdot dx \\ &= 2 \cdot 2\,(0 \cdot 01) \\ &= 0.04 \end{aligned}$$

Example 5. Find the approximate values using differentials, of cube root of 0.009.

Solution

Let $y = x^{1/3}$

Take $x = 0.008$ and $dx = 0.001$ as the nearest number to 0.009 is 0.008 whose cube root can be determined. Then, $x + dx = 0.009$

Now

$$\begin{aligned} \Delta y &= (x + \Delta x)^{1/3} - x^{1/3} \\ &= (0.009)^{1/3} - (0.008)^{1/3} \\ &= (0.009)^{1/3} - 0.2 \end{aligned}$$

$$\therefore \quad \Delta y + 0.2 = (0.009)^{1/3} \qquad \ldots (1)$$

and

$dy = dx = \cdot dx$

$= x^{-2/3}(0.001)$

$= (0.008)^{-2/3}(0.001)$

$= . (0.001)$

$= =$

$= 0.008$

Hence, $(0.009)^{1/3}$ is approximately equal to $0.2 + 0.008 = 0.208$.

Example 6. A circular metal plate expands under heating so that it increases by 2%. Find the approximate increase in the area of the plate if the radius of the plate before heating is 10 cm.

Solution

Let S be the area and r be the radius of the circular plate, then

$S = \pi r^2 \Rightarrow \ = 2\pi r$

Given $dr = 2\%$ of $10 = \ = 0.2$

ΔS is approximately equal to dS and

$dS = dr = 2\pi r \cdot dr = 2\pi \times 10 \times 0.2 = 4\pi$ sq. cm.

Hence, approximate increase in the area of the plate is 4π sq. cm.

Example 7. If $y = x^4 - 10$ and if x changes from 2 to 1.99, what is the exact and approximate change in y?

Solution

Let $y = x^4 - 10$, then $\Delta y = [(1.99)^4 - 10] - (2^4 - 10) = -0.3176$

is the exact change in y

Again

$= \ = 4x^3$

At $x = 2$, $\ = 4(2)^3 = 32$

As x changes from 2 to 1.99, we have

$x = 2$ and $x + \Delta x = 1.99$

$\Rightarrow \quad \Delta x = -0.01$

Now, $dy = \ \Delta x \ = 32(-0.01) = -0.32$

Hence, approximate change in the value of y is -0.32.

Example 8. If $y = \sin x$ and x changes from to . What is the approximate change in y?

Solution

Let $y = \sin x$. Then $\ = \cos x$

As x changes from to , we have

$x =$ and $x + \Delta x = \ \Rightarrow \ \Delta x = \ - x$

$\therefore \quad \Delta x = \ - \ .$

Now

$dy = \ \Delta x = \cos x \cdot \Delta x = \cos \ = 0$

Hence, approximate change in y is zero, i.e. there is no change in y.

Example 9. Find the approximate increase in the surface area of a cube if the edge increases from 10 to 10.01 cm. Calculate the percentage error in the use of differential approximation also compare the two values.

Solution

Let a side of a cube $x = 10$ cm

$\Delta x \ = dx = 10.01 \text{ cm} - 10 \text{ cm} = 0.01 \text{ cm}$

Surface area = $S = 6x^2$

Now, $= \ = 12x$

$dS = 12x \times dx = 12 \times 10 \times 0.01 = 1.2 \text{ cm}^2$

The approximate increase in area = 1.2 cm^2

Again, $S = 6x^2$,

$S + \Delta S = 6(x + \Delta x)^2$

$\therefore \quad \Delta S = 6(x + \Delta x)^2 - 6x^2$

$= 6(10.01)^2 - 6 \times (10)^2$

$= 1.2006$

The actual increase in area = 1.2006 cm^2

Error = Actual increase in area – Approximate increase in area

$= 1.2006 - 1.2$

$= 0.0006$

% Error = $\times 100 = 0.0001$ %

The comparison of two areas

$= \ =$

i.e. $ds : \Delta S = 0.995 : 1$.

Example 10. Find the approximate change necessary in the radius of a spherical pot in order to increase the volume by 10 cubic cms if the radius of the pot is 20 cms.

Solution

We have, $v = \pi r^3$, $r = 20$, $\Delta v = 10$, then $dv \approx 10$.

Since $dv = \pi. 3r^2 \, dr = 4\pi r^2 dr$, we have

$10 = 4\pi \cdot 20^2 \cdot dr$

or, $dr = \ = 0.001999 = 0.002$ cm

Hence, the required change in r is approximately 0.002 cm.

Exercise 14.4

1. Find the differential dy in each of the followings:

a. $y = x^3 + 2$ b. $y = 2t^2 + t + 1$ c. $y = (x + a)^2$

2. In each of the followings, find Δy and dy:

a. $y = x^3 + 3$, for $x = 2$ and $\Delta x = 0.1$

b. $y =$, for $x = 4$ and $\Delta x = 0.41$

3. If $y = x^2 - 3x$, find $\Delta y - dy$ in terms of x and Δx.

4. What is the exact change in the value of $y = x^2$ when x changes from 10 to 10.1? What is the approximate change in y?

5. The edge of a cube increases from 10 cm to 10.025 cm. Find the approximate increments in the volume and the surface area of the cube.

6. Use differentials to approximate the change in x^3 as x changes from 5 to 5.01.

7. A circular copper plate is heated so that its radius increases from 5 cm to 5.06 cm. Find the approximate increases in area and also the actual increase in area.

8. Find the approximate increase in the volume of a sphere when its radius increases from 2 to 2.1. Find also the actual increase in volume and compare these two volumes.

9. Find the approximate change in the volume of a cube of side x cm caused by increasing the sides by 1%.

10. If the radius of a sphere changes from 3 cm to 3.01 cm, find the approximate increase in its volume.

11. If the radius of a circle is increased from 5 cm to 5.1 cm, find the approximate increase in area.

Answers

1. a. $dy = 3x^2\,dx$ b. $dy = (4t + 1)\,dt$ c. $dy = 2(x + a)\,dx$
2. a. 1.261, 1.2 b. 0.105, 0.1
3. $(\Delta x)^2$ 4. $\Delta x = 2.01$, $dy = 2$ 5. 7.5 cm^3, 3 cm^2 6. 0.75
7. 0.6π, 0.6036π 8. 9. 3%
10. 0.36π cm^3 11. π cm^2

Multiple Choice Questions

1. If $y = f(x)$, then the approximate change in y is
   - a. $\Delta y = f(x + \Delta x) - f(x)$
   - b. $dy = f'(x)\,\Delta x$
   - c. $\Delta y - dy$
   - d. None of them
2. For the function $y = f(x)$, percentage error in the estimate is given by
   - a. $\Delta \times 100$
   - b. $\times 100$
   - c. $\Delta \times 100$
   - d. None of them
3. If $y = f(x) = x^2 + 1$, then the actual change in y when $x = 1$ and $\Delta x = dx = 0.1$ is
   - a. 0.21
   - b. 0.2
   - c. 0.01
   - d. 0.5
4. If the radius of circle is increased from 10 to 10.1 cm, then the approximate increase in area is
   - a. 100 π cm^2
   - b. 102.01 π cm^2
   - c. 0.01 π cm^2
   - d. 2 π cm^2
5. If the radius of a sphere changes from 2 to 2.1 cm, then the approximate increase in its surface area is

a.	$16\pi\ cm^2$	b.	$1.6\pi\ cm^2$
c.	$17.64\pi\ cm^2$	d.	$1.64\pi\ cm^2$

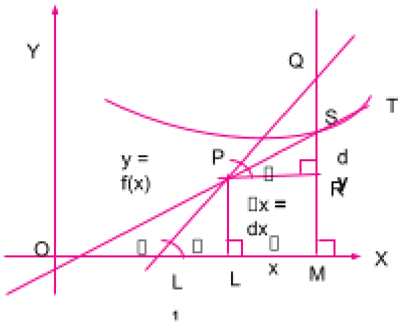
Answers

1	2	3	4	5					
b	c	a	d	b					

14.8Tangents and Normals

To find equations of tangents and normals, it is necessary to understand the geometrical meaning of the derivative

Let $y = f(x)$ be a function represented by the curve as in the adjoining figure. Let $P(x, y)$ be a fixed point and $Q(x + \Delta x, y + \Delta y)$ be a neighbouring point on the curve. From P and Q, draw PL and QM perpendiculars to the x-axis. Also, draw perpendicular PR on QM. From the figure, $QR = \Delta y$ and $PR = LM = \Delta x$



Slope of the secant PQ = tanθ = = ,

where $\angle QL_1M = \angle QPR = \theta$

The ratio is called difference quotient, where

=

Now, when Q → P, $\Delta x \to 0$, $\Delta y \to 0$, $\theta \to \varphi$, PT will be tangent at P. Thus, the slope of tangent PT is

= = tan φ, if the limit exists

By the definition of derivative, $f'(x) = \tan\varphi$ is the slope of the tangent PT

Hence, the slope of the tangent to the curve $y = f(x)$ at a point of tangency (x, y) is given by the value of the derivative of the function $f(x)$ with respect to x, at any point x.

NOTE

1. The derivative of $f(x)$ at $x = a$ is written as

   $f'(a) =$

2. The slope of the tangent to a curve is sometimes called the slope of the curve
3. If the tangent is parallel to x axis, then $\varphi = 0$, then = 0
4. If the tangent is perpendicular to x axis, then $\varphi = 90°$

∴ = tan φ = tan 90° = ∞, i.e. = 0

Example 1. Find the slope of the curve of the function $f(x) = x^2 + x$ at $x = 3$.

Solution

The slope of the curve $y = f(x) = x^2 + x$ at $x = 3$ is

= = $2x + 1$

∴ At $x = 3$, = $2 \cdot 3 + 1 = 7$.

Equations of Tangents and Normals

Let $y = f(x)$ be the equation of the curve and $P(x_1, y_1)$ be any point on the curve. If a line passes through $P(x_1, y_1)$, then the equation of the line with slope m is

$y - y_1 = m(x - x_1)$.

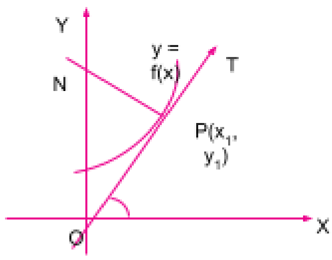
This line becomes the tangent at $P(x_1, y_1)$ if

$m =$

Hence, the equation of the tangent to the curve

$y \quad = f(x)$ at $P(x_1, y_1)$ is

$y - y_1 = \ (x - x_1)$ . . . (1)



The normal at the point $P(x_1, y_1)$ is the line passing through $P(x_1, y_1)$ and perpendicular to the tangent at P.

If m_1 is the slope of the normal at $P(x_1, y_1)$, then

$m_1 \quad = -1$

or, $\quad m_1 = -$

Hence, the equation of normal to the curve $y = f(x)$ at $P(x_1, y_1)$ is

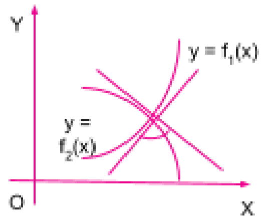
$y - y_1 = - \quad (x - x_1)$ . . . (2)

Tangent at the Origin

Tangent at any point (x_1, y_1) is $y - y_1 = \ (x - x_1)$ and hence the tangent at (0, 0) is $y = \ x$.

Tangents at the origin are obtained by equating the lowest degree terms to zero in the equation of the curve.

Angle of Intersection of Two Curves



The angle of intersection of two curves at a point of intersection is the angle between the tangents to the two curves at the point. If m_1 and m_2 are the values of for the two curves at a point of their intersection, then θ, the angle of intersection of the curves is given by

$\tan \theta = \pm$

The two curves will cut orthogonally or at right angle if $m_1 m_2 = -1$.

Illustrative Examples

Example 1. Find the equations of the tangent and normal to the curve $y = 2x^2 - 3x - 1$ at (1, – 2).

Solution

Equation of the curve is $y = 2x^2 - 3x - 1$

∴ $\quad = 4x - 3$

This is the slope of the tangent at (x, y).

Now, the slope at the point (1, – 2) is $m = 4 \times 1 - 3 = 1$

∴ Equation of the tangent at (1, – 2) is

$y - y_1 = m(x - x_1)$

or, $y + 2 = 1\ (x - 1)$

or, $x - y - 3 = 0$

Slope of the normal $(m_1) = -1$

Equation of the normal at (1, – 2) is

$y + 2 = -1(x - 1)$

∴ $x + y + 1 = 0$.

Example 2. Find the tangents at the origin to the curves:

a. $x^2y^2 = a^2(x^2 - y^2)$ b. $y^2 = 4ax$ c. $x^3 + y^3 = 3axy$

Solution

As the curve passes through origin, equating to zero the lowest degree terms in each case, we get

a. $a^2(x^2 - y^2) = 0$

or, $y = \pm x$

b. $4ax = 0$

or, $x = 0$ i.e. y axis

c. $3axy = 0$

or, $x = 0, y = 0$ i.e. x-axis and y-axis.

Example 3. Find the equation of the tangent and normal to the curve $y = x^2 + 4x + 1$ at the point whose abscissa is 3.

Solution

Equation of the curve is $y = x^2 + 4x + 1$

We have to find out the equation of the tangent at $(3, y)$. Therefore, the ordinate y of the point whose abscissa is 3 is

$y = 3^2 + 4 \times 3 + 1 = 22$

The given point on the curve is (3, 22)

Now

$= 2x + 4$

The slope of the tangent at the point (3, 22) is

$= 2 \times 3 + 4 = 10$

Equation of the tangent at (3, 22) is

$y - 22 = 10\ (x - 3)$

or, $10x - y - 8 = 0$

Now, slope of the normal at (3, 22) is –

Equation of the normal at (3, 22) is

$y - 22 = \ (x - 3)$

or, $10(y - 22) = -(x - 3)$

$\therefore\ x + 10y - 223 = 0.$

Example 4. Determine the points on the curve $2y = (3 - x^2)$ at which the tangent is parallel to the line $x + y = 0$.

Solution

Equation of the curve is

$2y = 3 - x^2$ . . . (1)

Slope of the line $x + y = 0$ is – 1

Differentiating (1) w.r.t. x, we get

$2 = -2x$

or, $= -x$

Since the tangent is parallel to the given line, we have

Slope of tangent = Slope of line = – 1

or, $-x = -1$

$\therefore\ x = 1$

Putting $x = 1$ in (1), we get

$2y = (3 - 1)$

$\therefore\ y = 1$

Hence, the required point is (1, 1).

Example 5. Find the angle of intersection of curves $xy = 6$ and $x^2y = 12$.

Solution

Given

$xy = 6$ . . . (1)

and $x^2y = 12$

$\therefore \quad y =$ . . . (2)

From (1) and (2), we get

$x^2 \ = 12$

or, $\quad x = 2$

Putting $x = 2$ in (1), we get $y = 3$.

$\therefore$ The point of intersection is (2, 3)

Differentiating (1) w.r.t. x, we get

$y + x \ = 0$

$\therefore \quad m_1 = \ = \ =$ at the point (2, 3)

Differentiating (2) w.r.t. x, yields

$2xy + x^2 \ = 0$

$\therefore \quad m_2 \quad =$

$= \ = -3$ at (2, 3)

Let θ be the angle between the two curves, then

tan θ $=$

$=$

$=$

$\therefore$ The angle between the two curves (θ) = $\tan^{-1}$.

Example 6. Find the equations of horizontal and vertical tangents to the curve $x^2 + y^2 - xy - 27 = 0$.

Solution

The given curve is

$x^2 + y^2 - xy - 27 = 0$ . . . (1)

Differentiating (1) w.r.t. x, we get

$$2x + 2y \quad - x \quad - y = 0$$

$\therefore \quad =$

For the horizontal tangent, i.e. the tangent parallel to the x axis

$$= 0$$

$\therefore \quad y = 2x \qquad \qquad \qquad \qquad \qquad \qquad . . . (2)$

Solving (1) and (2), the points of horizontal tangents are (3, 6) and (– 3, – 6)

Hence, the equations of the horizontal tangents are

$$y - 6 = 0(x - 3)$$

$\therefore \quad y = 6$

and $y + 6 = 0(x + 3)$

$\therefore \quad y = -6$

For the vertical tangent, i.e., the tangent parallel to the y axis

$$= 0$$

or, $\quad x = 2y \qquad \qquad \qquad \qquad \qquad \qquad . . . (3)$

Solving (1) and (3), the points of vertical tangents are (6, 3) and (– 6, – 3)

Hence, the equations of the vertical tangents are

$$y - 3 = \quad (x - 6)$$

$\therefore \quad x = 6$

and $y + 3 = \quad (x + 6)$

$\therefore \quad x = -6.$

Example 7. Find the equation of the normal line to the curve $x^2 + y^2 = 16$ which makes an angle of 30° with the positive direction of x axis.

Solution

Equation of the curve is

$x^2 + y^2 = 16 \qquad \qquad \qquad \qquad \qquad \qquad . . . (1)$

Differentiating (1) w.r.t. x, we get

$$2x + 2y \quad = 0$$

or, $\quad = -$

$\therefore$ Slope of the tangent at (x, y) is – and the slope of the normal is .

Since the normal makes an angle of 30° with the x-axis, Slope of the normal = tan 30° =

or, =

or, $x = y$ . . . (2)

From (1) and (2), we get

$3y^2 + y^2 = 16$

or, $y = \pm 2$ . . . (3)

From (2) and (3), we conclude that there are two points on the curve where the normal makes an angle 30° with the x axis. The points are (2, 2) and (– 2, – 2)

The equation of normal at (2, 2) is

$y - 2 = (x - 2)$

or, $y - 2 = x - 2$

∴ $x - y = 0$

The equation of normal at (– 2, – 2) is

$y - (-2) =$ $[x - (-2]$

or, $y + 2 = x + 2$

∴ $x - y = 0$.

Example 8. Show that + = 1 touches the curve $y = b$e at the point where the curve crosses the axis of y.

Solution

The point where the curve $y = be$ cuts the y axis, i.e., $x = 0$ is $y = b\, e^0 = b$

Now

= – e

At $x = 0$

= – e^0 = –

The equation of the tangent at the required point (0, b) is given by

$y - b = -$ $(x - 0)$

or, $ay - ab = -bx$

or, = 1

∴ + = 1.

Example 9. Show that the tangents to the curve $x^3 + y^3 - 3axy = 0$ at the points where it meets the parabola $y^2 = ax$ are parallel to y-axis.

Solution

Equation of the curve is

$x^3 + y^3 - 3axy = 0$ . . . (1)

Differentiating (1) w.r.t. x, we get

$3x^2 + 3y^2 \quad - 3a \quad = 0$

or, $(3y^2 - 3ax) = 3ay - 3x^2$

∴ = = is the slope of the tangent at (x, y)

If the tangent is parallel to y axis at (x, y), then

= 0

or, $y^2 - ax = 0$

∴ $y^2 = ax$ is the parabola

Hence, the tangents at the common points of the given curve and the parabola $y^2 = ax$ are parallel to y axis.

Exercise 14.5

1. Find the slope and the inclination with the x-axis of tangent of

a. $y = -3x - x^4$ at $x = -1$.

b. $x^2 + y^2 = 25$ at $(0, 5)$.

2. Show that the tangents to the curve $y = 2x^3 - 3$ at the points $x = 2$ and $x = -2$ are parallel.

3. At what angles does the curve $y(1 + x) = x$ cut the x-axis?

4. a. Find the equation of the tangent to the curve $y = -5x^2 + 6x + 7$ at the point $(1, 8)$.

b. Find the equation of the normal to $y = 2x^3 - x^2 + 3$ at $(1, 4)$.

5. Find the equation of the tangent and normal to the given curves at the given points.

a. $y = x^3$ at $(2, 8)$.

b. $y = x^3 - 2x^2 + 4$ at $(2, 4)$.

c. $x^2 - y^2 = 7$ at $(4, 3)$.

d. $x^2 + 3xy + y^2 = 5$ at $(1, 1)$.

6. Find the equation of the tangent line to the curve $y = x^2 + 4x - 16$ which is parallel to the line $3x - y + 1 = 0$.

7. Find the points on the circle $x^2 + y^2 = 16$ at which the tangents are parallel to the (a) x axis (b) y axis.

8. Find the angle of intersection of the following curves.

a. $y^2 = x^3$ and $y = 2x$

b. $x^2 + 4y^2 = 8$ and $x^2 - 2y^2 = 4$

c. $y = x^2$ and $6y = 7 - x^3$ at (1, 1).

9. Find the equation of the tangent and normal to the parabola $y^2 = 4ax$ at $(at^2, 2at)$.

Answers

1. a. b. 0, 0 3. π
4. a. $4x + y - 12 = 0$ b. $x + 4y = 17$
5. a. $12x - y - 16 = 0; x + 12y - 98 = 0$ b. $4x - y - 4 = 0; x + 4y = 18$
 c. $4x - 3y = 7; 3x + 4y = 24$ d. $x + y = 2; x - y = 0$
6. $12x - 4y = 65$
7. a. (0, 4), (0, – 4) b. (4, 0), (– 4, 0)
8. a. $\tan^{-1}$ b. 0 c.
9. $ty = x + at^2; tx + y = 2at + at^3$

Multiple Choice Questions

1. The slope of the tangent to the curve of $f(x) = x^2$ at $x = 1$ is

a. 1 b. 2
c. 3 d. None of them

2. The equation of the normal to the curve $y = x^3 - 2x$ at the point (1, – 1) is

a. $x + y = 2$ b. $x - y = 2$
c. $x + y = 0$ d. $x - y = 0$

3. The inclination with the x-axis of tangent of $x^2 + y^2 = 16$ at (0, 4) is

a. π b. 0
c. π d. π

4. The equation of the tangent to the curve $x^2 - y^2 = 7$ at (4, 3) is

a. $4x - 3y = 7$ b. $4x + 3y = 7$
c. $3x - 4y = 24$ d. $3x + 4y = 24$

5. The tangents at the origin to the curve $x^2y^2 = a^2(x^2 - y^2)$ is

a. $y = 0$ b. $x = 0$
c. $x = 0, y = 0$ d. $y = x, y = -x$

Answers

1	2	3	4	5					
b	c	b	a	d					

14.9L' Hospital's Rule

We first discuss about indeterminate forms.

Indeterminate Forms

The forms , ∞∞, 0 × ∞, ∞ – ∞, 0^0, ∞^0, 1^∞ are known as indeterminate forms.
In this chapter, we will know the methods of evaluating the limits of the indeterminate forms through the use of differentiation and expansion in series.

Indeterminate Form

L' Hospital's Rule: If φ(x) and ψ(x) and their derivatives φ'(x) and ψ'(x) are continuous at $x = a$ and if φ(a) = ψ(a) = 0, then

φψ= φψ= φψ provided ψ'(a) ≠ 0.

Proof: By Taylor's theorem

φ(x) = φ(a) + (x – a) φ'(a) + φ"(a) + · · ·

ψ(x) = ψ(a) + (x – a) ψ'(a) + ψ"(a) + · · ·

∴ φψ= φφφ · · ·ψψψ · · ·

Now, φ(a) = ψ(a) = 0

∴ φψ= φφ · · ·ψψ · · ·

∴ φψ= φψ= φψ.

NOTE If φ(x) and ψ(x) are functions such that φψtakes the indeterminate form and the functions φ "(x) and ψ "(x) satisfy the conditions of the L'Hospital's theorem, then

φψ= φψ= · · · = φψ.

The Indeterminate form ∞∞

If φ(x) = ∞ and ψ(x) = ∞, then

φψ= φψ

the same rule as that for evaluating the indeterminate form .

NOTE We try to reduce the different forms of indeterminate forms either to the form or to the form ∞∞, so that the theorem can be applied to evaluate the limit of the function.

Illustrative Examples

Example 1. Evaluate , using L'Hospital's rule.

Solution

= ·

=

=

= .

Example 2. Evaluate ∞, using L'Hospital's rule.

Solution

∞ $\infty\infty$

$=$ ∞ $\infty\infty$

$=$ ∞

$=$

$=$.

Example 3. Evaluate , using L'Hospital's rule.

Solution

$=$

$=$

$=$

$=$

$=$

$=$.

Example 4. Evaluate , using L'Hospital's rule.

Solution

$\infty\infty$

$=$ $\cdot$

$=$

$= - \cdot \sin x^2$

$= -1 \cdot 0$

$= 0.$

Example 5. Evaluate , using L'Hospital's rule.

Solution

$=$

$=$ $\cdot$ $\cdot$

$=$

$=$

$=$ $=$.

Example 6. Evaluate , using L'Hospital's rule.

Solution

$=$

$=$

$=$

$=$

=
=
=
=
= –
= – .

Exercise 14.6

1. Evaluate by using L'Hospital's rule:

a.
b.
c.
d.
e. π π
f.
g.
h. .

2. Evaluate by using L'Hospital's rule:

a. ∞
b. πππ
c. π
d. π
e. ∞ (*n* being a positive integer)
f. .

Answers

1. a. b. na^{n-1} c. d. e. 1
f. 2 g. 2 h. 2

2. a. b. 3 c. 0 d. 3 e. 0
f. 1

Multiple Choice Questions

1. Which of the following is not an indeterminate form?

a. ∞∞
b.
c. ∞ + ∞
d. ∞^0

2. The value of →∞is

a. 2
b. 0
c. 1
d. None of them

3. The value of →is

a. 2
b. 0
c. 1
d. None of them

4. The value of →πππis

a. 3
b.
c. 0
d. π

5. The value of →is

a. 0

b. – 1

c. 1

d. None of them

Answers

1	2	3	4	5					
c	b	a	b	c					

14.10 Mean Value Theorem

Intervals

A number x is said to belong to

1. a closed interval $[a, b]$ if $a \leq x \leq b$
2. an open interval (a, b) if $a < x < b$

Upper and Lower Bound

Suppose a function $f(x)$ is bounded in $a \leq x \leq b$. If M and m are the upper and lower bounds, then M has the properties:

1. $f(x) \leq M$ for all x in $a \leq x \leq b$
2. $f(x) > M - \in$, $\in > 0$ for at least one value of x in $a \leq x \leq b$

and m has the properties:

1. $f(x) \geq m$ for all $a \leq x \leq b$
2. $f(x) < m + \in$, $\in > 0$, for at least one value of x in $a \leq x \leq b$.

NOTE If a function $f(x)$ is continuous in a closed interval $[a, b]$, then $f(x)$ is bounded and attains its bounds which then becomes the maximum and minimum values of the function in $[a, b]$.
In other words, if $f(x)$ is continuous in the closed interval $[a, b]$, then there exist real numbers m and M such that $m \leq f(x) \leq M$ for all $x \in [a, b]$ and $f(c) = m$, $f(d) = M$ for $c, d \in [a, b]$.

We now state and prove two standard theorems, namely Rolle's and Lagrange's mean value theorem. The proofs of these theorems are not in the syllabus.

Rolle's Theorem

If $f(x)$ be a function defined in $[a, b]$ such that

1. $f(x)$ is continuous in $[a, b]$
2. $f(x)$ is differentiable in (a, b)
3. $f(a) = f(b)$

then there exists at least one point $c \in (a, b)$ such that $f'(c) = 0$

Proof: Since $f(x)$ is continuous in $[a, b]$, it is bounded and attains its bounds at least once in $[a, b]$. Let m and M be the lower and upper bounds of $f(x)$ in $[a, b]$

Case I: If $M = m$, then $f(x) = M = m$ for every value of x in the interval, i.e. $f(x)$ is constant in $[a, b]$

$\therefore$ $f'(x) = 0$, for all x in $[a, b]$.

Thus the theorem is true in this case.

Case II: If $M \neq m$, i.e., $f(x)$ is not constant in $[a, b]$.

Let the upper bound $M \neq f(a) = f(b)$. Suppose $f(c) = M$ for c in (a, b). The number c, being different from a and b belongs to the open interval (a, b) and the function which is differentiable in the open interval, is derivable at $x = c$ i.e., $f'(c)$ exist in (a, b).

$\therefore$ $f'(c) =$ . . . (i)

Since $f(c)$ is the greatest value

$f(c) > f(c + h)$ for positive as well as negative values of h

$\therefore$ < 0 if $h > 0$ . . . (ii) and > 0 if $h < 0$ . . . (iii)

From (ii) and (iii), making $h \to 0$ through positive values and through negative values, we get

$f'(c) \leq 0$ and $f'(c) \geq 0$

$\therefore$ $f'(c) = 0$

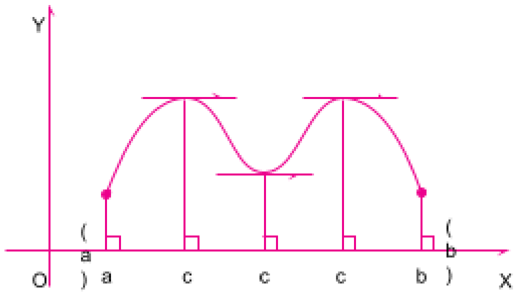
Similarly, if the lower bound m is attained at c, $f'(c) = 0$.

Failure of Rolle's Theorem

If any of the conditions of Rolle's theorem be not satisfied, then the Rolle's theorem will not be true.

Geometrical Interpretation of the Rolle's Theorem

If the graph of $y = f(x)$ continuously drawn between two points $x = a$ and $x = b$ such that tangent can be drawn at each point on the curve between $x = a$ and $x = b$ and $f(a) = f(b)$, then there exists at least one point $c \in (a, b)$ on the curve at which the tangent is parallel to the x-axis.



Illustrative Examples

Example 1. Verify Rolle's theorem for the function

a. $f(x) = x^2 - 5x + 7$ in the interval [2, 3].

b. $f(x) = \sin x$ in $[0, \pi]$.

Solution

a. We have

$f(x) = x^2 - 5x + 7$

i. Here, $f(x)$ being a polynomial function is continuous on $\mathbb{R}$, so it is continuous in [2, 3]

ii. Also, $f'(x) = 2x - 5$ which exists for all $x \in (2, 3)$. So $f(x)$ is differentiable in (2, 3)

iii. $f(2) = f(3) = 1$

Hence, there exist at least one point $c \in (a, b)$ such that $f'(c) = 0$. We have to determine this c

Now

$f'(c) = 0$ implies

$2c - 5 = 0$

or, $c = \ \in (2, 3)$

Hence, the Rolle's theorem is verified.

b. Here, $f(x)$ satisfies the followings:

i. $f(x)$ is continuous in $[0, \pi]$

ii. $f'(x) = \cos x$ which exists for all $x \in (0, \pi)$. So $f(x)$ is differentiable on $(0, \pi)$

iii. $f(0) = f(\pi) = 0$

Hence, there exist one point $c \in (0, \pi)$ such that $f'(c) = 0$. We have to find this c.

Now

$f'(c) = 0$ implies

$\cos c = 0$

or, $c = \pi \in (0, \pi)$.

Hence, the Rolle's theorem is verified.

Example 2. Show that there is no real number p for which the equation $x^2 - 3x + p = 0$ has two distinct roots in $[0, 1]$.

Solution

Suppose that there is a real number p for which the given equation has two distinct roots α and β in $[0, 1]$ where $\alpha < \beta$

Consider $f(x) = x^2 - 3x + p$ in $[\alpha, \beta]$

Since α and β are roots of $f(x)$, $f(\alpha) = f(\beta) = 0$. Also, $f(x)$ being polynomial function is continuous and differentiable on (α, β), the conditions of Rolle's theorem are satisfied. Hence, there exist one point $c \in (\alpha, \beta)$ such that $f'(c) = 0$

Now

$f'(x) = 2x - 3$

$f'(c) = 0$

or, $2c - 3 = 0$

or, $c = \notin (0, 1)$

Also, $(\alpha, \beta) \subset (0, 1)$ is not true, a contradiction to the supposition

Hence, there is no real number p having two distinct roots in $[0, 1]$.

Lagrange's Mean Value Theorem

Let $f(x)$ be a function defined in $[a, b]$ such that

1. $f(x)$ is continuous in $[a, b]$
2. $f(x)$ is differentiable in (a, b),

then there exists at least one $c \in (a, b)$ such that

$$f(b) - f(a) = (b - a) f'(c)$$

Proof: Consider the function

$$\varphi(x) = f(x) + Ax \qquad \ldots \text{(i)}$$

where A is a constant to be so chosen that

$$\varphi(a) = \varphi(b).$$

Thus $\varphi(a) = f(a) + Aa$ . . . (ii)

and $\varphi(b) = f(b) + Ab$ . . . (iii)

From (ii) and (iii)

$$\varphi(a) = \varphi(b)$$

or, $f(a) + Aa = f(b) + Ab$

or. A = –

Therefore from (i), we have

$\varphi(x) = f(x) - \cdot x$.

Now, $\varphi(x)$ is continuous in $[a, b]$ and derivable in (a, b) and $\varphi(a) = \varphi(b)$. Thus, $\varphi(x)$ satisfies Rolle's theorem. Hence, there exist a point $c \in (a, b)$ such that $\varphi'(c) = 0$,

i.e., $f'(c) - \; = 0$

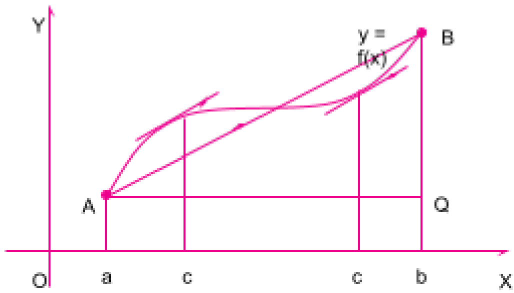
or, $f'(c) =$

or, $f(b) - f(a) = (b - a) f'(c)$

Hence, the theorem is proved.

NOTE This theorem is the generalization of Rolle's theorem and is usually known as "the first mean value theorem" or "the law of the mean." It is named after the French Mathematician Joseph Louis Lagrange (1736 – 1813).

Geometrical Interpretation of Lagrange's Theorem



On the graph of $y = f(x)$, let A and B be the end points at $x = a$ and $x = b$, then

slope of the chord AB =

=

and the slope of the tangent to the curve at $x = c$ is $f'(c)$

Now, $f'(c) =$ means

the slope of the chord is equal to the slope of the tangent

Thus, if a curve is continuously drawn between two points A and B such that the tangent can be drawn at each point of the curve between A and B, then there exist at least one point of the curve where the tangent is parallel to the chord AB.

NOTE It is usual to take the interval as $(a, a + h)$ and the value of c as $a + \theta h$, where $0 < \theta < 1$.
Thus the mean value theorem can be written as, in another form,
$f(a + h) - f(a) = hf'(a + \theta h)$, where $b - a = h$.

Example 3. Verify Lagrange's Mean value theorem for $f(x) = x^3 - x^2 - 5x + 3$ in [0, 4].

Solution

As a polynomial function, $f(x)$ is continuous in [0, 4] and differentiable in (0, 4).

Hence, there exists c in (0, 4) such that

$= f'(c)$ . . . (1)

Now

$f'(x) = 3x^2 - 2x - 5$ . . . (2)

$f(4) = 4^3 - 4^2 - 5 \cdot 4 + 3 = 31$

$f(0) = 3$

$f'(c) = 3c^2 - 2c - 5$ [from (2)]

Substituting these values in (1), we get

$= 3c^2 - 2c - 5$

or, $7 = 3c^2 - 2c - 5$

or, $3c^2 - 2c - 12 = 0$

or, $c = \pm = \pm$

∴ $c = \in (0, 4)$. [Taking +ve sign]

Hence, the theorem is verified.

Exercise 14.7

1. Verify Rolle's theorem for each of the followings:

a. $f(x) = x^2; x \in [-1, 1]$

b. $f(x) = (x - 2)(x - 3)(x - 4); x \in [2, 4]$

c. $f(x) = x(x - 2)^3; x \in [0, 2]$

d. $f(x) = \sin x; x \in [0, 2\pi]$

e. $f(x) = ; x \in [0, \pi]$

f. $f(x) = ; x \in [-5, 5]$

2. Verify Rolle's theorem for the function $f(x) = \sin x$, $x \in [0, \pi]$. Also find a point on the curve represented by given function where the tangent is parallel to the x-axis.

3. Verify Lagrange's Mean Value Theorem:

a. $f(x) = x(x - 2); x \in [1, 2]$

b. $f(x) = x(x - 1)(x - 2); x \in$

c. $f(x) = (x - 1)(x - 2)(x - 3); x \in [1, 4]$

d. $f(x) = e^x; x \in [0, 1]$

e. $f(x) = ; x \in [1, 4]$

f. $f(x) = \cos x; x \in \pi$

g. $f(x) = \ln x; x \in [1, e]$

h. $f(x) = ; x \in [2, 4]$

4. If the mean value theorem is $f(b) - f(a) = (b - a) f'(x_1)$, find x_1 when $f(x) = x(x - 1)(x - 2)$; $x \in$.

5. a. Using Lagrange's mean value theorem, find the point on the curve $f(x) = x(x - 2)$, the tangent at which is parallel to the chord joining the points $(1, -1)$ and $(4, 8)$.

b. Examine whether the function $f(x) = x^2 - 6x + 1$ satisfies Lagrange's mean value theorem. If it satisfies, then find the coordinates of the point at which the tangent is parallel to the chord joining the points A$(1, -4)$ and B$(3, -8)$.

6. Is Lagrange's Mean Value Theorem applicable to the functions defined below?

a. $f(x) = x \cos$ for $x \neq 0$

$= 0$ for $x = 0$, in the interval $[-1, 1]$.

b. $f(x) = 4 - (6 - x)^{2/3}$ in $[5, 7]$.

Answers

2. π **4.** $x_1 = 0.23$

5. a. b. $(2, -7)$

6. a. Not applicable, as $f'(x)$ does not exist at $x = 0$, where $0 \in (-1, 1)$

b. Not applicable, $f'(x)$ does not exist at $x = 6$, $6 \in (5, 7)$

Multiple Choice Questions

1. "If $f(x)$ is continuous in $[a, b]$, derivable in (a, b) and $f(a) = f(b)$, then there exists at least one point $c \in (a, b)$ such that $f'(c) = 0$." This is the statement of

a. L'Hospital rule
b. Rolle's theorem
c. Lagrange's mean value theorem
d. None of them

2. "If $f(x)$ is continuous in $[a, b]$ and differentiable in (a, b), then there exists at least one point $c \in (a, b)$ such that $f(b) - f(a) = (b - a) f'(c) = 0$." This is the statement of

a. L'Hospital rule
b. Rolle's theorem
c. Lagrange's mean value theorem
d. Generalized mean value theorem

3. If $f(x) = \sin x$ is defined on $[0, \pi]$, then the value of $c \in (0, \pi)$ for which $f'(c) = 0$ is

a. 0
b. π
c. π
d. π

4. In the graph of a continuous curve $y = f(x)$ whose end points are at $x = a$ and $x = b$, the expression is the slope of

a. secant line
b. tangent line
c. any line
d. all of them

5. The point on the curve $f(x) = x^2 - 2$ where the tangent is parallel to the x-axis is

a. (0, 0)
b. (0, – 2)
c. (, 0)
d. (0,)

Answers

1	2	3	4	5					
b	c	d	a	b					

Miscellaneous Exercise

1. Discuss the continuity and discontinuity of the following functions:

a. $f(x) = \cdot$, at $x = 0$

b. $f(x) = |x| + |x - 1|$ at $x = 0, 1$

c. $f(x) =$, at $x = 0$

d. $f(x) =$, at $x = 0$.

2. Find $f(0)$ if the function $f(x) =$, $x \neq 0$ is continuous at $x = 0$.

3. Examine the continuity of the function $f(t) =$ ππππ, at $t =$π.
4. If $f(x) = \neq$, then prove that $f(x)$ is continuous at $x = 0$.
5. If $f(x) =$. Is $f(x)$ differentiable at $x = 1$ and 2?
6. If $x^y + y^x = 2$, find .
7. If $e^x + e^y = x + y$, find .
8. If $y = xx^{x \cdots}$, prove that = .
9. If $y = \cdots$, prove that = .

   Hint: $y =$.
10. If $y = (\cos x)^{\cos x \cos x \cdots}$, prove that = .
11. Find the equation of the normal to the parabola $y^2 = 4ax$ in the form $y = mx - 2am - am^3$, where m is the slope of the normal.
12. Find the equation to the tangent to $x^3 = ay^2$ at $(4am^2, 8am^3)$.
13. Find the equation of the tangent to the curve

    a. $y = \cot^2 x - 2 \cot x + 2$ at $x =$

    b. $y = 2 \sin x + \sin 2x$ at $x =$.

14. Evaluate:

a. π (1 – sin x) tan x

b. (1 – x) tan π

c. sin x ln x^2

d. x^2 ln (x^2)

15. Evaluate:

a.

b.

c.

d. π (sec x – tan x)

e. π

f.

g.

Answer

1. a. Continuous b. Continuous c. Continuous d. Discontinuous
2. 1
3. Continuous
5. Continuous at x = 1, and 2
6. – · , x^y ln x + ·)
7.
11. $y = mx - 2am - am^3$
12. $y = 3mx + 8am^3 - 12am^3$
13. a. $2x - 2y + 2 - \pi = 0$ b. $2y = 3$
14. a. 0 b. π c. 0 d. 0
15. a. 0 b. c. d. 0 e. – ∞
f. g. 0